INTRODUCTION TO FUZZY PROJECT PLANNING WITH FLEXIBLE EXTENDED ADDITION

HEINRICH J. ROMMELFANGER
Institute of Statistics and Mathematics, J.W. Goethe-University Frankfurt
D-60054 Frankfurt am Main, Mertonstraße 17-23
E-mail: Rommelfanger@wiwi.uni-frankfurt.de

In case of modeling vague data in project planning by fuzzy sets or fuzzy intervals, the formulas for determining earliest starting dates and finishing dates can be extended on fuzzy durations. It should however be noticed that the extended subtraction is not the inversion of the extended addition. Therefore, Rommelfanger [9] develops formulas of computation, which are based on extended addition and ranking relations between fuzzy intervals. Numerical examples illustrate that fuzzy earliest and latest dates as well as fuzzy slack times can be calculated efficiently in this way. Furthermore, critical paths can be recognized directly. Due to the aggregation of fuzzy intervals with the usual extended addition, based on the minimum operator, the duration will get fuzzier. In order to permit the fitting of duration, a flexible extended addition, based on the Yager’s T-norm $T_p$, is introduced. The parameter $p$ can be changed in the inter-active process for an optimal project plan.

1 Introduction

Network analysis is a successful instrument for planning, coordinating and controlling complex projects, depending on the time element. It has as well been proved in large-scale projects. However, one has to make sure that the person responsible for planning has an exact conception of the operation structure and will be able to give an exact temporal evaluation of the operational elements. Especially the duration of the jobs is quite often not known at the planning moment, but it must be estimated. In most cases these valuations are not estimations in the stochastic sense, they are rather based on personal experiences, which had been made in former projects under more or less similar conditions. Sometimes there only exist verbal expert statements about the duration, as for example "about 4 hours are usual" or "clearly more than two, but less than three hours". Since the fuzzy set theory provides the possibility to model vague and linguistic formulated durations mathematically, it seems obvious to apply the fuzzy set theory on network analysis.

In former papers about the application of fuzzy sets in project planning, see e.g. [1], [3], [4], it was shown that vague durations can be modeled by fuzzy sets and that the formulas of deterministic models for the calculation of earliest and latest starting dates can be transferred on fuzzy durations. The authors of the above papers however ignored the fact that the extended subtraction is not the inversion of the extended addition. Consequently the retrograde calculation for fixing the latest starting dates and slack times lead to false results and a critical path was no longer
found. Rommelfanger [9] indeed showed that intelligible fuzzy starting dates and slack times can be calculated, if mathematical operations are applied that are based on extended addition and order relations between fuzzy numbers. Furthermore, as analogous to classical network analysis, critical paths can be recognized. So far literature does not mention the problem that in case of extended addition the durations get fuzzier, since an equalization of the time divergences between the different jobs is not taken into account. Therefore, in this paper we will discuss whether by means of the flexible extended addition, based on the Yager T-Norm $T_p$, a more realistic overall duration could be calculated. Moreover, it should be discussed how the values of the parameter $p$ could be defined.

2. Modeling vague data by fuzzy sets

Though this point is a basic criteria for the acceptance of fuzzy models, the problem of practical determination of membership values is hardly mentioned in literature. In case of practical application it is most unlikely that a decision maker will be able to describe the exact shape of the membership function. The expense would be too high and the definition has to be adjusted continuously to the changes in information. Therefore, in literature examples almost exclusively use fuzzy numbers or fuzzy intervals of the L-R type with simple reference functions. Quite often basic figures will be given to the decision maker, which will then be adjusted to the individual requirements by changing few parameters. Especially in case of modeling fuzzy intervals with linear reference functions the definition of arguments for smaller membership values is often very difficult.

