Sliding Mode Controller Design: An input-output approach

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In this presentation, we propose a **linear** based sliding mode control scheme for trajectory tracking in internally, or externally, perturbed switched input-output dynamical systems. The class of systems treated with this approach spans the class of minimum phase systems, whether linear, nonlinear, mono-variable or multi-variable, continuous, or discrete-time, nature with, or without, control input delays and including, or not, fractional derivatives. So, it naturally applies to *differentially flat* systems.

Here we specialize to the implications of the approach in sliding mode controller design for nonlinear SISO systems.
Sliding mode controller design has been dominated by the state space formulation of dynamic systems, i.e., sliding surfaces are *endogenous* entities (Utkin 1977, Edwards and Spurgeon 1998, Utkin *et al* 2001, etc.).

Exogenous sliding surfaces exhibit the advantage of creating sliding regimes on a measurable, exogenous (first order) state space. Its zero dynamics coincides with the average closed loop system responses. It, thus, makes available all the arsenal of control design methods for input-output descriptions of dynamic systems.

We combine exogenous sliding modes with a robust linear input-output controller design method based on a simplified linear perturbed model of the nonlinear system.
Let $y^*(t) = [y^*(t), \ldots, y^*(t)]^{(n-1)}(t)]$. Consider the following smooth SISO nonlinear trajectory tracking control system ($y \in \mathbb{R}, u \in \mathbb{R}$),

\[
\begin{align*}
y^{(n)} &= \phi(y, \dot{y}, \ldots, y^{(n-1)}) + u\psi(y, \dot{y}, \ldots, y^{(n-1)}), \quad u \in [0, 1] \\
\dot{\zeta} &= \vartheta(\zeta, u, y, y^*(t)), \quad \zeta \in \mathbb{R}^m \\
u &= u^*(t) + \theta(\zeta, y, y^*(t))
\end{align*}
\]

Assume the dynamical feedback controller above has some set, $P$, of desirable properties.
Dynamic output feedback controller for input-output nonlinear system
Suppose we would like to take advantage of the properties, \( P \), of the above average design in controlling the switched system

\[
y^{(n)} = \phi(y, \dot{y}, ..., y^{(n-1)}) + u\psi(y, \dot{y}, ..., y^{(n-1)}), \quad u \in \{0, 1\}
\]

Naturally, we would like to deem the previously designed controller as an *average dynamic output feedback controller* for the switched system and bestow, via a corresponding sliding mode controller, all the features, and properties, of the average closed loop response of the obtained design.

An *exogenous*, relative degree 1, sliding surface for the switched system may be conformed by setting:

\[
\sigma = \int_0^t \left[ u^*(\tau) + \theta(\zeta(\tau), y(\tau), y^*(\tau)) - u \right] d\tau
\]
The equivalent control, $u_{eq}$, obtained under the *invariance condition*: $\dot{\sigma} = 0$, clearly coincides with the average controller design,

$$ u_{eq} = u^*(t) + \theta(\zeta, y, y^*(t)) $$

and the *zero dynamics*, or *ideal sliding dynamics*, associated with, $\sigma = 0$, $\dot{\sigma} = 0$, is constituted by

$$
\begin{align*}
    y^{(n)} &= \phi(y, \dot{y}, ..., y^{(n-1)}) + u_{eq}(t)\psi(y, \dot{y}, ..., y^{(n-1)}), \quad u_{eq} \in [0, 1] \\
    \dot{\zeta} &= \vartheta(\zeta, u_{eq}, y, y^*(t)), \quad \zeta \in \mathbb{R}^\nu \\
    u_{eq} &= u^*(t) + \theta(\zeta, y, y^*(t))
\end{align*}
$$

