Higher Order Exponential Time Differencing Scheme for System of Coupled Nonlinear Schrödinger Equations

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Abstract

The coupled nonlinear Schrödinger equations are highly used in modeling the various phenomena in nonlinear fiber optics, like propagation of pulses. Efficient and reliable numerical schemes are required for analysis of these models and for improvement of the fiber communication system. In this paper, we introduce a new version of the Cox and Matthews third order exponential time differencing Runge-Kutta (ETD3RK) scheme based on the (1, 2)-Padé approximation to the exponential function. In addition, we present its local extrapolation form to improve temporal accuracy to the fourth order. The developed scheme and its extrapolation are seen to be strongly stable, which have ability to damp spurious oscillations caused by high frequency components in the solution. A computationally efficient algorithm of the new scheme, based on a partial fraction splitting technique is presented. In order to investigate the performance of the novel scheme we considered the system of two and four coupled nonlinear Schrödinger equations and performed several numerical experiments on them. The numerical experiments showed that the developed numerical scheme provide an efficient and reliable way for computing long-range solitary solutions given by coupled nonlinear Schrödinger equations and conserved the conserved quantities mass and energy exactly, to at least five decimal places.

Keywords: Coupled nonlinear Schrödinger equations, local extrapolation, exponential time differencing, vector solitons, Padé approximation

1. Introduction

The system of coupled nonlinear Schrödinger equations (CNLSE) has been found to be relevant in many scientific applications, especially in the area of hydrodynamics and fiber optics and has been the subject of intensive research over the past decades [1, 2]. In nonlinear optics, the CNLSE model an optical soliton which is a special solitary wave that not only maintains its shape after wave interaction but travels long distance without any optical loss. This form of soliton is created by balancing the anomalous group velocity dispersion with the fiber nonlinearity, called self-phase modulation and offers unmodulated transfer of pulses from one place to another over a long distance [3]. Elastic and inelastic collisions among/between solitons can occur in CNLS systems. In elastic

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collision the solitons can balance their intensities and shapes after interacting with each other, known as shape preserving collision, which can maintain the quality and capacity of a fiber communication system. On the other hand, for the inelastic collision, the solitons undergo shape-changing with intensity redistribution, which can help to exploit the applications in signal amplification and soliton switching (see [4] and references therein).

Efficient and reliable numerical schemes are essential to develop an understanding of the phenomena related to the CNLSE, which do not have an analytical solution. The difficulties in evaluating analytical solutions of nonlinear dispersive wave equations have stimulated a growing interest in the development of numerical schemes in recent years [5]. Many authors developed different numerical schemes based on finite difference for solving system (1) for \( N = 1 \) and 2 including convergence, stability, and conservation.

Among these authors, Ismail and Alamri [6] solved coupled nonlinear Schrödinger equations using an implicit midpoint rule, together with Dogulu approximation, to make the scheme fourth order accurate in space. Kurtinaitis and Ivanauska [7] employed explicit, implicit, Hopsotch type and Crank-Nicolson type finite difference scheme to simulate the dynamics of three components involving coupled nonlinear Schrödinger equations. Sheng et al. [8] developed a new conservative finite difference scheme with semi-discretization and an application of the quartic spline approximation to solve generalized nonlinear Schrödinger equation. A linearly implicit conservative scheme was proposed by Ismail and Taha [9] to solve coupled nonlinear Schrödinger equations. Sun et al. [10] derived a new six point method of coupled nonlinear Schrödinger equations with a symplectic scheme. Ismail [11] solved coupled nonlinear Schrödinger equations utilizing fourth order explicit schemes including the Runge-Kutta scheme. Sanz-Serna and Verwer [12] discussed conservative and non-conservative schemes for the solution of nonlinear Schrödinger equation and pointed out that three of them performed poorly due to nonlinear blow-up that was noticed for all of them. Moreover, Fei et al. [13] developed a new conservative scheme, which is linearly implicit for solving Schrödinger type equations, and at the same time proved the scheme is convergent and stable.

Recently, de la Hoz and Vadillo [14] utilized an exponential time differencing, fourth-order Runge-Kutta (ETDRK4) scheme derived by Cox and Matthews [15] and modified by Kassam and Trefethen [16] for solving one and two dimensional nonlinear Schrödinger equations. Moreover, Hederi et al. [17] utilized an exponential time differencing (ETD) scheme for solving NLS and higher order NLS equations with a fourth-order Runge-Kutta type method with smoothing, as developed by Khaliq et al. [18] and found that a ETD scheme is highly efficient and stable for long-distance soliton computations.

In this paper, our main objective was to introduce a novel version of a third order ETD3RK scheme developed by Cox and Matthews [15] along with its extrapolation for solving system (1) for \( N = 2 \) and 4 because little numerical work has been reported in literature on the system of coupled nonlinear Schrödinger equations, utilizing exponential time differencing schemes. To be precise, the CNLS system under consideration as mentioned in [19] is as follows:

\[
\frac{\partial \Psi_n}{\partial t} + \alpha_n \frac{\partial^2 \Psi_n}{\partial x^2} + \left( \sum_{m=1}^{N} \sigma_{nm} |\Psi_m|^2 \right) \Psi_n = 0, \quad -\infty < x < \infty \\
\quad n = 1, 2, ..., N
\]
where \( \Psi_n, n = 1, 2, \ldots, N \) are complex valued wave amplitudes, \( i \) is an imaginary number, and \( x, t \) represent space and time variables respectively. The parameters \( \alpha_n, n = 1, 2, \ldots, N \) are group velocity dispersion (GVD) coefficients, \( \sigma_{nn}, n = 1, 2, \ldots, N \) are self-phase modulation (SPM) coefficients, which are also known as the Landau constants, and \( \sigma_{nm}, n \neq m \) are cross phase modulations or wave-wave interaction coefficients.

Depending on the values of the above mentioned coefficients, the system can be either integrable or nonintegrable. The system (1) converts to the familiar nonlinear cubic Schrödinger equation for \( N = 1 \), which plays a vital role in the nonlinear Physics, and it is totally integrable. For \( N = 2 \) with the case where \( \alpha_1 = \alpha_2 = 1, \sigma_{11} = e = \sigma_{22} \), Manakov [20] showed that the system is integrable by the inverse scattering method. However, the works for \( N \geq 3 \) are rare [19], even though the system (1) is of particular interest in Physics. In this paper we present numerical results including conserved quantities like energy and mass both for integrable and non-integrable cases of system (1) for \( N = 2 \) and 4 by utilizing developed scheme.

The rest of the paper will be arranged as follows. In section 2, we will present the procedure of an ETD scheme along with an overview of basic time stepping schemes. Section 3, will include a modified time stepping scheme, its partial fraction form, its algorithm and stability analysis of the scheme. Extrapolation form of the developed scheme will be included in section 4. The spatial discretization approach for a system of four coupled nonlinear Schrödinger equations will be presented in section 5. Propagation and collision of solitons will be analyzed and discussed with numerical experiments in section 6. The last section will contain brief conclusions.

2. An ETD Procedure and Basic Time Stepping Scheme

Let us consider the following semi-linear initial boundary value problem:

\[
\begin{align*}
\frac{\partial u}{\partial t} + Au &= F(u,t) \quad \text{in} \quad \Omega, \quad t \in (0, \bar{t}] = J \\
u &= v \quad \text{on} \quad \partial \Omega, \quad t \in J, \quad u(.,0) = u_0 \in \Omega. \\
\end{align*}
\]

(2)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^d \) with Lipschitz continuous boundary. \( A \) represents a linear operator, \( u = u(t) \) represent a group of physical or biological species, \( u = v \) represent boundary condition and \( F \) is a nonlinear reaction function. Let \( k = t_{n+1} - t_n \) be the time step size, then following the same derivation in [21] we come up with the following recurrence formula:

\[
u(t_{n+1}) = e^{-kA}u(t_n) + k \int_0^1 e^{-kA}(1 - \tau))F(u(t_n + \tau k), t_n + \tau k)d\tau
\]

(3)