In order to approximately characterize membership functions of a fuzzy duration we recommend the following approach:

At first the decision maker fixes some prominent membership values, relates them to special meanings and specifies to each of these special membership degrees a crisp interval $D^\alpha_{ij}$. Suitable criteria for the duration $D^\alpha_{ij}$ may be:

- usual duration $D^\alpha_{ij} = [d_{ij}, d_{ij}]$, i.e. the most probably expected duration,
- "expected duration" $D^\lambda_{ij} = [d_{ij}^\lambda, d_{ij}^\lambda]$, i.e. data of the interval $D^\lambda_{ij}$ should be taken into consideration,
- "realistic" duration $D^\varepsilon_{ij} = [d_{ij}^\varepsilon, d_{ij}^\varepsilon]$, i.e. data out of the interval $D^\varepsilon_{ij}$, are quite unrealistic and should be neglected. Such durations may occur, but they have little chance of realization.

These selected intervals can be interpreted as $\alpha$-level set $D^\alpha_{ij}$ of the fuzzy interval $\tilde{D}_{ij}$, where $\lambda$ has to be a medium value, e.g. $\lambda = 0.5$, and $\varepsilon$ has to be a small value, e.g., $\varepsilon = 0.05$. 
A suitable approach to the piecewise linear membership function of $\tilde{D}_{ij}$ is then the polygon line connecting the points $(d_{ij}^\varepsilon,\varepsilon), (d_{ij}^\lambda,\lambda), (d_{ij}^\lambda,1), (d_{ij},1), (d_{ij},\lambda), (d_{ij}^\varepsilon,\varepsilon)$, see Figure 1.

![Figure 1: Membership function of $\tilde{D}_{ij} = (d_{ij}^\varepsilon,\varepsilon), (d_{ij}^\lambda,\lambda), (d_{ij}^\lambda,1), (d_{ij},1), (d_{ij},\lambda), (d_{ij}^\varepsilon,\varepsilon)$](image)

Off course the specification of the interval limits $d_{ij}^\varepsilon, d_{ij}^\lambda, d_{ij}^\lambda, d_{ij}$ and $d_{ij}$ is difficult in practical use. But, compared with deterministic network and the use of time intervals, here the duration is not strictly defined with crisp borders, but softly with interim, so that small mistakes could be easily tolerated.

If we use the symbols $\delta_{ij}^\varepsilon, \delta_{ij}^\lambda, \delta_{ij}^\lambda$ and $\delta_{ij}$ to describe the spreads of $\tilde{D}_{ij}$ according to the membership level $\varepsilon$ and $\lambda$, fuzzy number $\tilde{D}_{ij}$ of the "$\varepsilon$-$\lambda$ type" can be abbreviated as $\tilde{D}_{ij} = (d_{ij},\delta_{ij}^\varepsilon, \delta_{ij}^\lambda, \delta_{ij}^\lambda, \delta_{ij})^\lambda,\varepsilon$.

3. Calculation of earliest and latest starting dates of events in fuzzy CPM-project planning

In order to define a CPM-project plan (Critical Path Method) we refer to the classical procedure of analyzing projects by breaking them down into jobs, events and order relations. Each job is assigned to an initial event and an end event, in which the project must exhibit a well-defined state. Putting the events in order by means of the classification pre-event and after-event we can describe the arrangement of jobs.

Denoting the set of events by $E$ and the set of jobs by $V$ an event $e_1 \in E$ is called pre-event of $E_2 \in E$, if there exists a job $v$ with the initial event $e_1$ and the end event $e_2$. Then $e_2$ is called after-event of $e_1$. For an event $e$, the set of pre-event is symbolized by $P_e$ and $S_e$ symbolize the set of after-events.

The project structure can be visualized by a directed graph plan without cycles, where the nodes correspond to the events and the edges (acres) to the jobs.
For simplifying reasons the nodes are numbered by natural numbers in increasing order on condition that 

\[ i < j \quad \text{for all } i \in P_j \quad \text{and} \quad j < k \quad \text{for all } k \in S_j \]

Then a job with the initial-event i and the end-event j may be symbolized by \((i, j)\) and its duration by \(D_{ij}\).