A necessary, and sufficient, condition for the existence of a sliding regime on $\sigma = 0$, is constituted by:

$$ 0 < u_{eq}(t) < 1 $$
Indeed, letting $u_{av} = u^*(t) + \theta(\zeta, y, y^*(t))$,

$$\sigma \dot{\sigma} = \sigma(u_{av} - u) < 0$$

with

$$u = \frac{1}{2} (1 + \text{sign}\sigma)$$

$\Sigma - \Delta$-modulation for switched input synthesis.
Dynamic output feedback controller for switched input-output nonlinear system, implemented via $\Sigma - \Delta$ modulation.
Consider the following perturbed nonlinear SISO system

\[ y^{(n)} = \phi(y, \dot{y}, ..., y^{(n-1)}) + u\psi(y, \dot{y}, ..., y^{(n-1)}) + p(y, \dot{y}, ..., t), \quad u \in [0, 1] \]

We adopt the following *non-phenomenological* model of the tracking error system, \( e_y = y - y^*(t) \), in which two basic features are preserved: The order of the system, \( n \), and the input gain, \( \mu(t) = \psi(y(t), \dot{y}(t), ..., y^{(n-1)}(t)) \):

\[ e^{(n)}_y = \varphi(t) + u\mu(t) \]

where \( \varphi(t) \), and its time derivatives, are assumed to be uniformly absolutely bounded, with a finite number of similarly bounded time derivatives.
The control input gain, $\psi(y(t), \dot{y}(t), ..., y^{(n-1)}(t)) = \mu(t)$, may be:

- perfectly known (constant, output dependent) and hence it may be exactly canceled,
- it may be partially unknown, due to output and phase variables dependence, and, hence, it may be asymptotically canceled via (linear) estimation.
- completely unknown and, hence, it is canceled via fast, on-line, algebraic parameter estimation methods.
Similar approaches and viewpoints exist in the literature, for disturbance handling and gain cancelation, with various degrees of rigorous justification,

- Sliding Mode control \( \dot{\sigma} = Wu + \phi(x, t) \) (Utkin 1977)
- Disturbance Accommodation (C. D. Johnson 1971)
- Active Disturbance Cancelation (J. Han, 2009, Z. Gao, 2006)
- Intelligent PID control (M. Fliess and C. Join, 2006)

We advocate *linear* observers, with a self-updating internal polynomial model for \( \varphi(t) \), and *linear* estimated phase variable feedback with disturbance and gain cancelation via *linear* estimation. The key tool: high gain observers.
Let $e_0 = \hat{e}_y$, $e_j = \hat{e}_y^{(j)}$, a GPI linear observer,

\[
\begin{align*}
\dot{e}_0 &= e_1 + \lambda_{m+n-1}(e - e_0) \\
\dot{e}_1 &= e_2 + \lambda_{m+n-2}(e - e_0) \\
&\vdots \\
\dot{e}_{n-1} &= \mu(t)u + z_1 + \lambda_m(e - e_0) \\
\dot{z}_1 &= z_2 + \lambda_{m-1}(e - e_0) \\
&\vdots \\
\dot{z}_{m-1} &= z_m + \lambda_1(e - e_0) \\
\dot{z}_m &= \lambda_0(e - e_0)
\end{align*}
\]

The tracking error estimation error, $\tilde{e} = e - e_0$, satisfies:

\[
\tilde{e}^{(n+m)} + \lambda_{n+m-1}\tilde{e}^{(n+m-1)} + \cdots + \lambda_0\tilde{e} = \varphi^{(m)}(t)
\]
The appropriate choice of the constant coefficients, \( \{\lambda_{n+m-1}, \ldots, \lambda_0\} \), to have the roots of the dominant characteristic polynomial \( p_o(s) \),

\[
p_o(s) = s^{n+m} + \lambda_{n+m-1}s^{n+m-1} + \cdots + \lambda_0
\]

located sufficiently deep into the left half of the complex plane, guarantees that, for absolutely uniformly bounded, \( \varphi^{(m)}(t) \), the trajectories of \( \tilde{e} \), and of its time derivatives, asymptotically converge to a small as desired neighborhood of the origin of the estimation error phase space. Moreover, \( z_1 = \hat{\varphi}(t) \), converges towards a close estimate of the disturbance input \( \varphi(t) \).

