An approach for spherical harmonic analysis of non-smooth data

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Abstract

A method is proposed to evaluate the spherical harmonic coefficients of a global or regional, non-smooth, observable dataset sampled on an equiangular grid. The method is based on an integration strategy using new recursion relations. Because a bilinear function is used to interpolate points within the grid cells, this method is suitable for non-smooth data; the slope of the data may be piecewise continuous, with extreme changes at the boundaries. In order to validate the method, the coefficients of an axisymmetric model are computed, and compared with the derived analytical expressions. Numerical results show that this method is indeed reasonable for non-smooth models, and that the maximum degree for spherical harmonic analysis should be empirically determined by several factors including the model resolution and the degree of non-smoothness in the dataset, and it can be several times larger than the total number of latitudinal grid points. It is also shown that this method is appropriate for the approximate analysis of a smooth dataset. Moreover, this paper provides the program flowchart and an internet address where the FORTRAN code with program specifications are made available.

Keywords: Spherical harmonic analysis; Non-smooth data; Recursion relations; Programming

1. Introduction

Spherical harmonic analysis is an important tool in the study of geoscience and planetary science because it has a wide range of applications; it can be used in the interpolation, decomposition, filtering, and analysis of any data obtained on the regular, equiangular grids on a sphere. The fundamental theory of spherical harmonics and their decomposition and synthesis is described in classic texts such as Hobson (1931) and MacRobert (1967). The major remaining obstacle in using spherical harmonics for geodetic analysis is the rapid and precise numerical evaluation of high-order coefficients (see Sneeuw, 1994; Blais and Provins, 2002; Provins, 2003 for a review). This paper is concerned with the analysis of non-smooth observables using spherical harmonics of appropriate degree and order. In this context, by “non-smooth” we mean data which may be discontinuous in slope. Non-smooth data are in fact commonplace in planetary observations. In topographical data, for example, obvious slope

\textsuperscript{*}The code is available from server at http://www.iamg.org/CGEditor/index.htm.

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discontinuities exist at mountain ridges and the edges of basins and oceans. It has been found, however, that current programs (e.g. Spherepack3.1) provide too smooth a synthesis result for non-smooth observables. Accordingly, the aim of this paper is to develop an analysis method that is effective for non-smooth data.

The analysis procedure can be outlined as follows. The input data are first discretized onto grids (e.g., Gaussian grids or equally spaced grids), and then integrated over the surface area (see Eq. (6), below). The integration is separated into two parts: a latitudinal integration using Legendre polynomials, and a longitudinal integration involving harmonic functions. The former can be calculated using well-known recurrence relations, and the latter using a (fast) Fourier transform (e.g. Ricardi and Burrows, 1972; Colombo, 1981). Whereas there are several different analysis techniques, the number of the data points needed for a successful analysis is often relatively uniform. According to Kampes (1998) it takes \( N_{\text{max}} + 1 \) parallels and 2 \( N_{\text{max}} \) meridians to compute a spectrum to the degree \( N_{\text{max}} \).

Considering that many global/regional geoscience datasets and model results are given on regular grids (i.e., equiangular along longitude and latitude), this paper also uses a regular grid and proposes a simple and direct approach to the spherical harmonic expansion of non-smooth datasets in the space domain. This approach is valid for non-smooth data because it assumes that the input data can be interpolated using bilinear functions. This approach also allows integrations on the grid to be solved analytically using recursion relations. For the sake of completeness, all the fundamental formulas used in this paper are summarized in Section 2. In Section 3, our new integration strategy and its recursion relations are derived. Section 4 describes the algorithm and code. The criteria used to validate our method are given in Section 5, whereas Section 6 describes numerical results for three models. Conclusions are given in Section 7. Finally, Appendix A describes two other types of spherical harmonic expansion and their relation to the present work.