In deterministic network analysis the earliest date \(E_D_j\) for realizing the event j can be calculated as

\[
E_D_j = \begin{cases} 
\max \{E_D_i \odot D_j \mid i \in P_j\} & \text{für } P_j \neq \emptyset \\
E_D_0 & \text{für } P_j = \emptyset
\end{cases}
\]

where \(E_D_0\) symbolizes the earliest starting date of the project.

Then the earliest date for finishing the project is

\[
E_D_w = \max \{E_D_i \mid i \in E\}.
\]

If the numeration of the nodes was done correctly, \(w\) is the greatest natural number in this graph and equal to the number of nodes.

The symbols \(\odot\), \(\max\) and \(\min\) define the extended operations in the sense of Zadeh’s extension principle. For fuzzy intervals \(\tilde{A} = (\tilde{a}, \tilde{a}^\lambda; a^\varepsilon, \tilde{a}^\varepsilon; a^\mu, a^\mu)\) and \(\tilde{B} = (\tilde{b}, \tilde{b}^\lambda; b^\varepsilon, \tilde{b}^\varepsilon; b^\mu, b^\mu)\) of the \(\varepsilon\)-\(\lambda\)-type the extended addition can easily be obtained by using the formula, see [7]

\[
\tilde{A} \oplus \tilde{B} = (\tilde{a} + \tilde{b}, \tilde{a} + \tilde{b}^\varepsilon; a^\varepsilon + \tilde{b}, a^\varepsilon + \tilde{b}^\varepsilon; a^\mu + \tilde{b}, a^\mu + \tilde{b}^\varepsilon)
\]

Since the membership function of fuzzy intervals of the \(\varepsilon\)-\(\lambda\)-type are only approximately defined between the levels \(\varepsilon, \lambda\) and 1 it is sufficient to use the following approximations for maximization (\(\max\)) and minimization (\(\min\)).

\[
\max(\tilde{A}, \tilde{B}) = (\max_{\varepsilon}(a, b), \max_{\varepsilon}(a, b));
\]

\[
\min(\tilde{A}, \tilde{B}) = (\min_{\varepsilon}(a, b), \min_{\varepsilon}(a, b));
\]

This procedure has the additional advantage that always fuzzy intervals of the \(\varepsilon\)-\(\lambda\)-type are produced.

If somebody is not convinced by the approximations made above, more detailed approximation formulas, which refer to more membership levels, may be applied instead. Since the following calculated (event) dates are no longer of the \(\varepsilon\)-
λ-type, the calculation gets more difficult. A possible algorithm is the Vertex method of Dong and Wong [2].

In order to illustrate the procedure we take a look at the project plan in Figure 2 and the durations mentioned in Table 1.