“Clutching” of observer variables may be necessary to avoid the large initial “peaking”, typical of high-gain designs.
The linear feedback controller is readily synthesized to be,

\[
u = \frac{1}{\mu(t)} \left[ -z_1 s(t) - \sum_{j=0}^{n-1} \kappa_j \hat{e}_s^{(j)}(t) \right], \quad \varphi_s(t) = \begin{cases} \varphi(t) \sin^2 q \left( \frac{\pi t}{2\epsilon} \right) & t < \epsilon \\ \varphi(t) & t \geq \epsilon \end{cases} \quad q = 4
\]

The closed loop tracking error satisfies,

\[
e_y^n + \kappa_{n-1} e_y^{(n-1)} + \cdots + \kappa_0 e_y = [\varphi(t) - \varphi(t)] + \eta(t, \hat{e}_j(t)|_{j=1}^{n-1})
\]

The constant coefficients, \(\{\kappa_{n+m-1}, \ldots, \kappa_0\}\), are chosen to have the roots of the dominant characteristic polynomial, \(p_c(s)\),

\[
p_c(s) = s^n + \kappa_{n-1} s^{n-1} + \cdots + \kappa_0
\]

located far into the left half of the complex plane. For absolutely uniformly bounded disturbance signals, the trajectories of, \(e_y\), and of its time derivatives, converge to a small as desired neighborhood of the origin of the tracking error phase space.
Consider the normalized model of a Buck converter with time-varying load:

\[
\begin{align*}
\dot{x}_1 &= u - x_2 \\
\dot{x}_2 &= x_1 - \frac{x_2}{Q(\tau)} \\
y &= x_2
\end{align*}
\]

where \(x_1\) is the normalized inductor current, \(x_2\) is the normalized output voltage, \(Q(t)\) is the normalized time-varying load, which is completely unknown, and \(u\) is the average control input restricted to the interval \(u \in [0, 1]\). The normalized time is denoted by \(\tau\).
The input-output model satisfies the following time-varying linear differential equation

\[ \ddot{y} + \frac{\dot{y}}{Q(\tau)} + y \left(1 - \frac{\dot{Q}(\tau)}{Q^2(\tau)}\right) = u \]

It is desired to track a given reference signal \( y^*(\tau) \) regardless of \( Q(\tau) \).

We assume, thanks to the uncertainty in \( Q(\tau) \) and un-modeled sources and resistors, that the system is given, in a simplified manner, by the following perturbed linear input-output dynamics:

\[ \ddot{y} = u + \varphi(\tau) \]
Let $y_1 = y$, $y_2 = \dot{y}$. We propose the GPI observer:

\begin{align*}
\dot{\hat{y}}_1 &= \dot{\hat{y}}_2 + \lambda_4(y_1 - \hat{y}_1) \\
\dot{\hat{y}}_2 &= u + z_1 + \lambda_3(y_1 - \hat{y}_1) \\
\dot{z}_1 &= z_2 + \lambda_2(y_1 - \hat{y}_1) \\
\dot{z}_2 &= z_3 + \lambda_1(y_1 - \hat{y}_1) \\
\dot{z}_3 &= \lambda_0(y_1 - \hat{y}_1)
\end{align*}

and set, via the design coefficients, the roots of the dominant characteristic polynomial

$$p(s) = s^5 + \lambda_4 s^4 + \lambda_3 s^3 + \lambda_2 s^2 + \lambda_1 s + \lambda_0$$

We let $p(s) = p_d(s) = (s^2 + 2\zeta \omega_n s + \omega_n^2)^2(s + p)$, with $\zeta = 1$, $\omega_n = 10$. 
The observer-based average tracking controller is readily proposed as a p-d controller

\[ u = u^*(t) - k_1(\hat{y}_2 - \dot{y}^*(\tau)) - k_0(y - y^*(\tau)) - z_1 \]

where \( u^*(\tau) \) is the unperturbed system input reference signal, computed as \( u^*(\tau) = \ddot{y}^*(\tau) \) and \( z_1 = \hat{\phi}(\tau) \). The gains \( \{k_0, k_1\} \) are chosen so that the roots of \( s^2 + k_1 s + k_0 \) are located in the stable portion of the complex plane.