2. Summary of fundamental formulas

A continuous function \( f(\theta, \phi) \) defined on the surface of a sphere (where \((r, \theta, \phi)\) is the usual spherical coordinate system), can be expanded into the spherical harmonics (e.g. Wang and Guo, 2000, pp. 217–283):

\[
f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{l,m} Y_{l,m}(\theta, \phi),
\]

where \( Y_{l,m}(\theta, \phi) \) and \( f_{l,m} \) are the surface spherical harmonics of degree \( l \) and order \( m \) (\( 0 \leq |m| \leq l \)) and their coefficient, respectively.

\[
\begin{align*}
Y_{l,m}(\theta, \phi) &= (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\phi}, \\
Y_{l,-m}(\theta, \phi) &= (-1)^m (Y_{l,m}(\theta, \phi))^*.
\end{align*}
\]

(2)

where the \( P_l^m \) are associated Legendre polynomials of degree \( l \) and order \( m \). If we let \( x = \cos \theta \), then the associated Legendre polynomials can be related to the ordinary Legendre polynomials by

\[
P_{l,m}(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) = \frac{(1 - x^2)^{m/2}}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l,
\]

(3)

where \( P_l(x) \) is the Legendre polynomial of degree \( l \) and is given by

\[
P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l.
\]

(4)

Using the orthogonality of the spherical harmonics

\[
\int_{\sigma} Y_{l,m}(\theta, \phi) Y_{l,m'}^*(\theta, \phi) \, d\sigma = \delta_{ll'} \delta_{mm'}, \quad -l \leq m \leq l,
\]

(5)

where \( \sigma \) denotes the surface of a unit sphere and \( d\sigma = \sin \theta d\theta d\phi \), we can easily obtain from Eq. (1)

\[
\begin{align*}
f_{l,m} &= \int_{\sigma} f(\theta, \phi) Y_{l,m}^*(\theta, \phi) \, d\sigma, \\
f_{l,-m} &= (-1)^m f_{l,m}^*.
\end{align*}
\]

(6)

Before evaluating the spherical harmonics in Eq. (2), one needs to determine the associated Legendre polynomials as follows. For a given \( m \), one begins by computing polynomials for the first two \( l \)-values allowed: \( P_{m,m}(x) = (2m - 1)!! \times (1 - x^2)^{m/2} \) and \( P_{m+1,m}(x) = (2m + 1)!! \times P_{m,m}(x) \), where \( (2m - 1)!! = 1 \times 3 \times 5 \times \cdots \times (2m - 1) \) for \( m \geq 1 \) and \( (2m - 1)!! = 1 \) for \( m = 0 \). Next, the recursion relation \( P_{l,m}(x) = [(2l - 1)xP_{l-1,m}(x) - (l + m - 1) P_{l-2,m}(x)]/(l - m) \) can be used to move from \( P_{m,m} \) and \( P_{m+1,m} \) to the remaining \( P_{l,m} \). Finally, the spherical harmonics and their coefficients can be obtained from Eqs. (2) and (6). For low-order spherical harmonic expansions, this procedure is straightforward. For high-order expansions, however, this procedure may fail numerically due to
underflow from the factor \((l - m)!/(l + m)!\) in Eq. (2). Likewise, \(P_{lm}(x)\) may become so large that overflow occurs. To avoid this, we perform computations using the quantity:

\[
T_{lm}(x) = \sqrt{\frac{(l - m)!}{(l + m)!}} P_{lm}(x).
\]  

(7)

Similar treatments using normalized associated Legendre function can be found in previous works (e.g. Blais and Provins, 2002). Accordingly, Eq. (2) can be rewritten as

\[
\begin{aligned}
Y_{l,m}(\theta, \phi) &= (-1)^m \sqrt{\frac{2l+1}{4\pi}} T_{lm}(x) e^{im}\phi, \\
Y_{l-m}(\theta, \phi) &= (-1)^m Y_{l,m}(\theta, \phi)^*.
\end{aligned}
\]