Table 1: Duration of jobs, earliest and latest dates of events

<table>
<thead>
<tr>
<th>(i,j)</th>
<th>Name</th>
<th>( \tilde{D}_{ij} )</th>
<th>ESD_{ij} = ED_i</th>
<th>LFD_{ij} = LD_j</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>A_1</td>
<td>(4, 5 ; 1, 1 ; 1.5, 2)</td>
<td>(0, 0 ; 0, 0 ; 0, 0)</td>
<td>(11.5, 13 ; 2.5, 4 ; 2.5, 4.5)</td>
</tr>
<tr>
<td>0-2</td>
<td>A_2</td>
<td>(10, 11 ; 1, 2 ; 1.5, 3)</td>
<td>(0, 0 ; 0, 0 ; 0, 0)</td>
<td>(10, 11 ; 1, 2 ; 1.5, 3)</td>
</tr>
<tr>
<td>1-3</td>
<td>Sham</td>
<td>(0, 0 ; 0, 0 ; 0, 0)</td>
<td>(4, 5 ; 1, 1 ; 1.5, 2)</td>
<td>(12.5, 13.5 ; 2, 3 ; 2, 4)</td>
</tr>
<tr>
<td>1-4</td>
<td>B_1</td>
<td>(6, 6 ; 0, 0 ; 0, 0)</td>
<td>(10, 11 ; 1, 2 ; 1.5, 3)</td>
<td>(17.5, 19 ; 2.5, 4 ; 3.5, 6)</td>
</tr>
<tr>
<td>2-3</td>
<td>Sham</td>
<td>(0, 0 ; 0, 0 ; 0, 0)</td>
<td>(12.5, 13.5 ; 2, 3 ; 2, 4)</td>
<td>(12.5, 13.5 ; 2, 3 ; 2.5, 4)</td>
</tr>
<tr>
<td>2-5</td>
<td>B_3</td>
<td>(12, 13 ; 2, 3 ; 3, 5)</td>
<td>(10, 11 ; 1, 2 ; 1.5, 3)</td>
<td>(22, 24 ; 3, 5 ; 4, 4)</td>
</tr>
<tr>
<td>3-4</td>
<td>B_2</td>
<td>(5, 5.5 ; 0.5, 1 ; 1.5, 2)</td>
<td>(10, 11 ; 1, 2 ; 1.5, 3)</td>
<td>(17.5, 19 ; 2.5, 4 ; 3.5, 6)</td>
</tr>
<tr>
<td>4-5</td>
<td>C_2</td>
<td>(4.5, 5 ; 0.5, 1 ; 1)</td>
<td>(15, 16.5 ; 1.5, 3 ; 3.5)</td>
<td>(22, 24 ; 3, 5 ; 4, 4)</td>
</tr>
<tr>
<td>4-6</td>
<td>C_1</td>
<td>(3, 3.5 ; 0.5, 1 ; 0.5, 1)</td>
<td>(10, 11 ; 1, 2 ; 1.5, 3)</td>
<td>(31, 36 ; 5, 7.5 ; 8, 13)</td>
</tr>
<tr>
<td>5-6</td>
<td>D_1</td>
<td>(8, 9.5 ; 1, 2 ; 1.5, 2.5)</td>
<td>(15, 16.5 ; 1.5, 3 ; 3.5)</td>
<td>(31, 36 ; 5, 7.5 ; 8, 13)</td>
</tr>
<tr>
<td>5-7</td>
<td>D_2</td>
<td>(10, 13 ; 2, 3 ; 2.5, 4)</td>
<td>(22, 24 ; 3, 5 ; 4.5, 8)</td>
<td>(32, 37 ; 5, 8 ; 7, 12)</td>
</tr>
<tr>
<td>5-8</td>
<td>E_3</td>
<td>(4, 4.5 ; 0.5, 1 ; 1.5, 2)</td>
<td>(22, 24 ; 3, 5 ; 4.5, 8)</td>
<td>(32, 37 ; 5, 8 ; 7, 12)</td>
</tr>
<tr>
<td>6-8</td>
<td>E_1</td>
<td>(9, 10 ; 1, 2 ; 1.2)</td>
<td>(22, 24 ; 3, 5 ; 4.5, 8)</td>
<td>(32, 37 ; 5, 8 ; 7, 12)</td>
</tr>
<tr>
<td>7-8</td>
<td>E_2</td>
<td>(8, 9 ; 1, 1.5 ; 2, 3)</td>
<td>(22, 24 ; 3, 5 ; 4.5, 8)</td>
<td>(32, 37 ; 5, 8 ; 7, 12)</td>
</tr>
</tbody>
</table>