We set

\[ k_1 = 2\zeta_c\omega_{nc}, \quad k_0 = \omega_n^2 \]

with \( \zeta_c = 1, \omega_{nc} = 1 \).
We used as a model for the unknown variations of the load the following Duffing chaotic system response.

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= a\xi_1 - b\xi_1^3 - c\xi_2 + E\cos(2\pi f \tau)
\end{align*}
\]

with

\[a = 1.0, \quad b = 1.0, \quad c = 1.5, \quad E = 1, \quad f = \frac{1}{2\pi}.\]

The initial conditions were set to be: \(\xi_1(0) = 0.1, \xi_2(0) = 0\). The time-varying load, \(Q(t)\), and its first order time derivative, were then allowed to be:

\[
Q(t) = 0.1\xi_1(t) + 0.4, \quad \dot{Q}(t) = 0.1\xi_2(t)
\]
Average performance of observer based gpi controller for a stabilization task in a normalized buck converter with unknown time-varying load.
Simulated, unknown, normalized time-varying load $Q(\tau)$ and inset showing the perturbation estimation, due to load variations, affecting the simplified average model.
Sigma-Delta modulation implementation of robust output feedback stabilizing controller for uncertain time-varying load.
Sigma-Delta modulation implementation of robust output feedback trajectory tracking controller for uncertain time-varying load.
Sigma-Delta modulation implementation of robust output feedback trajectory tracking controller for uncertain time-varying load with sudden variation.
Conclusions

- Disturbance accommodation, Active Disturbance cancelation, Intelligent PID, robust GPI-observer based controller, are all closely related robust output feedback controller design schemes for input-output dynamic systems. Flatness significantly simplifies the design issues reducing the problem to a linear problem.

- $\Sigma - \Delta$ modulation implements, via an exogenous sliding regime, all the relevant features of any average feedback controller design on an input-switched system.

- Many laboratory examples of robust GPI-observer based control are becoming available, including systems with input delays. (Robots: D. Librado, Penduli: M. Ramírez, Power Electronics, Induction Motors, Mobile Robots: A. Luviano, J. Cortes, C. García)
Final Remarks

The presence of zero mean noises is handled via appropriate filtering of the tracking error through iterated integration according to an intuitive zero mean noise integration model. This can be accomplished at the observer level. Let $\tilde{e}_p$ denote the estimate of $\int^{(p)}(e_y)$,

\begin{align*}
\dot{e}_{-p} &= e_{-p+1} + \lambda_{m+n+p-1}(\int^{(p)}(e_y) - e_{-p}) \\
\dot{e}_{-p+1} &= e_{-p+2} + \lambda_{m+n+p-2}(\int^{(p)}(e_y) - e_{-p}) \\
&\vdots \\
\dot{e}_{-1} &= e_0 + \lambda_{m+n}(\int^{(p)}(e_y) - e_{-p}) \\
\end{align*}

The observer estimation error $\tilde{e} = (\int^{(p)}(e_y) - e_{-p})$, satisfies

$$
\tilde{e}^{(m+n+p)} + \lambda_{m+n+p-1}\tilde{e}^{(m+n+p-1)} + \cdots + \lambda_0\tilde{e} = \varphi^{(m)}(t)
$$
Consider the perturbed linear system with perfect observations \( y \)

\[
\ddot{y} = u + \xi(t) + \zeta(t)
\]

where \( \zeta \) is a zero mean noise with unknown statistics and \( \xi(t) \) is an unknown but bounded perturbation input to the system, possibly a state dependent one.
We induce some filtering via the following augmented state space model

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_1 \\
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u + \xi(t) + \zeta(t) \\
y &= x_1
\end{align*}
\]

and regard as an auxiliary output the second integral of the original output \( y \), i.e., \( \eta = x_1 \). The problem now reduces to control the perturbed auxiliary output dynamics for \( \eta = x_1 \)

\[
\eta^{(4)} = u + \xi(t) + \zeta(t)
\]
Simulations

Performance of extended GPI observer-based controller in the case of noisy input.