The formula (3) is an exact and different order ETD schemes that come into picture from how one approximates the integral. The ETD schemes solve the linear part exactly and then explicitly approximates the integral part by polynomial approximation. Cox and Matthews [15] developed time stepping method based on the same approach by using Newton’s Backward difference formula, which gives a Runge-Kutta type higher order approximations. This scheme has the advantage of constructing a family of higher order numerical schemes with potentially good performance, except for some computational difficulties, as pointed out by Kassam and Trefethen [16] and Khaliq and et
Several schemes based on Runge-Kutta time stepping for the case, when $A$ is scalar or a diagonal, were established by Cox and Matthews but here we consider only the following third order scheme called ETD3RK:

$$u_{n+1} = e^{-kA}u_n + \frac{1}{k^2}(-A)^{-3}(F(u_n, t_n)[-4 + kA + e^{-kA}(4 + 3kA + k^2A^2)])$$

$$+ 4F(a_n, t_n + \frac{k}{2})[2 - kA + e^{-kA}(-2 - kA)] + F(b_n, t_{n+1})[-4 + 3kA - k^2A^2]$$

$$+ e^{-kA}(4 + kA))$$ \hspace{1cm} (4)

where,

$$a_n = e^{-\frac{kA}{2}}u_n - (A)^{-1}(e^{-\frac{kA}{2}} - I)F(u_n, t_n)$$

$$b_n = e^{-kA}u_n - (A)^{-1}(e^{-kA} - I)\left(2F(a_n, t_n + \frac{k}{2}) - F(u_n, t_n)\right)$$

Implementation of this scheme can be easy, when the linear operator is a diagonal matrix with no zero entry on the diagonal. But, for the case of linear operator being a non-diagonal matrix, this process can not be applied, when $A$ has eigenvalues close to zero, since it is necessary to compute $(-A)^{-1}$, $(-A)^{-2}$ and $e^{-kA}$. Kassam and Trefethen \[16\] have shown that ETD schemes higher than second suffer from numerical instability for eigenvalue tends to zero due to a cancellation error.

3. Modified Time Stepping Scheme

3.1. The ETD-Padé(1,2) Scheme

In this section we present a new version of the third order scheme ETD3RK developed by Cox and Matthews utilizing third order $(1,2)$-Padé approximation of $e^{-z}$. An advantage we find in using $(1,2)$ Padé approximation to ETD3RK is that the factors $A^{-1}$ and $A^{-3}$ cancel out.

We use the notation $R_{r,s}(z)$ for $(r,s)$—Padé approximation to $e^{-z}$ and $\tilde{R}_{r,s}(z)$ for $(r,s)$—Padé approximation to $e^{-\frac{z}{2}}$. The $(r + s)^{th}$ order rational Padé approximation to $e^{-z}$ is defined as:

$$R_{r,s}(z) = \frac{P_{r,s}(z)}{Q_{r,s}(z)}$$

where

$$P_{r,s}(z) = \sum_{j=0}^{r} \frac{(s + r - j)!r!}{(s + r)!j!(r - j)!}(-z)^j$$

and

$$Q_{r,s}(z) = \sum_{j=0}^{s} \frac{(s + r - j)!}{(s + r)!j!(s - j)!}(-z)^j.$$

Definition 3.1.1.

A rational approximation $R_{r,s}(z)$ of $e^{-z}$ is said to be $A$—acceptable, if $|R_{r,s}(z)| < 1$, whenever $\Re(z) < 0$ and $L$—acceptable if, in addition, $|R_{r,s}(z)| \to 0$ as $\Re(z) \to -\infty$.

The rational Padé approximation $R_{r,s}(z)$ to $e^{-z}$ is:
• $A$-acceptable if $r = s$. 
• $L$-acceptable if $r = s - 1$ or $s - 2$.

We have employed $(1, 2)$-Padé approximation to $e^{-z}$ in Eqn. (4), which is given by $R_{1,2}(z) = \frac{1 - \frac{z}{2}}{1 + \frac{z}{2} + \frac{k}{2}}$ to construct a $L$-stable method we call ETD-Padé$(1, 2)$. Plugging $R_{1,2}(kA)$ in Eqn. (4), we get the following ETD Scheme:

$$u_{n+1} = R_{1,2}(kA)u_n + P_1(kA)F(u_n, t_n) + P_2(kA)F(a_n, t_n + \frac{k}{2}) + P_3(kA)F(b_n, t_{n+1}),$$  

(5)

where

$$R_{1,2}(kA) = (6I - 2kA)(6I + 4kA + k^2A^2)^{-1},$$
$$P_1(kA) = k(6I + 4kA + k^2A^2)^{-1},$$
$$P_2(kA) = k(6I + 4kA + k^2A^2)^{-1},$$
$$P_3(kA) = k(I + kA)(6I + 4kA + k^2A^2)^{-1}.$$ 

In addition we have :

$$a_n = \tilde{R}_{1,2}(kA)u_n + \tilde{P}_1(kA)F(u_n, t_n)$$
$$b_n = R_{1,2}(kA)u_n + \tilde{P}_2(kA)(2F(a_n, t_n + \frac{k}{2}) - F(u_n, t_n))$$

with

$$\tilde{R}_{1,2}(kA) = 4(6I - kA)(24I + 8kA + k^2A^2)^{-1},$$
$$\tilde{P}_1(kA) = k(12I + kA)(24I + 8kA + k^2A^2)^{-1},$$
$$\tilde{P}_2(kA) = k(6I + kA)(6I + 4kA + k^2A^2)^{-1}.$$ 

3.2. The Partial Fraction Form of Modified Time Steeping Scheme

The scheme (5) introduced in 3.1 consists of high order matrix polynomial to invert and this can cause computational inaccuracies, if these matrices have high condition numbers. Moreover, round off error in computing the power of the matrices can also produce bad approximations [22]. To overcome this difficulty we utilized partial fraction decomposition, as suggested by Khaliq et al. [18, 23]. This decomposition scheme is very important in alleviating ill-conditioned problems because only an Implicit Euler type solver is required.

To compute $u_{n+1}$ for the new scheme, we employ:

$$R_{1,2}(z) = 2\Re\left(\frac{w_1}{z - c_1}\right)$$

and corresponding $\{P_j(z)\}_{j=1}^3$ takes the form:

$$P_j(z) = 2k\Re\left(\frac{w_{j1}}{z - c_1}\right); j = 1, 2, 3,$$
where $c_1$ is the pole of $R_{1,2}$ as well as of $P_j$ with corresponding weights $w_1$ and $w_{j,1}$ respectively. To evaluate $a_n$ and $b_n$ we employ:

$$
\tilde{R}_{1,2}(z) = 2\Re\left(\frac{\tilde{w}_1}{z-c_1}\right) \text{ and corresponding } \tilde{P}_1(z) \text{ and } \tilde{P}_2(z)\text{ as }
$$

$$
\tilde{P}_1(z) = 2k\Re\left(\frac{\tilde{\Omega}}{z-c_1}\right) \text{ and } \tilde{P}_2(z) = 2k\Re\left(\frac{\tilde{\Omega}}{z-c_1}\right),
$$

where $\tilde{c}_1$ is pole of $\tilde{R}_{1,2}$ as well as of $\tilde{P}_1(z)$ with corresponding weights $\tilde{w}_1$ and $\tilde{\Omega}$ respectively.

### 3.3. Algorithm

Here, we present a description of the algorithm by implementing the partial fraction splitting technique. The fundamental purpose of the solution procedure is to efficiently implement the scheme on a serial machine.

**Step 1.** Solve

$$(kA - c_1 I)R_a = w_1 u_n + k\Omega F(u_n, t_n),$$

for $R_a$ and then compute $a_n$ as: $a_n = 2\Re(R_a)$.

**Step 2.** Solve

$$(kA - c_1 I)R_b = w_1 u_n + k\Omega(2F(a_n, t_n + \frac{k}{2}) - F(u_n, t_n),$$

for $R_b$ and then compute $b_n$ as: $b_n = 2\Re(R_b)$.

**Step 3.** Solve

$$(kA - c_1 I)R_u = w_1 u_n + kw_{1,1}F(u_n, t_n) + kw_{2,1}F(a_n, t_n + \frac{k}{2}) + kw_{3,1}F(b_n, t_{n+1}),$$

for $R_u$ and then compute $u_{n+1}$ as: $u_{n+1} = 2\Re(R_u)$.

In order to implement this third order scheme in its partial fraction form, we have computed poles and corresponding weights for $R_{1,2}(z)$ and $\{P_j(z)\}_{j=1}^3$, which are as follows:

$$
c_1 = -2.0 + i1.4142135623730950488,
$$

$$
w_1 = -1.0 - i3.5355339068654752443,
$$

$$
w_{11} = -i0.35355339068654752443,
$$

$$
w_{21} = -i1.4142135627461900977,
$$

$$
w_{31} = 0.5 + i0.35355339068654752443,
$$

and for $\tilde{R}_{1,2}(z)$, $\tilde{P}_1(z)$ and $\tilde{P}_2(z)$ are as:

$$
\tilde{c}_1 = -4.0 + i2.8284271247461900976,
$$

$$
\tilde{w}_1 = -2.0 - i7.0710678112309504881,
$$

$$
\tilde{\Omega} = 0.5 - i1.4142135622461900976.
$$

$$
\tilde{\bar{\Omega}} = 0.5 - i1.4142135622461900976,
$$

where $i = \sqrt{-1}$. 