Based on the definition that the given finishing date \( LD_w \) for the project is just kept, Rommelfanger [9] propose the extended addition for the calculation of the latest date \( LD_i \) of the event i:
In order to avoid absurd results it is not sufficient to calculate the latest starting date $LS_{ij}$ of the job $(i, j)$ according to the formula

$$LS_{ij} = \Delta_{ij} - \Delta_{ij} - \Delta_{ij} - \Delta_{ij} - \Delta_{ij} \quad (5)$$

where $\Delta_{ij}$ is defined as fuzzy interval $\Delta_{ij} = \varepsilon_{ij} \lambda_{ij} \tau_{ij}$. Rommelfanger [9] recommends the application of the following formulas. They are based on the so-called $\varepsilon$-preference, see [8, p. 76], using the $\lambda$ instead of $\varepsilon$ due to the definition of the membership level

$$\tilde{\tau}_{ij} = \text{Max}(\tilde{\tau}_{ij} - \tau_{ij}, 0) \quad (6)$$

$$\tilde{\tau}_{ij} = \text{Max}(\tilde{\tau}_{ij} - \tau_{ij}, 0) \quad (7)$$

$$\tilde{\tau}_{ij} = (\Delta_{ij} - \Delta_{ij}) - \text{Max}(\tilde{\tau}_{ij} - \tau_{ij}, 0) \quad (8)$$

$$\tilde{\tau}_{ij} = \text{Max}(0, (\tilde{\tau}_{ij} - \tau_{ij}) - \text{Max}(0, (\Delta_{ij} - \Delta_{ij}) - \text{Max}(\tilde{\tau}_{ij} - \tau_{ij}, 0)) \quad (9)$$

$$\tilde{\tau}_{ij} = \text{Max}(0, (\tilde{\tau}_{ij} - \tau_{ij}) - \text{Max}(0, (\Delta_{ij} - \Delta_{ij}) - \text{Max}(\tilde{\tau}_{ij} - \tau_{ij}, 0)) \quad (10)$$

The latest finishing date $\tilde{\Delta}_{ij}$ will then be defined as

$$\tilde{\Delta}_{ij} = \begin{cases} \min\{LS\tilde{\Delta}_{ij} \mid j \in S_{i} \} & \text{if } S_{i} \neq \emptyset \\ \Delta_{iw} & \text{if } S_{i} = \emptyset \end{cases} \quad (12)$$

Furthermore for the calculation of slack times the extended addition should be applied instead of the extended subtraction. In order to avoid absurd results calculations should be made in accordance with the above mentioned formulas.

The complete slack time $\tilde{CST}_{ij}$ of a job $(i, j)$, i.e. the maximum interval, about which the beginning of a job can be postponed without risking the final date, can be calculated as

$$(\tilde{E}_{i} \oplus \tilde{D}_{j}) \oplus \tilde{CST}_{ij} = \tilde{L}_{ij} \quad (13)$$
The free fuzzy slack time $\widetilde{FS}_{ij}$ of a job $(i, j)$, i.e. the maximum interval, about which the beginning of a job $(i, j)$ can be postponed under the condition that all following jobs will be able to start as soon as possible, is defined as

$$(\widetilde{ED}_i \oplus \widetilde{D}_j) \oplus \widetilde{FS}_{ij} = \widetilde{ED}_j$$

(14)

