Reversibility and Potentiality of Exclusion Processes on Countable Discrete Groups

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Abstract

Transition invariant exclusion processes on discrete groups is discussed in this article, as examples of spatially extended systems. We show that it is reversible if and only if its entropy production vanishes, or iff its speed functions field has some bounded potential, which is expressed in terms of physical states rather than mathematical states in probability space. Moreover, we conclude that the only possible bounded potential is the constant one.

KEY WORDS: Reversibility; Potentiality; Entropy production; Exclusion process

1 Introduction

The question whether a steady system is in equilibrium, correspondingly, whether a stationary stochastic process modelling the system is reversible, has attracted much interest from physicists as well as mathematicians. In fact, the theory of reversibility of Markov processes has been discussed for quite a long time since Kolmogorov. Many equivalent conditions for reversibility have been proved, such as the detailed balance, the vanishing of entropy production and the existence of potential. Here, we recommend [2] for the systematic conclusions of this theory.

For example, suppose that $X$ is an irreducible and positive-recurrent stationary Markov chain with continuous time, which has the finite state space $S$, the transition density matrix $Q = (q_{ij})_{i,j \in S}$ and the invariant probability distribution $\Pi = (\pi_i)_{i \in S}$. Then the following statements are equivalent:

(i) The Markov chain $X$ is reversible.

(ii) The Markov chain $X$ is in detailed balance, that is, $\pi_i q_{ij} = \pi_j q_{ji}, \forall i, j \in S$.

(iii) The entropy production rate $e_p = \frac{1}{2} \sum_{i,j \in S} (\pi_i q_{ij} - \pi_j q_{ji}) \log \frac{\pi_i q_{ij}}{\pi_j q_{ji}}$ vanishes.

(iv) The transition density matrix $Q$ of $X$ satisfies the Kolmogorov cyclic condition:

\[ q_{i_1i_2} q_{i_2i_3} \cdots q_{i_{s-1}i_s} q_{i_si_1} = q_{i_1i_s} q_{i_{s-1}i_{s-2}} \cdots q_{i_3i_2} q_{i_2i_1}, \]

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for any directed cycle \( c = (i_1, \ldots, i_s) \).

The last equation can be regarded as the existence of potential, i.e.
\[
\sum_{k=1}^{s-1} \log \frac{q(i_k, i_{k+1})}{q(i_{k+1}, i_k)} = \log \pi_{i_s} - \log \pi_{i_1},
\]
where the left side is regarded as the "force" in the sense of probability.

Therefore, we ask the same question to a interacting particle system. Interacting particle system is a large and growing field of probability theory which is devoted to the rigorous analysis of certain types of models that arise in statistical physics, biology, economics, and other fields. Its concept of reversibility was put forward in [3, 4], but only a few results about it were stated there. Chapter 11 of [1] deals with the reversibility of two important classes of particle systems, spin processes and exclusion processes where the relationship among reversibility, potentiality and Gibbs measures were discussed.

Recently, Zhang, F.X.[5] defined the entropy production density of exclusion processes on countable discrete groups, and showed that it vanishes if and only if the process is reversible. This is the first time to express the equivalent conditions of reversibility in terms of the physical states \( x, y \) rather than mathematical states \( \xi, \eta \) in probability space.

For example, consider the exclusion process \( \{\xi_t : t \geq 0\} \) on the discrete circle \( \mathbb{Z}_N = \mathbb{Z} (mod N) \). Suppose the transition probability matrix on \( \mathbb{Z}_N \) is transition invariant, i.e. \( p(x, y) = p(x + z, y + z) \) for any \( x, y, z \in \mathbb{Z}_N \). Take the invariant measure as the product measure with density \( \alpha[3] \). Then each state is a function on \( \mathbb{Z}_N \) with image 0 or 1, the invariant measure is
\[
\pi_{\xi} = \alpha \sum_{x \in \mathbb{Z}_N} \xi(x) (1 - \alpha)^{N - \sum_{x \in \mathbb{Z}_N} \xi(x)},
\]
and the Q-matrix is
\[
q_{\xi\eta} = \begin{cases} p(x, y), & \xi(x) = \eta(y) = 1, \xi(y) = \eta(x) = 0, \xi(z) = \eta(z), \forall z \neq x, y, \\ 0, & \text{otherwise}, \end{cases}
\]
for any \( \eta \neq \xi \). The EPR(entropy production rate) of \( \{\xi_t\} \) is easily computed as
\[
epr = \sum_{\xi, \eta} \pi_{\xi} q_{\xi\eta} \log \frac{\pi_{\xi} q_{\xi\eta}}{\pi_{\eta} q_{\eta\xi}} = N \alpha (1 - \alpha) \sum_x p(0, x) \log \frac{p(0, x)}{p(x, 0)}.
\]
Then the site-average EPR named the entropy production density is defined as
\[
epd = \frac{\text{epr}}{N} = \alpha (1 - \alpha) \sum_x p(0, x) \log \frac{p(0, x)}{p(x, 0)} = \frac{1}{2} \alpha (1 - \alpha) \sum_x [p(0, x) - p(x, 0)] \log \frac{p(0, x)}{p(x, 0)}.
\]
Each term in the summand is nonnegative, so epd vanishes if and only if \( p(0, x) = p(x, 0), \forall x \), or iff \( \{\xi_t\} \) is reversible.