3.4. Stability Analysis of ETD-Padé(1, 2) Scheme

The stability analysis of ETD-Padé(1, 2) scheme is analyzed by an approach discussed in [14, 15, 24]. Following [15], we consider the non-linear ODE:

\[ u_t = -cu + F(u), \]  

where \( F(u) \) is the non-linear part and there exist fixed point \( u_0 \) such that \( -cu_0 + F(u_0) = 0 \). Linearizing about fixed point \( u_0 \) leads to:

\[ u_t = -cu + \lambda u, \]  

where \( u \) is perturbation of \( u_0 \), \( c \) is the diffusion in \( x \) direction and \( \lambda = F'(u_0) \). The fixed point \( u_0 \) is stable if \( \text{Re}(\lambda - c) < 0 \), see [15].

The boundaries of the stability region (a family of curves for different values of \( ck \), where \( k \) is time step) based on equation (7) are presented below for third order ETD-Padé(1, 2) scheme. Nie et al. [25] noticed that the quantity \( ck \) actually represents CFL number involving the ratio of temporal and spatial grid size for discretization of the Eq. (2).

To obtain the stability regions, we applied Eq. (5) to (7), which leads to recurrence relation involving \( u_n \), and \( u_{n+1} \). By letting \( r = \frac{u_{n+1}}{u_n} \), \( x = \lambda k \), and \( y = -ck \) and using Maple 15 computer algebra package, we come up with the following amplification factor:

\[ r(x, y) = c_0 + c_1 x + c_2 x^2 + c_3 x^3. \]  

where

\[
\begin{align*}
c_0 &= 1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + \frac{1}{36}y^4 + O(y^5), \\
c_1 &= 1 + y + \frac{1}{2}y^2 + \frac{1}{8}y^3 - \frac{19}{864}y^4 + O(y^5), \\
c_2 &= \frac{1}{2} + \frac{1}{2}y + \frac{1}{6}y^2 - \frac{55}{864}y^3 - \frac{151}{1296}y^4 + O(y^5), \\
c_3 &= \frac{1}{6} + \frac{5}{72}y - \frac{11}{216}y^2 - \frac{443}{5184}y^3 - \frac{491}{7776}y^4 + O(y^5). \end{align*}
\]

In general, the parameters \( c \) and \( \lambda \) may both be complex-valued. The stability region of ETD-Padé(1, 2) scheme is four dimensional and therefore difficult to represent it [15]. The boundaries of the stability region are obtained by substituting \( r = e^{i\theta} \) into the Eq. (8) and solve for \( x \), but unfortunately we do not know the explicit expression for \( |r| = 1 \). We will only be able to plot it. In order to plot two-dimensional stability region, the most common idea is to use complex \( x \)-plane, assuming \( c \) to be fixed and real [24] or that both \( c \) and \( \lambda \) are pure imaginary [26].

In the Fig. 1 we present the picture of the stability regions of ETD-Padé(1, 2) scheme with different values of \( y = 0, -0.9, -5, -10, -18 \), in complex \( x \)-plane. According to Beylkin et al. [24] for method to be useful, it is important that stability regions grow as \( ck \) becomes larger. We observed that the stability region tends to third order Runge-Kutta scheme as \( y \to 0 \), and as \( y \to -\infty \), the stability region grows. This result gives an indication of the stability of the ETD-Padé(1, 2) scheme.
Fig. 1. Stability regions (interior of closed curves) of ETD-Padé(1,2) scheme in the complex $x-$plane with several negative $y$.

4. Extrapolation of ETD-Padé(1, 2) Scheme

This section includes the local extrapolation of the developed scheme based on the procedure proposed by Lawson and Morris [27] to improve the temporal accuracy and enhance the accuracy to fourth order.

Let us consider Eq. (5) in the following form:

$$u(t + k) = R_{1,2}(kA)u(t) + P_1(kA)F(u(t), t) + P_2(kA)F(a(t), t + \frac{k}{2}) + P_3(kA)F(b(t), t + k),$$

(9)

where

$$a(t) = \tilde{R}_{1,2}(kA)u(t) + \tilde{P}_1(kA)F(u(t), t)$$

$$b(t) = R_{1,2}(kA)u(t) + \tilde{P}_2(kA)(2F(a(t), t + \frac{k}{2}) - F(u(t), t))$$

and all other functions are same as in (5).

Now writing the 9 over the double time step 2k and denote the result with $u^{(1)}(t + 2k)$ we have:

$$u^{(1)}(t+2k) = R_{1,2}(2kA)u(t) + P_1(2kA)F(u(t), t) + P_2(2kA)F(a(t), t+k) + P_3(2kA)F(b(t), t+2k),$$

(10)

where

$$a(t) = \tilde{R}_{1,2}(2kA)u(t) + \tilde{P}_1(2kA)F(u(t), t)$$

$$b(t) = R_{1,2}(2kA)u(t) + \tilde{P}_2(2kA)(2F(a(t), t + k) − F(u(t), t))$$
Alternatively, if \([9]\) is computed over two single time steps and denote the result with \(u^{(2)}(t + 2k)\), we have:

\[
\begin{align*}
  u^{(2)}(t + 2k) &= (R_{1,2}(kA))^2 u(t) + \\
  &\quad R_{1,2}(kA) \left( P_1(kA)F(u(t), t) + P_2(kA)F(a(t), t + \frac{k}{2}) + P_3(kA)F(b(t), t + k) \right) + \\
  &\quad P_1(kA)F(u(t + k), t + k) + P_2(kA)F(a(t + k), t + \frac{3k}{2}) + P_3(kA)F(b(t + k), t + 2k),
\end{align*}
\]  

\(\text{(11)}\)

where

\[
\begin{align*}
  a(t + k) &= \tilde{R}_{1,2}(kA)u(t + k) + \tilde{P}_1(kA)F(u(t + k), t + k) \\
  b(t + k) &= R_{1,2}(kA)u(t + k) + \tilde{P}_2(kA) \left( 2F(a(t + k), t + \frac{3k}{2}) - F(u(t + k), t + k) \right)
\end{align*}
\]

Consequently \(10\) and \(11\) are two alternative ETD-Padé(1, 2) schemes for evaluating solution at time \(t + 2k\). Neither \(10\) nor \(11\) is \(O(k^4)\) accurate. However, if we combine the expansion of \(u^{(1)}\) and \(u^{(2)}\) by taking \(\frac{8}{7}u^{(2)}\) and subtracting \(\frac{1}{7}u^{(1)}\), we get:

\[
u(t + 2k) = \frac{8}{7}u^{(2)} - \frac{1}{7}u^{(1)} + O(k^5)
\]

Thus

\[
u(t + 2k) \approx \frac{8}{7}u^{(2)} - \frac{1}{7}u^{(1)}.
\]  

\(\text{(12)}\)

and which makes the final solution fourth order accurate both in time and space.

Fig. 2 displays the behaviour of the function \(e^{-z}\), Padé approximation \(R_{1,2}(z)\) and extrapolation of (1, 2)-Padé, which is given by:

\[
E_{1,2}(z) = \frac{8}{7} \left( \frac{1 - \frac{2}{3}z}{1 + \frac{4}{3}z + \frac{2}{3}z^2} \right)^2 - \frac{1}{7} \left( \frac{1 - \frac{2}{3}z}{1 + \frac{4}{3}z + \frac{2}{3}z^2} \right).
\]

As can be seen from Fig. 2 that the (1, 2)-Padé approximation, possessing growth factor, which tends to zero asymptotically but has a small negative values for \(z > 3\). This indicates that small oscillations could theoretically appear but these should be damped with increasing \(t\). On the other hand the extrapolation of (1, 2)-Padé produces a growth factor, which tends to zero monotonically and satisfies the maximum modulus theorem \([28]\) for \(L\)-acceptability given in definition 3.1.1.