<table>
<thead>
<tr>
<th>$(i,j)$</th>
<th>Name</th>
<th>$\widetilde{D}_{ij}$</th>
<th>$\widetilde{CS}_{ij}$</th>
<th>$\widetilde{FS}_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>A$_1$</td>
<td>(4, 5 ; 1, 1 ; 1.5, 2)</td>
<td>6,8 ; 1.5, 3 ; 1, 2.5</td>
<td>0,0 ; 0, 0 ; 0,0</td>
</tr>
<tr>
<td>0-2</td>
<td>A$_2$</td>
<td>(10, 11 ; 1, 2 ; 1.5, 3)</td>
<td>0,0 ; 0, 0 ; 0,0</td>
<td>0,0 ; 0, 0 ; 0,0</td>
</tr>
<tr>
<td>1-3</td>
<td>Sham</td>
<td>(0, 0 ; 0, 0 ; 0, 0)</td>
<td>8,8,5 ; 1,2 ; 0.5, 2.0</td>
<td>6,6 ; 0, 1 ; 0,1</td>
</tr>
<tr>
<td>1-4</td>
<td>B$_1$</td>
<td>(6, 6 ; 0, 0 ; 0, 0)</td>
<td>7,5,8 ; 1.5, 3 ; 2.4</td>
<td>5,5,5 ; 0.5, 2 ; 1.5, 3</td>
</tr>
<tr>
<td>2-3</td>
<td>Sham</td>
<td>(0, 0 ; 0, 0 ; 0, 0)</td>
<td>2.5, 2.5 ; 1,1 ; 0.5, 1</td>
<td>0,0 ; 0, 0 ; 0,0</td>
</tr>
<tr>
<td>2-5</td>
<td>B$_3$</td>
<td>(12, 13 ; 2, 3 ; 3, 5)</td>
<td>0,0 ; 0, 0 ; 0,0</td>
<td>0,0 ; 0, 0 ; 0,0</td>
</tr>
<tr>
<td>3-4</td>
<td>B$_2$</td>
<td>(5,5,5 ; 0.5, 1 ; 1.5, 2)</td>
<td>2.5, 2.5 ; 1, 1 ; 0.5, 1</td>
<td>0,0 ; 0, 0 ; 0,0</td>
</tr>
<tr>
<td>4-5</td>
<td>C$_2$</td>
<td>(4.5, 5 ; 0.5, 1 ; 1.2)</td>
<td>2.5, 2.5 ; 1,1 ; 0.5, 1</td>
<td>2.5, 2.5 ; 1, 1 ; 0.5, 1</td>
</tr>
<tr>
<td>4-6</td>
<td>C$_1$</td>
<td>(3,3 ; 0.5, 1 ; 0.5, 1)</td>
<td>13, 20 ; 3,3,5 ; 4.5,7</td>
<td>12,14 ; 2,3 ; 2.5, 4.5</td>
</tr>
<tr>
<td>5-6</td>
<td>D$_1$</td>
<td>(8,9.5 ; 1, 2 ; 1.5, 2.5)</td>
<td>1, 2.5 ; 1, 1 ; 1.1, 5</td>
<td>0,0 ; 0, 0 ; 0,0</td>
</tr>
<tr>
<td>5-7</td>
<td>D$_2$</td>
<td>(10, 13 ; 2, 3 ; 2.5, 4)</td>
<td>0,0 ; 0, 0 ; 0,0</td>
<td>0,0 ; 0, 0 ; 0,0</td>
</tr>
<tr>
<td>5-8</td>
<td>E$_3$</td>
<td>(4, 4 ; 0.5, 1 ; 1.5, 2)</td>
<td>6,8,5 ; 1.5,1.5 ; 1.5, 2.5</td>
<td>14,18 ; 2,5, 3.5 ; 3.5</td>
</tr>
<tr>
<td>6-8</td>
<td>E$_1$</td>
<td>(9, 10 ; 1, 2 ; 1.2)</td>
<td>1, 2.5 ; 1, 1 ; 2.2, 5</td>
<td>1, 2.5 ; 1, 1 ; 2, 2.5</td>
</tr>
<tr>
<td>7-8</td>
<td>E$_2$</td>
<td>(8,9 ; 1, 1.5 ; 2, 3)</td>
<td>0,0 ; 0, 0 ; 0,0</td>
<td>0,0 ; 0, 0 ; 0,0</td>
</tr>
</tbody>
</table>

Table 2: Complete and free fuzzy slack times

With reference to the deterministic network analysis jobs with a complete slack time $\widetilde{CS}_{ij} = (0., 0 ; 0, 0 ; 0 , 0)$ are called critical jobs, because the extension of this job results automatically in a corresponding extension of the project in the same amount. A path that is solely composed of critical jobs - except for the starting event - is called critical path. In the given numerical example, see Table 2, the critical path connects the events „0“„2“„5“„7“„8“.