In this paper, we continue on Zhang's work to consider the equivalence of reversibility and potentiality of exclusion processes on groups. The main result is proved in section 2:
Theorem 1.1 \(\{\xi_t\}\) is a transition invariant exclusion process on groups. The following statements are equivalent:
1. \(\{\xi_t\}\) is reversible;
2. The generator \(A\) is symmetric, i.e. \(\int fA g d\mu = \int gA f d\mu\), where \(\mu\) is the invariant measure of \(X\);
3. It is in detailed balance: i.e. \(p(x, y) = p(y, x)\) for any \(x, y\);
4. The entropy production density vanishes;
5. Its field of speed functions has a bounded potential;
6. Its field of speed functions has a constant potential.

And finally, we apply theorem 1.1 to amenable Cayley graphs in section 3.

2 Reversibility and Potentiality of Exclusion Processes on Groups

First, we recall the definition of the exclusion process in the lattice gas interpretation.

Let \(S\) be a countable set, \(N \geq 1\) is the maximum number of particles on each site, and \(X = \{0, 1, \ldots, N\}^S\). For any \(\xi \in X\), \(\xi(x)\) denotes the number of particles on site \(x\) for any \(x \in S\). Let \(\{p(x, y)\}\) be an irreducible transition probability matrix on \(S\).

Each particle at \(x\) waits for an exponential time with parameter one, then jumps to site \(y\) with probability \(p(x, y)\). The exclusion rule is that the jumping will be successful if \(\xi(y) \leq N - 1\) at that time. Let \(D\) be the set of all functions on \(X\) taking value in \(\{0, 1, 2, \ldots, N\}\) and each depends only on finitely many coordinates, and denote that

\[
\xi^{x,y}(z) = \begin{cases} 
\xi(x) - 1 & \text{if } z = x, \\
\xi(y) + 1 & \text{if } z = y, \\
\xi(z) & \text{otherwise,}
\end{cases}
\]

when \(\xi(x) \geq 1\) and \(\xi(y) \leq N - 1\).

An exclusion process \(\{\xi(t) : t \geq 0\}\) on \(S\) with transition rate \(\{p(x, y)\}\) is a Feller process whose infinitesimal generator acts on \(D\) as

\[
Af(\xi) = \sum_{x, y \in S: \xi(x) \geq 1, \xi(y) \leq N - 1} p(x, y)(f(\xi^{x,y}) - f(\xi)).
\]

Suppose \(S\) is a group, which we would like to denote by \(G\). Suppose \(\{p(x, y)\}\) is (left) translation invariant, i.e., \(p(x, y) = p(zx, zy)\) for any \(x, y, z \in G\). For any \(\alpha > 0\), let \(\alpha_k = \frac{\alpha^k}{Z(\alpha)}\), \(k = 0, 1, \ldots, N\), where \(Z(\alpha) = \sum_{l=0}^N \alpha^l\) is a normalization constant. Let \(\nu_\alpha\) be the product measure with marginal \(\nu_\alpha(\{\xi(x) = k\}) = \alpha_k\), \(k = 0, 1, \ldots, N\). A similar proof of Theorem VIII.2.1 in Ref. [3] shows that \(\nu_\alpha\) is an invariant measure of \(\{\xi(t)\}\).