In order to apply the extrapolation algorithm \(12\) in computationally efficient way, we apply the algorithm in section (3.2) over double time step to get \(u^{(1)}\) and over two single steps to get \(u^{(2)}\).
5. Spatial Discertization

For space discretization, we utilized Numorov/Douglas approximation [6] to approximate space derivative and achieve fourth order accuracy in space, under the assumptions of $0 \leq t \leq T_{\text{max}}$, and the solution of system (1) for $N = 4$ (system of four coupled nonlinear Schrödinger equations) is negligible outside the range $X_L \leq x \leq X_R$. We presented a spatial discretization procedure for system (1) for $N = 4$. We set $\alpha_n = \frac{1}{\mu}$, $\sigma_{nn} = \sigma$, $n = 1, 2, 3, 4$, and $\sigma_{nm} = e$, $n \neq m$ in system (1) for $N = 4$ so that, for our numerical study we considered the following system:

$$
\begin{align*}
\mathcal{I} & \partial \Psi_1 \partial t + \frac{1}{\mu} \partial^2 \Psi_1 \partial x^2 + \left[ \sigma (|\Psi_1|^2) + e (|\Psi_2|^2 + |\Psi_3|^2 + |\Psi_4|^2) \right] \Psi_1 = 0, \\
\mathcal{I} & \partial \Psi_2 \partial t + \frac{1}{\mu} \partial^2 \Psi_2 \partial x^2 + \left[ \sigma (|\Psi_2|^2) + e (|\Psi_1|^2 + |\Psi_3|^2 + |\Psi_4|^2) \right] \Psi_2 = 0, \\
\mathcal{I} & \partial \Psi_3 \partial t + \frac{1}{\mu} \partial^2 \Psi_3 \partial x^2 + \left[ \sigma (|\Psi_3|^2) + e (|\Psi_1|^2 + |\Psi_2|^2 + |\Psi_4|^2) \right] \Psi_3 = 0, \\
\mathcal{I} & \partial \Psi_4 \partial t + \frac{1}{\mu} \partial^2 \Psi_4 \partial x^2 + \left[ \sigma (|\Psi_4|^2) + e (|\Psi_1|^2 + |\Psi_2|^2 + |\Psi_3|^2) \right] \Psi_4 = 0,
\end{align*}
$$

with initial conditions

$$
\Psi_n(x, 0) = g_n(x), \quad n = 1, 2, 3, 4.
$$

and no flux boundary conditions

$$
\frac{\partial \Psi_n(x, t)}{\partial x} = 0, \quad n = 1, 2, 3, 4 \quad \text{at} \quad x = x_L, x_R, \text{ for } t \geq 0.
$$
Decomposing complex functions $\Psi_n$ and $g_n$, $n = 1, 2, 3, 4$ into their real and imaginary parts as:

$$
\begin{align*}
\Psi_1(x, t) &= u_1(x, t) + iu_2(x, t) \\
\Psi_2(x, t) &= u_3(x, t) + iu_4(x, t) \\
\Psi_3(x, t) &= u_5(x, t) + iu_6(x, t) \\
\Psi_4(x, t) &= u_7(x, t) + iu_8(x, t)
\end{align*}
$$

(16)

where $u_i(i = 1, \ldots, 8)$, $g_{nR}, g_{nI}$, $n = 1, 2, 3, 4$ are real functions. Plugging Eq. (16) into system (13), we get the following system of PDEs.

$$
\begin{align*}
\frac{\partial u_1}{\partial t} + \frac{1}{\mu} \frac{\partial^2 u_2}{\partial x^2} + z_1 u_2 &= 0, & \frac{\partial u_2}{\partial t} - \frac{1}{\mu} \frac{\partial^2 u_1}{\partial x^2} - z_1 u_1 &= 0 \\
\frac{\partial u_3}{\partial t} + \frac{1}{\mu} \frac{\partial^2 u_4}{\partial x^2} + z_2 u_4 &= 0, & \frac{\partial u_4}{\partial t} - \frac{1}{\mu} \frac{\partial^2 u_3}{\partial x^2} - z_2 u_3 &= 0 \\
\frac{\partial u_5}{\partial t} - \frac{1}{\mu} \frac{\partial^2 u_6}{\partial x^2} - z_3 u_6 &= 0, & \frac{\partial u_6}{\partial t} - \frac{1}{\mu} \frac{\partial^2 u_5}{\partial x^2} - z_3 u_5 &= 0 \\
\frac{\partial u_7}{\partial t} - \frac{1}{\mu} \frac{\partial^2 u_8}{\partial x^2} - z_4 u_8 &= 0, & \frac{\partial u_8}{\partial t} - \frac{1}{\mu} \frac{\partial^2 u_7}{\partial x^2} - z_4 u_7 &= 0
\end{align*}
$$

(17)

with initial condition

$$
\begin{align*}
u_1(x, 0) &= g_{1R}(x), & u_2(x, 0) &= g_{1I}(x) \\
u_3(x, 0) &= g_{2R}(x), & u_4(x, 0) &= g_{2I}(x) \\
u_5(x, 0) &= g_{3R}(x), & u_6(x, 0) &= g_{3I}(x) \\
u_7(x, 0) &= g_{4R}(x), & u_8(x, 0) &= g_{4I}(x)
\end{align*}
$$

and no flux boundary condition

$$
\frac{\partial u_n}{\partial x} = 0 \quad n = 1, 2, \ldots, 8 \quad \text{at} \quad x = x_L, x_R, \quad \text{for} \quad t \geq 0.
$$

where

$$
\begin{align*}
z_1 &= \sigma \left( u_1^2 + u_2^2 \right) + e \left( u_3^2 + u_4^2 + u_5^2 + u_6^2 + u_7^2 + u_8^2 \right) \\
z_2 &= \sigma \left( u_3^2 + u_4^2 \right) + e \left( u_1^2 + u_2^2 + u_5^2 + u_6^2 + u_7^2 + u_8^2 \right) \\
z_3 &= \sigma \left( u_5^2 + u_6^2 \right) + e \left( u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_7^2 + u_8^2 \right) \\
z_4 &= \sigma \left( u_7^2 + u_8^2 \right) + e \left( u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2 \right)
\end{align*}
$$

(18)

The matrix-vector form of system (17) can be written as:

$$
\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\mu} A \frac{\partial^2 \mathbf{u}}{\partial x^2} + B(\mathbf{u})\mathbf{u} = 0,
$$

(19)
where

\[ A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 
\end{bmatrix} \] (20)

\[ u = \begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 \\
u_6 \\
u_7 \\
u_8 
\end{bmatrix}, \quad B(u) = \begin{bmatrix}
0 & z_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & z_2 & 0 & 0 & 0 & 0 \\
0 & 0 & -z_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & z_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -z_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & z_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -z_4 & 0 
\end{bmatrix}. \]

In order to solve system (19) numerically, we discretized the domain Ω with a uniform grid of step size length \( h \) such that \( x_m = x_L + mh, \ m = 0, 1, 2, ..., N \), where \( N \) is the total number of grid points and partitioned time domain with time step size length \( k \) such that \( t_n = nk, \ n = 0, 1, 2, ... \). Let us approximate an exact solution \( u(x_m, t) \) with \( U_m(t) \) and second order spatial derivative \( u_{xx} \) by using the Numorov approximation or the Douglas approximation, which is given by:

\[
\frac{\partial^2 u}{\partial x^2} = \frac{1}{h^2} (1 + \frac{1}{12} \delta_x^2)^{-1} \delta_x^2 u_m + O(h^4),
\] (21)

Plugging Eqn. (21) in Eqn. (19) and dropping error terms we get the semidiscrete system:

\[
\frac{\partial U_m}{\partial t} + \frac{A}{\mu h^2} (1 + \frac{1}{12} \delta_x^2)^{-1} \delta_x^2 U_m + B(U_m)U_m = 0; \ m = 0, 1, 2, ..., N.
\] (22)

After imposing the Neumann boundary conditions and applying the following definitions:

\[ u = [u_1^t, u_2^t, u_3^t, ..., u_N^t]^t; \]

and

\[ u_i = [U_{1,i}, U_{2,i}, U_{3,i}, U_{4,i}, U_{5,i}, U_{6,i}, U_{7,i}, U_{8,i}]^t, \quad i = 1, 2, 3, ..., N \]

Now we can expressed the system (22) as:

\[
P\dot{u} = [S + PD(u)]u,
\] (23)

where the dot(.) denotes differentiation with respect to time, \( P, S \), and \( D(u) \) are a block-tridiagonal symmetric, a block-tridiagonal skew symmetric and a block-diagonal skew symmetric matrices re-
spectively given by:

\[
P = \frac{1}{12} \begin{bmatrix}
5I & I & 0 & \cdots & \cdots & 0 \\
I & 10I & I & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & I & 10I & I \\
0 & \cdots & \cdots & 0 & I & 5I
\end{bmatrix},
\]

\[
S = \frac{-1}{\mu h^2} \begin{bmatrix}
-A & A & 0 & \cdots & \cdots & 0 \\
A & -2A & A & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & A & -2A & A \\
0 & \cdots & \cdots & 0 & A & -A
\end{bmatrix},
\]

\[
D(u) = -\text{diag}[B_1(u_1), B_2(u_2), \ldots, B_N(u_N)],
\]

where 0 and I are \((8 \times 8)\) zero and identity matrices respectively and \(A\) given by Eq. (20) and

\[
B_i(u_i) = \begin{bmatrix}
0 & (z_1)_i & 0 & 0 & 0 & 0 & 0 & 0 \\
-(z_1)_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & (z_2)_i & 0 & 0 & 0 \\
0 & 0 & - (z_2)_i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & (z_3)_i & 0 \\
0 & 0 & 0 & 0 & - (z_3)_i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & (z_4)_i \\
0 & 0 & 0 & 0 & 0 & 0 & - (z_4)_i & 0
\end{bmatrix}.
\]

To implement the purposed so called ETD-Padé\((1, 2)\) scheme, we express Eq. (23) as:

\[
\dot{u} + S_1 u = D(u)u,
\]

where \(S_1 = -P^{-1} S\).