(8)

To compute \(T_{lm}(x)\), we use the recursion relation

\[
T_{lm}(x) = \sqrt{\frac{(2l - 1)(2l - 1)}{(l-m)(l+m)}} x T_{l-1,m}(x) - \sqrt{\frac{(l-m-1)(l+m-1)}{(l-m+1)(l+m)}} T_{l+1,m}(x),
\]

(9)

where \(l = m + 2, m + 3, \ldots, l_{\text{max}}\) (the maximum degree required). Following Gather (1977), this recursion starts with the computation of \(T_{m,m}(x)\) and \(T_{m+1,m}(x)\) using the Rodriguez formula:

\[
\begin{aligned}
T_{m,m}(x) &= \sqrt{\frac{(2m-1)!!}{2m!!}} (1 - x^2)^{m/2}, \\
T_{m+1,m}(x) &= \sqrt{\frac{(2m+1)!!}{2m!!}} x (1 - x^2)^{m/2},
\end{aligned}
\]

(10)

where \((2m)!! = 1 \times 2 \times 4 \times \cdots \times (2m)\) for \(m \geq 1\), \((2m)!! = 1\) for \(m = 0\) and \((1 - x^2)^{m/2} = 1\) when \(x = \pm 1\) and \(m = 0\).  

For high values of \(l\) and \(m\), the recursion Eq. (9) may become unstable. Based on numerical tests, however, the recursion Eq. (9) remains stable up to a degree of 2200 for double precision calculations. This is similar to the situation of the normalized associated Legendre functions (e.g. Wenzel, 1998). To improve stability beyond this point, quadruple precision calculations are necessary.

### 3. Integration strategy

Assume that the data are given on the domain \(0_a \leq \theta \leq 0_b, \quad 0_a \leq \phi \leq 0_b\), where \(0_a \geq 0\), \(0_b \leq 180^\circ\), \(0_a \geq 0\), and \(0_b \leq 360^\circ\). Assume further that the data mesh is equally spaced along both longitude and latitude, and the sampling intervals are \(\Delta \theta\) and \(\Delta \phi\). Let an arbitrary grid cell be marked by \(E_{ij}(i = 1, 2, \ldots, i_{\text{max}}; j = 1, 2, \ldots, j_{\text{max}})\), with its four nodes labeled by \(P_{i,j-1}, P_{i+1,j}, P_{i,j} \) and \(P_{i-1,j}\). The node rows are numbered from 0 at \(\theta = \theta_a\) to \(i_{\text{max}}\) at \(\theta = \theta_b\), and the node columns from 0 at \(\phi = \phi_a\) to \(j_{\text{max}}\) at \(\phi = \phi_b\). It will be shown below that under such global/regional grids the formulation is quite simple. For regional gridded data (\(0_a > 0, \theta_b < 180^\circ, \phi_a > 0, \phi_b < 360^\circ\)), it should be noted that the coefficients obtained by this method do not include contributions from data outside of the given range. Under these assumptions, Eq. (6) can be generally rewritten as

\[
\begin{aligned}
\int_{-1}^{1} f(\theta, \phi) Y_{l,m}(\theta, \phi) d\phi d\theta &= \sum_{i=1}^{i_{\text{max}}} \sum_{j=1}^{j_{\text{max}}} A_{ij} \phi_j f_i(x_i, \phi_j) \Delta \phi + B_{ij} \phi f_i(x_i, \phi_j) \Delta \phi + C_{ij} x f_i(x_i, \phi_j) \Delta \phi + D_{ij} \phi f_i(x_i, \phi_j) \Delta \phi \\
\int_{-1}^{1} f(\theta, \phi) Y_{l,m}(\theta, \phi) d\phi d\theta &= \sum_{i=1}^{i_{\text{max}}} \sum_{j=1}^{j_{\text{max}}} A_{ij} \phi_j f_i(x_i, \phi_j) \Delta \phi + B_{ij} \phi f_i(x_i, \phi_j) \Delta \phi + C_{ij} x f_i(x_i, \phi_j) \Delta \phi + D_{ij} \phi f_i(x_i, \phi_j) \Delta \phi.
\end{aligned}
\]

(11)

where the summation is over all grid cells within the domain \(0_a \leq \theta \leq 0_b, \phi_a \leq \phi \leq \phi_b\).