Denote by \(e\) the unit element of \(G\), assume that \(p(e, x) > 0 \Leftrightarrow p(x, e) > 0\), \(\forall x \in G\).

To assure that the EPD(entropy production density) is finite, we also assume that

\[
\sum_{x \in G} p(e, x) |\log \frac{p(e, x)}{p(x, e)}| < \infty.
\] (1)
Let \( \{ \xi(t) : t \geq 0 \} \) be an exclusion process on \( G \) with translation invariant transition rate \( \{ p(x, y) \} \) and initial measure \( \nu_\alpha \).

The definition of reversibility comes from [3].

**Definition 2.1** The probability measure \( \mu \) on \( X \) is said to be reversible for the process with semigroup \( S(t) \) if it satisfies

\[
\int f S(t) g d\mu = \int g S(t) f d\mu
\]

for all \( f, g \in C(X) \).

The definition of EPD comes from [5], which has already been explained in the introduction.

**Definition 2.2** The EPD of \( \{ \xi_t \} \) is defined as

\[
\text{epd} := \frac{1}{2}(1 - \alpha_0)(1 - \alpha_N) \sup_{B \subset G, |B| < \infty} \left\{ \sum_{x \in G} \frac{|B \cap Bx^{-1}|}{|B|} (p(e, x) - p(x, e)) \log \frac{p(e, x)}{p(x, e)} \right\},
\]

where \( Bx^{-1} = \{yx^{-1} : y \in B\} \).

**Remark 2.3** As in [5], if \( G = \mathbb{Z}^d \), take \( B_n = \{ x \in \mathbb{Z}^d : |x| \leq n \} \), then one can directly define that \( \text{epd} = \lim_{n \to \infty} \frac{\text{epd}_{B_n}}{|B_n|} = \frac{1}{2}(1 - \alpha_0)(1 - \alpha_N) \sum_{x \in \mathbb{Z}^d} (p(e, x) - p(x, e)) \log \frac{p(e, x)}{p(x, e)} \).

**Definition 2.4** Given the speed functions \( \{ p(x, y) \} \), if there exists a function \( \{ V(x) : x \in G \} \) so that

\[
\sum_{i=0}^{n-1} \log \frac{p(x_i, x_{i+1})}{p(x_{i+1}, x_i)} = V(x_n) - V(x_0), \ \forall x_0, x_1, \cdots, x_n \in G
\]

then we say that the field of speed functions \( \{ p(x, y) \} \) has a potential \( \{ V(\cdot) \} \).

Here, we refer to [1][Chapter 7, 11] for the general field theory and potential theory.

**Proof of theorem 1.1:**

1) \( \Leftrightarrow \) 2): See proposition II.5.3 in [3].

2) \( \Leftrightarrow \) 3): From theorem 1.1 in [5].

3) \( \Leftrightarrow \) 4): Since each term in the expression of epd is nonnegative, and \( |p(e, x) - p(x, e)| \log \frac{p(e, x)}{p(x, e)} = 0 \) if and only if \( p(e, x) = p(x, e) \).

4) \( \Rightarrow \) 5): We can take \( V(x) = \text{constant} \) in definition 2.4, which is bounded.

5) \( \Rightarrow \) 6): This is the heart of the proof. It means that if the bounded potential exists, then it must be the constant one.

For instance, we first consider the case of \( \mathbb{Z}^2 \). As we all know, \( \mathbb{Z}^2 \) is an Abelian (commutative) group with finite generator \( x \) and \( y \), and every site of \( \mathbb{Z}^2 \) can be expressed as \( x^k y^h \), for some integer \( k \) and \( h \).
When $k$ and $h$ are positive, take $x_0 = e$, $x_i = x^i$, $i = 1, 2, \cdots, k$, and $x_{j+k} = x^ky^j$, $j = 1, 2, \cdots, h$, $n = k + h$ in (2), one can get that

$$\sum_{i=0}^{n-1} \log \frac{p(x_i, x_{i+1})}{p(x_{i+1}, x_i)} = \sum_{i=0}^{k-1} \log \frac{p(x^i, x^{i+1})}{p(x^{i+1}, x^i)} + \sum_{j=0}^{h-1} \log \frac{p(x^ky^j, x^ky^{j+1})}{p(x^ky^{j+1}, x^ky^j)} = V(x^ky^h) - V(e).$$