The application of fuzzy networks does not only have the advantage that the planning becomes more exact, but by this kind of modeling information costs can be reduced as well. If first of all only the available information or such information that could be gathered at low costs will be applied in the project plan, one only has to get information about the jobs that compose the critical path to improve the planning. In addition, further information about the jobs with very low slack times could be obtained.
4. Flexible extended addition

So far literature does not mention the problem that in case of the extended addition, based on the minimum operator, the earliest starting dates get fuzzier with each proceeding. Equalization between extensions and reductions of the usual duration does not take place. In case of this „inactive“ addition it is supposed that one always has to take into account that at once for all jobs the maximum time interval may be required or on the other hand only the minimum time interval may be needed. This is however contrary to the general experience.

A first step to get out of this dilemma could be to get along without the $\varepsilon$-level, by using only values with a membership value greater than or equal to $\lambda$. According to the definition of the membership value this is surely a reasonable proceeding. The addition of the $\lambda$-cuts

$$D_{ik} + D_{jk} = [d_{ij} - \delta_{ij}^\lambda, d_{ij} + \delta_{ij}^\lambda] + [d_{jk} - \delta_{jk}^\lambda, d_{jk} + \delta_{jk}^\lambda]$$

$$= [(d_{ij} + d_{jk}) - (\delta_{ij}^\lambda + \delta_{jk}^\lambda), (d_{ij} + d_{jk}) - (\delta_{ij}^\lambda + \delta_{jk}^\lambda)]$$

illustrates once again that an equalization between reductions and extensions of the duration does not take place.

Under formal aspects one could simply use another extended addition, like the extended addition based on the Yager's $T$-norm $T_p$

$$T_p(u, v) = \max\{0, 1 - [(1 - u)^p + (1 - v)^p]^{p^{-1}}\} \quad u, v \in [0, 1], \quad p > 0 \quad (15)$$

Rommelfanger and Keresztfalvi [10] recommended the application of this flexible extended addition in LP-models with fuzzy coefficients.

Then, the sum of $n$ fuzzy intervals of type $\tilde{A}_i = (a_i, \bar{a}_i; \alpha^\lambda_i, \alpha^\varepsilon_i; \alpha^\lambda, \alpha^\varepsilon)^\lambda,\varepsilon$ is again a fuzzy interval of $\lambda$-$\varepsilon$-type.

$$\tilde{A} = \tilde{A}_1 \oplus \tilde{A}_2 \oplus \cdots \oplus \tilde{A}_n = (\bar{a}, \bar{a}; \alpha^\lambda, \alpha^\varepsilon; \alpha^\lambda, \alpha^\varepsilon)^\lambda,\varepsilon,$$

with the parameters

$$\bar{a} = \sum_{j=1}^{n} a_j \quad \bar{a} = \sum_{j=1}^{n} \bar{a}_j$$

$$\alpha^\lambda(p) = [(\alpha^\lambda_i)^q + \cdots + (\alpha^\lambda_i)^q]^{p^{-1}} \quad \alpha^\varepsilon(p) = [(\alpha^\varepsilon_i)^q + \cdots + (\alpha^\varepsilon_i)^q]^{p^{-1}} \quad (16)$$

$$\bar{\alpha}^\lambda(p) = [(\bar{a}_i)^q + \cdots + (\bar{a}_i)^q]^{p^{-1}} \quad \bar{\alpha}^\varepsilon(p) = [(\bar{a}_i)^q + \cdots + (\bar{a}_i)^q]^{p^{-1}} \quad (17)$$

where $q = \frac{p}{p-1} \geq 1$.