Note: The spatial descritization of system \([1]\) for \(N = 2\) can be found in the paper of Ismail et al. \([6]\).

5.1. Conserved Quantities

In this subsection, we present some of the conservation laws, which are most prominent for testing performance of numerical schemes for CNLSE. One can find a lot of possibilities in defining conservation laws, but in this paper we use the standard ones given in \([6, 8]\). We only focus on two conserved properties, namely mass and energy in this work.

Mass conservations \([6]\):

\[
I_i = \int_{-\infty}^{\infty} |\Psi_i|^2 \, dx, \quad i = 1, 2, \ldots, N.
\]

Energy conservation:

The conservation of the energy can be computed via spectral norm \(\| \cdot \|_2\) [8]. For given \(0 \leq \epsilon \ll 1\), the conservation can be obtained by:

\[
\| u \|_2 - c | \leq \epsilon(t - t_0), \quad t > t_0,
\]

where \(c\) represents the energy of an analytical solution and is constant.
For any given \( u, v \in \mathbb{R}^{2N} \), we define an inner product

\[
\langle u, v \rangle = u^T v = \sum_{k=1}^{2N} u_k v_k.
\]

It follows that, for \( u = u(t) \in \mathbb{R}^{2N} \), \( t_0 < t \leq T \), a discretized version of (26) can be defined as:

\[
\| u \|_2 = \sqrt{h \langle u, u \rangle} = \sqrt{h \sum_{j=1}^{N} u_j^2}.
\] (27)

Now let us prove that the solutions of system (13) satisfies the mass conservation laws i.e:

\[
\int_{-\infty}^{\infty} |\Psi_i|^2 \, dx = \text{constant}, \quad i = 1, 2, ..., N.
\] (28)

If we multiply the first equation of the system (13) with \( \Psi_1 \) and its complex conjugate by \( \overline{\Psi_1} \) and subtract the latter from the former, we get

\[
\Psi_1 \left[ i \frac{\partial \Psi_1}{\partial t} + \frac{1}{\mu} \frac{\partial^2 \Psi_1}{\partial x^2} + \left[ \sigma (|\Psi_1|^2) + e (|\Psi_2|^2 + |\Psi_3|^2 + |\Psi_4|^2) \right] \Psi_1 \right] - \Psi_1 \left[ -i \frac{\partial \Psi_1}{\partial t} + \frac{1}{\mu} \frac{\partial^2 \Psi_1}{\partial x^2} + \left[ \sigma (|\Psi_1|^2) + e (|\Psi_2|^2 + |\Psi_3|^2 + |\Psi_4|^2) \right] \overline{\Psi_1} \right] = 0
\] (29)

and this gives

\[
i \left[ \frac{\Psi_1}{\partial t} + \Psi_1 \frac{\partial \Psi_1}{\partial t} \right] + \frac{1}{\mu} \left[ \frac{\partial^2 \Psi_1}{\partial x^2} - \Psi_1 \frac{\partial^2 \overline{\Psi_1}}{\partial x^2} \right] = 0
\]

or

\[
i \frac{\partial}{\partial t} (\Psi_1 \overline{\Psi_1}) + \frac{1}{\mu} \frac{\partial}{\partial x} \left( \frac{\partial \Psi_1}{\partial x^2} - \Psi_1 \frac{\partial \overline{\Psi_1}}{\partial x^2} \right) = 0.
\] (30)

Integrating (30) both side with respect to \( x \) in \(-x_L \leq x \leq x_R\), we arrive at

\[
i \frac{\partial}{\partial t} \int_{-x_L}^{x_R} |\Psi_1|^2 \, dx + \frac{1}{\mu} \left( \frac{\partial \Psi_1}{\partial x^2} - \Psi_1 \frac{\partial \overline{\Psi_1}}{\partial x^2} \right) = 0
\]

Imposing the boundary condition (15) for \( n = 1 \) we get

\[
\frac{\partial}{\partial t} \int_{-x_L}^{x_R} |\Psi_1|^2 \, dx = 0,
\]

thus we obtain

\[
I_1 = \int_{-x_L}^{x_R} |\Psi_1|^2 \, dx \text{ is constant}
\] (31)

For \( \Psi_i, \ i = 2, 3, 4 \) the mass conservation laws are derived in a similar manner. Conservation of \( I_i, \ i = 1, 2, 3, 4 \) implies \( L^2 \) boundedness of the solution and no blow is expected during the simulation of the scheme [6].
6. Numerical Results and Discussions

In order to investigate the performance of the developed scheme, we consider solving system (1) for \( N = 2 \) and \( 4 \) with appropriate initial and boundary conditions. The accuracy of the scheme is measured in terms of \( L_\infty \) and \( L_2 \) error norms and rate of convergence of the developed scheme is calculated by the formula:

\[
P = \frac{\log(L_2(|\Psi_1|^n)/L_2(|\Psi_1|^{n+1}))}{\log(k_n/k_{n+1})}
\]

6.1. System of Two Nonlinear Schrödinger Equations

6.1.1. Single Soliton

In this section, we solve system (1) for \( N = 2 \) with \( \alpha_1 = \alpha_2 = \mu \) as given in [6], which is given below:

\[
i\frac{\partial \Psi_1}{\partial t} + \frac{1}{\mu} \frac{\partial^2 \Psi_1}{\partial x^2} + (\sigma_{11}|\Psi_1|^2 + \sigma_{12}|\Psi_2|^2)\Psi_1 = 0 \quad x_L < x < x_R
\]

\[
i\frac{\partial \Psi_2}{\partial t} + \frac{1}{\mu} \frac{\partial^2 \Psi_2}{\partial x^2} + (\sigma_{21}|\Psi_1|^2 + \sigma_{22}|\Psi_2|^2)\Psi_2 = 0 \quad x_L < x < x_R
\]

(32)

together with the initial condition:

\[
\Psi_1(x, 0) = \sqrt{\frac{\mu\alpha}{1 + e}} \text{sech}(\sqrt{\mu\alpha}x) e^{ivx}
\]

\[
\Psi_2(x, 0) = \sqrt{\frac{\mu\alpha}{1 + e}} \text{sech}(\sqrt{\mu\alpha}x) e^{ivx},
\]

(33)

boundary conditions:

\[
\frac{\partial \Psi_1(x, t)}{\partial x} = \frac{\partial \Psi_2(x, t)}{\partial x} = 0, \text{ at } x = x_L, x_R \text{ for } t \geq 0.
\]

(34)

and the parameters \( \sigma_{11} = \sigma_{22} = 1, \sigma_{12} = \sigma_{21} = e \), where \( \alpha, v, e \) are constants. We followed the note discussed in section 5 for spatial discretization of system (32). Following Wadati [29], the analytical solution to this problem with \( \mu = 2 \) is given by:

\[
\Psi_j(x, t) = \sqrt{\frac{2\alpha}{1 + e}} \text{sech} \left( \sqrt{2\alpha}(x - vt) \right) e^{i\{vx-[\frac{1}{2}\alpha]t\}}, \quad j = 1, 2.
\]

(35)