Moreover, by the transition invariance,

$$p(x^i, x^{i+1}) = p((x^i)^{-1}x^i, (x^i)^{-1}x^{i+1}) = p(e, x),$$

$$p(x^{i+1}, x^i) = p((x^i)^{-1}x^{i+1}, (x^i)^{-1}x^i) = p(x, e),$$

and

$$p(x^ky^j, x^ky^{j+1}) = p((x^ky^j)^{-1}x^ky^j, (x^ky^j)^{-1}x^ky^{j+1}) = p(e, y),$$

$$p(x^ky^{j+1}, x^ky^j) = p((x^ky^j)^{-1}x^ky^{j+1}, (x^ky^j)^{-1}x^ky^j) = p(y, e),$$

so

$$V(x^ky^h) - V(e) = k \log \frac{p(e, x)}{p(x, e)} + h \log \frac{p(e, y)}{p(y, e)}.$$

When $h \equiv 0$, and if $p(e, x) > p(x, e)$, then $\lim_{k \to \infty} V(x^k) = +\infty$, which contra the boundedness of potential function $V(.)$. On the other hand, if $p(e, x) < p(x, e)$, then $\lim_{k \to \infty} V(x^k) = -\infty$, which is another contradiction.

Therefore, $p(e, x) = p(x, e)$. Following the same steps above when $k \equiv 0$, one can get that $p(e, y) = p(y, e)$, too.

The case of $Z^d$ is just as the same as $Z^2$, $d \geq 3$.

Now, imitate the proof above, we can prove that 5) $\Rightarrow$ 6) when $G$ is an arbitrary countable discrete group.

6) $\Rightarrow$ 3): For each $n$, $\sum_{i=0}^{n-1} \log \frac{p(x_i, x_{i+1})}{p(x_{i+1}, x_i)}$ always vanishes. So for any $x \in G$, take $n = 1$, then log $\frac{p(e, x)}{p(x, e)} = V(x) - V(e) = 0$, i.e. $p(e, x) = p(x, e)$.

\[\blacksquare\]

Remark 2.5 Following the same steps of the proof, similar results also hold in the multi-colored exclusion processes whose reversibility and EPD have already been discussed in [5].

3 Reversibility and Potentiality of Exclusion Processes on Amenable Cayley Graphs

Suppose that $G$ is a finitely generated group and $H$ is a finite set of generators. Without loss of generality, suppose that $e \notin H$, and $x \in H \iff x^{-1} \in H$, $\forall x \in G$.

Define the set of edges as $E = \{ \{x, xh\} : x \in G, h \in H\}$. Then $\{G\}$ and $E$ compose a graph. Such graphs induced from finitely generated groups are called Cayley graphs, denoted by $(G, E)$.

Furthermore, we assume that $(G, E)$ is amenable, which means

$$\inf \{ |\partial B|/|B| : B \subset G, |B| < \infty \} = 0,$$

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where $\partial B = \{x \in B : N(x) \cap B^c \neq \emptyset\}$, and $N(x) = \{y \in G, (x, y) \in E\}$ is the set of neighbors of $x$, for any $x \in G$. In [5], Zhang F.X. has proved that its entropy production density can be expressed as

$$\text{epd} = \frac{1}{2} (1 - \alpha_0)(1 - \alpha_N) \sum_{x \in G} (p(e, x) - p(x, e)) \log \frac{p(e, x)}{p(x, e)}.$$  

(3)

Applying theorem 1.1, one can get another theorem that

**Theorem 3.1** \{\xi_t\} is a transition invariant exclusion process on the amenable Cayley graph $(G, E)$. The following statements are equivalent:
1. \{\xi_t\} is reversible;
2. The generator $A$ is symmetric, i.e. $\int f A g d\mu = \int g A f d\mu$, where $\mu$ is the invariant measure of $X$;
3. It is in detailed balance: i.e. $p(x, y) = p(y, x)$ for any $x, y \in G$;
4. The entropy production density (3) vanishes;
5. Its field of speed functions $(p(x, y))$ has a bounded potential;
6. Its field of speed functions has a constant potential.

**Remark 3.2** Applying theorem 1.1 to a Cayley graph relies on the associating group since a Cayley graph may be induced from different groups. The key point is under which group $(p(x, y))$ is translation invariant.

**References**