The question is however, which $p \in ]1, +\infty[ \$ shall be chosen or what kind of consequences has the choosing of a special value $p$. A first information can be obtained by taking a look at the extremes:

\[ Flins2.doc \] submitted to World Scientific 2/14/2007 : 12:03 PM 8/10
• If $p \to +\infty$ or $q=1$ $T_p$ corresponds to the Min T-norm. We will then receive the usual extended addition, i.e.

\[ \alpha^V(+\infty) = \alpha^V_1 + \cdots + \alpha^V_n \quad \text{and} \quad \overline{\alpha}^V(+\infty) = \overline{\alpha}^V_1 + \cdots + \overline{\alpha}^V_n, \quad \nu = \lambda, \varepsilon. \]

• If $p=1$ or $q \to +\infty$ $T_p$ corresponds the Lukasiewicz T-norm $T_L$:

\[ T_1 (u, v) = T_L (u, v) = \max \{u, u+v-1\}. \]

In this case we will get the smallest ranges

\[ \alpha^V(1) = \max(\alpha^V_1, \cdots, \alpha^V_n) \quad \text{and} \quad \overline{\alpha}^V(1) = \max(\overline{\alpha}^V_1, \cdots, \overline{\alpha}^V_n), \quad \nu = \lambda, \varepsilon. \]

• If $p \in ]1, +\infty[ $ the ranges $\alpha^V(p)$ and $\overline{\alpha}^V(p)$, $\nu = \lambda, \varepsilon$, are strictly increasing functions in $p$.

For example, if $p = q = 2$, we get

\[ \alpha^V(2) = [(\alpha_1^V)^2 + \cdots + (\alpha_n^V)^2]^{0.5} \quad \overline{\alpha}^V(2) = [(\overline{\alpha}_1^V)^2 + \cdots + (\overline{\alpha}_n^V)^2]^{0.5}, \quad \nu = \lambda, \varepsilon, \]

i.e. if $p = 2$ the Euclidean norm is fulfilled.

Since it is difficult to define further values $p$ of the interval $]1, +\infty[$, we will only take a look at the 3 above mentioned values and try to assign them to application classes.

I. The choice $p \to +\infty$ seems to be adequate, if the duration of two jobs depend in the same way on environment conditions. This may occur, if two jobs will be carried out on the same machine and if the condition have influence on the working time. Another example is provided in case of construction works; the time necessary for the drying up of the masonry, the roughcast or the tiles, based on mortar is in all these cases positively correlated with the atmospheric humidity.

Using $p \to +\infty$, the earliest starting date of the job $D_2$ is

\[ E\overline{D}_5 = D_{02} \oplus D_{25} = (10, 11; 1, 2; 1.5, 3) \oplus (12, 13; 2, 3; 3, 5) \]
\[ = (22, 24; 3, 5; 4.5, 8), \quad \text{see Table 1}. \]

II. The parameter $p$ should be equated to 2, if there exists no relation between the jobs and therefore one can assume that extensions and reductions of the usual duration’s compensate one another by chance. Under formal aspects this corresponds to the independence of two normal random variables $X$ and $Y$, the standard deviation of which is defined as $\sigma (X + Y) = [\sigma^2(X) + \sigma^2(Y)]^{0.5}$.

Using $p = 2$, the earliest starting date of the job $D_2$ is

\[ E\overline{D}_5 = D_{02} \oplus D_{25} = (22, 24; \sqrt{5}, \sqrt{13}; \sqrt{11.25}, \sqrt{34}). \]

III. The extreme value $p = 1$ seems to be adequately, if it is possible to compensate at least partially delays of one job by influencing another job.

Using $p = 1$, the earliest starting date of the job $D_2$ is
$\text{ED}_5 = \tilde{D}_{02} \oplus \tilde{D}_{25} = (22, 24; 2, 3; 3, 5).$

In case of non-application of the extended addition, based on the min-T-norm, this step should be thought about carefully and explained in each single case. In doing so, one can refer to the critical jobs only, if no other aspects, e.g. interest losses in case of too early finished jobs, have to be taken into account.

5. Concluding remarks

Like all heuristic procedures the above mentioned statements need a profound empirical examination. Since manual calculations would be too complicated, at first adequate software has to be produced. An object-orientated programming in visual C$^+$ would be useful in order to give the user a survey about how to find an interactive solution.

References

8 Rommelfanger H., Entseheidung bei Unsicherheit - Fuzzy Decision Support-Systeme (Springer Verlag, Berlin, 1988)