The aim of the present numerical experiment is to verify numerically that the developed exponential time differencing scheme exhibits the expected third-order convergence in time. To test whether the developed scheme exhibits the expected convergence rates in time, we performed numerical test on system (32) for various values of the time step \( k \) and following values of the other parameters:

\[
x_L = -20, \quad x_R = 80, \quad h = 0.05, \quad T = 10, \quad v = 1.0, \quad \alpha = 1.0, \quad \mu = 2, \quad e = 2/3,
\]

which corresponds to linearly birefringent fibers and the computed results are given in Table 1. We may observe from Table 1 that the computed convergence rates agree well with the expected rate of the ETD-Padé(1, 2) scheme. We also displayed order of convergence of ETD-Padé(1, 2) in Fig. 3 using a log-log scale.
Table 1. The rate of convergence of ETD-Padé(1, 2) scheme calculated from $L_2$ errors.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$L_2$</th>
<th>Order (P)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.11119</td>
<td>-</td>
</tr>
<tr>
<td>0.05</td>
<td>0.01447</td>
<td>2.94</td>
</tr>
<tr>
<td>0.025</td>
<td>0.00178</td>
<td>3.02</td>
</tr>
<tr>
<td>0.0125</td>
<td>0.00021</td>
<td>3.08</td>
</tr>
</tbody>
</table>

In addition, we tested the accuracy of the scheme by looking at $L_\infty$ and $L_2$—errors and the mass and energy conservation properties discussed in equations 23 and 27 with the parameters $x_L = -20$, $x_R = 80$, $h = 0.1$, $k = 0.01$, $T = 50$, $v = 1.0$, $\alpha = 1.0$, $\mu = 2$, $\epsilon = 2/3$. The computed values of these quantities are presented in Table 2. According to exact solution (35), we have $\| \Psi_1 \|_2 \approx 1.302711$ and $I_1 \approx 1.6970563$, $t \geq 0$. The conserved values in Table 2 demonstrate that the proposed scheme conserves both the conserved quantities exactly, to at least five decimal places. As a consequence, we can see in Fig. 4 that the soliton will travel a long distance without changing shape and velocity. We may also notice from Fig. 4 that the phase of the soliton travels to the right at a constant velocity of 1. The good match between analytical solution and numerical solution at different time levels also emphasize that, when amplitude of the pulses are equal at different time levels, they should propagate with same shape and travel long distance with maintaining initial velocity. Notice that Fig. 4 (d) displays the behavior of a single soliton at a different time level up to $t = 50$. The small values of $L_\infty$ and $L_2$—errors clearly indicates that the developed scheme maintain good accuracy during the time evolution.
Table 2. Conserved quantities and computed errors of the single soliton with ETD-Padé(1, 2)

<table>
<thead>
<tr>
<th>T</th>
<th>( | \Psi_1 |_2 )</th>
<th>( L_1 )</th>
<th>( L_{\infty} )</th>
<th>( L_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.30271</td>
<td>1.69706</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>10</td>
<td>1.30271</td>
<td>1.69706</td>
<td>0.00010</td>
<td>0.00013</td>
</tr>
<tr>
<td>20</td>
<td>1.30271</td>
<td>1.69706</td>
<td>0.00003</td>
<td>0.00006</td>
</tr>
<tr>
<td>30</td>
<td>1.30271</td>
<td>1.69706</td>
<td>0.00027</td>
<td>0.00035</td>
</tr>
<tr>
<td>40</td>
<td>1.30271</td>
<td>1.69706</td>
<td>0.00069</td>
<td>0.00092</td>
</tr>
<tr>
<td>50</td>
<td>1.30271</td>
<td>1.69706</td>
<td>0.00127</td>
<td>0.00173</td>
</tr>
</tbody>
</table>

Similarly, we performed another numerical test on system (32) with different values of \( h \) and \( k \) and with the parameters \( x_L = -10, X_R = 10, T = 10, v = 0.5, \alpha = 1.0, \mu = 2, e = 2/3 \), to verify that the Extrapolation-Padé(1,2) demonstrates the expected fourth-order convergence in both directions, space and time. The computed results are in Table [3]. From Table [3] it can be easily seen that the computed rate of convergence is almost equal to the expected rate of Extrapolation-Padé(1,2). The log-log scale graph of the order of convergence is depicted in Fig. [5].

Fig. 4. (a)-(c) Numerical solution \( |\Psi_1| \) VS its exact solution (d) Surface plot of \( |\Psi_1| \) at different time \( t \) via ETD-Padé(1,2)
Table 3. Rate of Convergence of Extrapolation-Padé(1,2) calculated with $L_2$-error

<table>
<thead>
<tr>
<th>h</th>
<th>k</th>
<th>$L_2$</th>
<th>Order (P)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.2</td>
<td>0.32324</td>
<td>-</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>0.02163</td>
<td>3.90</td>
</tr>
<tr>
<td>0.05</td>
<td>0.05</td>
<td>0.00134</td>
<td>4.01</td>
</tr>
<tr>
<td>0.025</td>
<td>0.025</td>
<td>0.00011</td>
<td>3.65</td>
</tr>
</tbody>
</table>

Fig. 5. Log-log scale graph of $L_2$-error VS time step $k$.

To compare the Extrapolation-Padé(1,2) scheme in terms of the accuracy and computational efficiency, we performed two sets of numerical tests on system (32) with $\mu = 2$. In the first set of tests, we compared the accuracy with extrapolation scheme mentioned by Ismail et al. [6] by using the parameters $k = 0.05$, $h = 0.2$, $e = 2/3$, $v = 1$, and $\alpha = 1$ and recorded the result in Table 4. It can be easily seen from Table 4 that the Extrapolation-Padé(1,2) scheme is more accurate than the extrapolation scheme mentioned in the Ismail et al. paper [6]. Both of the extrapolation schemes are fourth-order accurate in time and were introduced to improve the temporal accuracy. In the same set of tests we compare ETD-Padé(1,2) and its extrapolation form to verify numerically that the extrapolation form demonstrates better accuracy than the original scheme. We recorded the computed results in Table 4. From Table 4, we see that the extrapolation form gives better accuracy than the original scheme, as expected and error by extrapolation form increases almost linearly until final time $t = 20$. So, we can say that extrapolation form of the scheme will more accurate than the original scheme and also helped to enhance the order of the scheme and make the scheme fourth-order accurate in time.
Table 4. Comparison of maximum error of solution $|\Psi_1|$ of system (32) with initial condition (33) for $k = 0.05$.

<table>
<thead>
<tr>
<th>T</th>
<th>ETD-Padé(1,2)</th>
<th>Extrapolation-Padé(1,2)</th>
<th>Extrapolation [6]</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.00098</td>
<td>0.00130</td>
<td>0.00132</td>
</tr>
<tr>
<td>8</td>
<td>0.00491</td>
<td>0.00212</td>
<td>0.00241</td>
</tr>
<tr>
<td>12</td>
<td>0.01233</td>
<td>0.00270</td>
<td>0.00375</td>
</tr>
<tr>
<td>16</td>
<td>0.03142</td>
<td>0.00319</td>
<td>0.00543</td>
</tr>
<tr>
<td>20</td>
<td>0.05097</td>
<td>0.00353</td>
<td>0.00723</td>
</tr>
</tbody>
</table>

In the second set of numerical tests, we compared accuracy and computational efficiency with an existing standard fourth-order explicit method called DRK4 utilized by Ismail [11] on the same single soliton problem (33) with the parameters $x_L = -20, x_R = 60, h = 0.1, v = 1.0, \alpha = 1.0,$ and $e = 1.0$, which corresponds to elliptic birefringent fibers. We fixed $k = 0.008$ and $k = 0.0125$ and measure computing times, the $L_\infty$ error and conserved quantity energy at different time levels corresponding to both values of $k$. The results are reported in Table 5. We observe from Table 5 that extrapolation form for $k = 0.008$ exactly gives similar accuracy to that of DRK4 at different times mentioned in the table except when $t = 28$. At $t = 28$ extrapolation-Padé(1,2) performs better than DRK4. Moreover, extrapolation-Padé(1,2) conserves energy exactly, to at least five decimal places, which is also the case for DRK4 mentioned by Ismail [11]. According to exact solution (35) with the above parameters, we have $\| \Psi_1 \|_2 \approx 1.189207$.

In terms of the computational efficiency, we may observe from Table 5 that DRK4 took less time when compared with extrapolation-Padé(1,2), which corresponded to $k = 0.008$. However, if we compare the computational efficiency in terms of accuracy of the schemes, we found that extrapolation of a developed scheme took less time than that of DRK4. To come to this conclusion, we produced the results of both the schemes with a larger time step $k = 0.0125$ via MATLAB 7.0 platforms based on an Intel Core i5-2410M 2.30GHz workstation. We found that an extrapolation-Padé(1,2) provides exactly the same accuracy and same energy values that are correct up to five decimal places given by DRK4 for $k = 0.008$ whereas, the DRK4 scheme blow up for $k = 0.0125$. 

Table 5. Comparison of accuracy and efficiency of an extrapolation form of ETD-Padé(1,2) and DRK4 schemes using $L_\infty$ error.

<table>
<thead>
<tr>
<th>T</th>
<th>Extrapolation-Padé(1,2) k=0.008</th>
<th>DRK4 k=0.008</th>
<th>Extrapolation-Padé(1,2) k=0.0125</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L_\infty$ $|\Psi_1|_2$ CPU-time</td>
<td>$L_\infty$ CPU-time</td>
<td>$L_\infty$ $|\Psi_1|_2$ CPU-time</td>
</tr>
<tr>
<td>4</td>
<td>0.00008 1.18921 24.76</td>
<td>0.00008 19.36</td>
<td>0.00008 1.18921 16.33</td>
</tr>
<tr>
<td>8</td>
<td>0.00014 1.18921 49.90</td>
<td>0.00014 37.69</td>
<td>0.00014 1.18921 32.06</td>
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<tr>
<td>12</td>
<td>0.00020 1.18921 74.74</td>
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</tr>
<tr>
<td>16</td>
<td>0.00026 1.18921 99.62</td>
<td>0.00026 75.35</td>
<td>0.00026 1.18921 64.13</td>
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<tr>
<td>20</td>
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<td>0.00032 94.16</td>
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<tr>
<td>24</td>
<td>0.00038 1.18921 148.78</td>
<td>0.00038 113.22</td>
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<tr>
<td>28</td>
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<td>0.00043 132.29</td>
<td>0.00043 1.18921 112.33</td>
</tr>
<tr>
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<td>0.00050 1.18921 196.59</td>
<td>0.00050 151.21</td>
<td>0.00050 1.18921 128.31</td>
</tr>
</tbody>
</table>

6.2. Interaction of Two Solitons

To analyze the interaction scenario of two solitons moving in opposite directions with different wave amplitude with the developed scheme, we solve system (32) with $\mu = 1$ together with initial condition [9]:

$$
\Psi_1(x,0) = \sqrt{2}r_1 \text{sech}(r_1 x + x_{10}) e^{i(v_1 x)} \\
\Psi_2(x,0) = \sqrt{2}r_2 \text{sech}(r_2 x + x_{20}) e^{i(v_2 x)},
$$

no flux boundary condition at $x = x_L, x_R \forall t \geq 0$ and the parameters $\sigma_{11} = \sigma_{22} = 1, \sigma_{12} = \sigma_{21} = e, v_1 = -v_2 = \frac{\pi}{4}, r_1 = 1.2$ and $r_2 = 1$.

We selected $h = 0.1$, $k = 0.01$, $x_L = -40$, $x_R = 40$ for all of our numerical computations.

The selections for the initial position parameters $x_{10}$ and $x_{20}$ can be totally arbitrary. They do not affect the collision outcome, as long as $x_{10} > 0$ and $x_{20} < 0$ are large enough [30]. In the first test we set $x_{10} = -x_{20} = 20$ and selected the control parameters $e = 1$ and $v = 1$. To investigate the performance of the purposed scheme we performed the interaction scenario of two solitons for a long time range up to $t = 100$ and reported the conserved quantities in Table 6 and illustrated the simulation of interaction at different times in Fig. 6 and 7. It is observed from the figures that the two waves moving in opposite directions collide and separate after the interaction, moving forward in the same directions with the same shape and velocity, as the initial one. It is clear from Table 6 that the developed scheme also conserves the conserved quantities exactly, to at least five decimal places. We may also observe from the Fig. 6 and 7 that the interaction is elastic and this is not strange, since the cross phase modulation (XPM) coefficient $e = 1$ gives Manakov's equation, which is totally integrable.
Table 6. Conserved quantities energy and mass for Interaction of two solitons utilizing ETD-Padé(1,2) scheme

<table>
<thead>
<tr>
<th>T</th>
<th>$| \Psi_1 |_2$</th>
<th>$I_1$</th>
<th>$| \Psi_2 |_2$</th>
<th>$I_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.19089</td>
<td>4.80000</td>
<td>2.00000</td>
<td>4.00000</td>
</tr>
<tr>
<td>20</td>
<td>2.19089</td>
<td>4.80000</td>
<td>2.00000</td>
<td>4.00000</td>
</tr>
<tr>
<td>40</td>
<td>2.19089</td>
<td>4.80000</td>
<td>2.00000</td>
<td>4.00000</td>
</tr>
<tr>
<td>60</td>
<td>2.19089</td>
<td>4.80000</td>
<td>2.00000</td>
<td>4.00000</td>
</tr>
<tr>
<td>80</td>
<td>2.19089</td>
<td>4.80000</td>
<td>2.00000</td>
<td>4.00000</td>
</tr>
<tr>
<td>100</td>
<td>2.19089</td>
<td>4.80000</td>
<td>2.00000</td>
<td>4.00000</td>
</tr>
</tbody>
</table>

Fig. 6. Two solitons interaction at different time $t$ via ETD-Padé(1,2)
For the second test we choosed the parameters $x_{10} = -x_{20} = 10$, $v = 1.6$, $e = 3$ and $t = 100$. The interaction scenario is depicted in Fig. 8. In the figure we plotted the contours of the interaction scenario. From the figure we observe the creation of new vector solitons, which can be happened, when the cross phase modulation (XPM) coefficient $e$ is larger and the collision velocity is moderate [30].

Fig. 8. Creation scenario of $|\Psi_1|$ and $|\Psi_2|$ with $v = 1.6$ and $e = 3.0$ via ETD-Padé(1,2)

In a similar way, we performed another test with the parameters $e = 0.3$ and $v = 0.4$ to the time $t = 100$. As we have observed, after the collision two solitons merge into one soliton. Typically, such collision scenario happens, when cross phase modulation $e$ is small and the collision is slow [30]. The interaction scenario is displayed in Fig. 9.

Fig. 9. Interaction of two solitons Manakov type at different time levels
From the above results, we see that a developed scheme is able to model all of the above mentioned interaction scenarios in line with analytical predictions for these interactions discussed by Yang [30].

6.3. System of Four Nonlinear Schrödinger Equations
6.3.1. Interaction of Four Solitons
To investigate the performance of a developed scheme and to analyze the interaction scenarios of four solitons with different wave amplitudes and different velocities, we solved the system (13) with $\mu = 1$ together with the initial condition:

$$
\Psi_1(x, 0) = \sqrt{2} r_1 \text{sech} (r_1 x + x_{10}) \ e^{iv_1 x} \\
\Psi_2(x, 0) = \sqrt{2} r_2 \text{sech} (r_2 x - x_{10}) \ e^{-iv_2 x} \\
\Psi_3(x, 0) = \sqrt{2} r_3 \text{sech} (r_3 x + x_{30}) \ e^{iv_3 x} \\
\Psi_4(x, 0) = \sqrt{2} r_4 \text{sech} (r_4 x - x_{30}) \ e^{-iv_4 x}
$$

and the boundary condition:

$$
\frac{\partial \Psi_n}{\partial x} = 0 \quad n = 1, 2, \ldots, 4 \quad \text{at} \quad x = x_L, x_R \ \forall \ t \geq 0.
$$

where $a_j$, $v_j$, $j = 1, 2, 3, 4$ are arbitrary constants and $x_{j0}$, $j = 1, 3$ are initial phase constants. In all of the numerical computations we choose $r_1 = 1.0$, $r_2 = 1.2$, $r_3 = 1.3$, $r_4 = 1.4$, $h = 0.1$, $v_1 = \frac{v}{8}$, $v_3 = v_4 = \frac{v}{4}$, $x_L = -40$, $x_R = 40$ and initial position parameters $x_{10} = 10$ and $x_{30} = 30$.

We set the control parameters $e = 1.0$ and $v = 1.0$ in the first set of computations and stored the solution obtained from DRK4 method with a very, very small time step size, $k = 0.001$ to get a reference solution for a system of four Nonlinear Schrödinger Equations with initial and boundary conditions mentioned above. Utilizing this reference solution we calculated the rate of convergence of a ETD-Padé(1,2) scheme using $L_2$-error for different values of $k$ at time $t = 10$. As can be seen from Table 7 and Fig. 10, the computed rate of convergence was almost equal to 3, which is the expected rate of ETD-Padé(1,2) scheme.

From the results obtained from two numerical tests related to the order of convergence of ETD-Padé(1,2), we can definitely say that our developed scheme is third order accurate in time.
Table 7. The rate of convergence of ETD-Padé(1, 2) scheme calculated from L₂ errors.

<table>
<thead>
<tr>
<th>k</th>
<th>L₂</th>
<th>Order (P)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.24686</td>
<td>-</td>
</tr>
<tr>
<td>0.025</td>
<td>0.03257</td>
<td>2.92</td>
</tr>
<tr>
<td>0.0125</td>
<td>0.00410</td>
<td>2.99</td>
</tr>
<tr>
<td>0.00625</td>
<td>0.00051</td>
<td>3.00</td>
</tr>
</tbody>
</table>

Fig. 10. The log-log graph of L₂ error Vs time step k

In the second set of computations, we set the same control parameters, as in the first set of computations. We performed the simulation of interaction scenario of four solitons to the time \( t = 100 \) with \( k = 0.01 \) to watch the performance of our developed scheme for long time simulation by observing the mass and energy conservation properties. The values of the conserved quantities mass and energy are given in Table 8 and amplitude of the pulses at different time \( t \) are depicted in Fig. 11. We present the interaction scenario of four solitons at different times in Fig. 12 with the help of a surface and contour plot. The values in Table 8 shows that the ETD-Padé(1,2) scheme conserves both the conserved quantities exactly, to at least five decimal places. It can be seen from Fig. 11 that the amplitude of the pulses at time \( t = 100 \) stayed exactly around the amplitude of the pulses at \( t = 0 \), where as the amplitude of the pulses at \( t = 40 \) reached at the highest point due to the collision occurred around \( t = 40 \). We also observe from Fig. 12 that the collision takes place at \( t \approx 40 \) and waves separate after the interaction with maintaining initial velocity and shape. Notice that Fig. 12 display that the interaction is elastic, which is indeed the case, since \( e = 1 \) indicates the model is completely integrable.
Table 8. Conserved quantities mass and energy for interaction of four solitons (37) using ETD-Padé(1,2)

<table>
<thead>
<tr>
<th>T</th>
<th>$|\psi_1|^2$</th>
<th>$|\psi_2|^2$</th>
<th>$|\psi_3|^2$</th>
<th>$|\psi_4|^2$</th>
<th>$I_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.00000</td>
<td>2.19089</td>
<td>2.28035</td>
<td>2.36643</td>
<td>4.00000</td>
</tr>
<tr>
<td>20</td>
<td>2.00000</td>
<td>2.19089</td>
<td>2.28035</td>
<td>2.36643</td>
<td>4.00000</td>
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<tr>
<td>40</td>
<td>2.00000</td>
<td>2.19089</td>
<td>2.28035</td>
<td>2.36643</td>
<td>4.00000</td>
</tr>
<tr>
<td>60</td>
<td>2.00000</td>
<td>2.19089</td>
<td>2.28035</td>
<td>2.36643</td>
<td>4.00000</td>
</tr>
<tr>
<td>80</td>
<td>2.00000</td>
<td>2.19089</td>
<td>2.28035</td>
<td>2.36643</td>
<td>4.00000</td>
</tr>
<tr>
<td>100</td>
<td>2.00000</td>
<td>2.19089</td>
<td>2.28035</td>
<td>2.36643</td>
<td>4.00000</td>
</tr>
</tbody>
</table>

Fig. 11. Interaction of four solitons obtained with ETD-Padé(1,2) at different time $t$
We selected the parameters $e = 2$, $v = 1.6$ and $t = 35$ in the third set of computations. The interaction scenario is displayed in Fig. 13. From the figure, we may observe the creation of new vector solitons, which can be the result of the large value of $e$ and moderate collision velocity.

In the fourth set of computations, we selected the parameters $e = 0.25$, which corresponds to
elliptically birefringent fibers, \( v = 0.8 \) and final time equal to \( t = 100 \). We see that the four solitons fusion into one soliton after the collision. This fusion scenario has been illustrated in Fig. 14.

![Fig. 14. Fusion scenario of four solitons with \( \epsilon = 0.25 \) and \( v = 0.8 \) via ETD-Padé(1,2)](image)

To verify numerically that the extrapolation form demonstrates better accuracy than the original scheme while dealing with a large system, we compared ETD-Padé(1,2) and its extrapolation for solving the system of four nonlinear Schrödinger equations. We have presented the computed results in Table 9. From the Table 9 we observe clearly that the extrapolation form shows better accuracy than the original scheme until a long time simulation, as expected. From the two different tests that computed the accuracy of developed scheme and its extrapolation, we can strongly say that extrapolation of the scheme is more accurate than the original scheme and enjoys a larger time step during the simulation.

Table 9. Comparison of maximum error of solution \(|\Psi_1| + |\Psi_2| + |\Psi_3| + |\Psi_4|\) of system (32) with \( \mu = 1, \ k = 0.01 \) and initial condition (37).

<table>
<thead>
<tr>
<th>T</th>
<th>Extrapolation-Padé(1,2)</th>
<th>ETD-Padé(1,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.00000</td>
<td>0.00034</td>
</tr>
<tr>
<td>10</td>
<td>0.00000</td>
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</tr>
<tr>
<td>15</td>
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<tr>
<td>20</td>
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<td>0.00544</td>
</tr>
</tbody>
</table>

From all the performed numerical tests for system of four nonlinear Schrödinger equations with ETD-Padé(1,2) scheme, we see that interaction scenarios are similar to that of the system of two nonlinear Schrödinger equations and all of them follow analytical prediction discussed by Yang [30] for system of two nonlinear Schrödinger equations.
6.4. Blow-up

According to Sanz-Serna et al. [12], in order for a numerical method for system (1) with \( N = 1 \) to be useful, it is necessary that it integrates \( x \)-independent version of system (1) accurately; failure to do so results in nonlinear blow-up of the solution in finite time. Keeping this in mind, in the present study system (1) with \( N = 1 \) and \( \alpha_1 = 0 \) is solved together with the following initial and boundary condition:

\[
\frac{i}{t} \frac{\partial \Psi_1}{\partial t} + 2|\Psi_1|^2\Psi_1 = 0, \tag{38}
\]

initial conditions:

\[
\Psi_1(x, 0) = e^{(2i(x+10))} \text{sech}(x + 10) + e^{(-2i(x-10))} \text{sech}(x - 10),
\]

and Neumann boundary condition:

\[
\frac{\partial \Psi_1}{\partial x}(x, t) = 0, \text{ at } x = \pm 25
\]

We selected \( h=0.1, k=0.01 \) and ran the simulation until \( t=1000 \) explicitly. During the simulation we did not observe any sign of blow and found that the developed scheme showed excellent numerical behaviors to simulate the dynamics of (38) with the mentioned initial and boundary condition and conserved the energy exactly, to at least three decimal places. We have depicted the simulation pattern of two solitons until \( t=1000 \) with \( k=0.01 \) in Fig. 15. As we can see from Fig. 15, the two solitons travel long distance without a significant change in shape and velocity. Although at this stage we do not have a formal analysis of blow-up of a developed scheme, we can say from the numerical results that the ETD-Padé(1,2) is a good choice for dealing with a system of coupled nonlinear Schrödinger equations.

![Fig. 15. Blow-up analysis of ETD-Padé(1,2) with diffusion free nonlinear Schrödinger equation (38)](image)
7. Conclusion

In this paper we have developed an exponential time differencing method and its extrapolation form for solving a system of two and four nonlinear Schrödinger equations. The numerical results reported that the developed scheme and its extrapolation exhibit the expected rate of convergence and the extrapolation of the developed scheme is more efficient and accurate than that of existing schemes mentioned in the paper [6][11]. We have studied the interaction of two and four solitons with different velocities and amplitudes and found that the scheme showed excellent numerical behaviors in simulated interaction, where no significant change occurred in the conserved quantities energy and mass at each time step. The numerical result indicated that the scheme did not notice any sign of blow up in finite time, but due to the lack of formal analysis of blow-up of the developed scheme at this stage we can not strongly say that the developed scheme is perfectly conservative. Finally, we recommend that the scheme is equally suitable for those nonlinear PDEs, which consist irregular initial data or discontinuity involving initial and boundary conditions, due to its ability to damp spurious oscillations, which is caused by high frequency components in the solution. As a conclusion, we can generalize that the scheme is useful to solve higher dimensional problems and this will be considered next.

Acknowledgements

The authors are grateful and appreciative for referee’s valuable and constructive suggestions, which improved original manuscript.

References