Next, the integration in Eq. (11) for a specific grid cell \(E_{ij}\) (with domain \(0_{ij-1} \leq \theta \leq 0_{ij}, \phi_{ij-1} \leq \phi \leq \phi_{ij}\)) needs to be evaluated. Allowing for the possibility that the gridded dataset may not represents a smooth quantity, the values of \(f(\theta, \phi)\) within the grid can be approximated by the bilinear function:

\[
f(\theta, \phi) = f(x, \phi) = A_{ij} x + B_{ij} \phi + C_{ij} x \phi + D_{ij} \phi.
\]

(12)

In (12), the four interpolation coefficients can be determined from the function values at the four nodes by solving a set of linear algebraic equations. The solution is

\[
\begin{aligned}
A_{ij} &= \left\{ \phi_{ij-1} f(x_{ij-1}, \phi_{ij}) - f(x_i, \phi_{ij}) \right\} / \Delta, \\
B_{ij} &= \left\{ x_{ij-1} f(x_i, \phi_{ij-1}) - f(x_i, \phi_{ij}) \right\} / \Delta, \\
C_{ij} &= \left\{ f(x_{ij-1}, \phi_{ij-1}) + f(x_i, \phi_{ij}) \right\} / \Delta, \\
D_{ij} &= \left\{ x_i \phi_j f(x_{ij-1}, \phi_{ij-1}) - x_{ij-1} \phi f(x_{ij-1}, \phi_{ij}) + x_{ij-1} \phi_{ij} f(x_i, \phi_{ij-1}) - x_i \phi f(x_i, \phi_{ij}) \right\} / \Delta.
\end{aligned}
\]

(13)
where \( \Delta = (x_i - x_{i-1})(\phi_j - \phi_{j-1}) \). Inserting Eqs. (12) and (8) into Eq. (11), we obtain

\[
f_{i,m} = (-1)^{m+1} \sqrt{\frac{2l+1}{4\pi}} \sum_{l=1}^{\infty} \int_{x_{i-1}}^{x_i} x T_{l,m}(x) \, dx \int_{\phi_{j-1}}^{\phi_j} \left( A_{ij} + C_{ij}\phi \right) e^{-im\phi} \, d\phi
\]

\[
+ (-1)^{m+1} \sqrt{\frac{2l+1}{4\pi}} \sum_{l=1}^{\infty} \int_{x_{i-1}}^{x_i} T_{l,m}(x) \, dx \int_{\phi_{j-1}}^{\phi_j} \left( D_{ij} + B_{ij}\phi \right) e^{-im\phi} \, d\phi, \quad 0 \leq m \leq l,
\]

(14)

where the integrations inside the braces have analytical solutions. We define

\[
SE(m, \phi_{j-1}, \phi_j, A_{ij}, C_{ij}) = \int_{\phi_{j-1}}^{\phi_j} \left( A_{ij} + C_{ij}\phi \right) e^{-im\phi} \, d\phi
\]

\[
= \begin{cases} 
[A_{ij}\phi + \frac{1}{2}C_{ij}\phi^2]_{\phi_{j-1}}, & m = 0, \\
\left[ A_{ij} + C_{ij}\phi \right]_{\phi_{j-1}} + C_{ij} \frac{\phi}{m} e^{-im\phi}_{\phi_{j-1}}, & 0 < m \leq l,
\end{cases}
\]

(15)

to simplify notation. The integrations within square brackets in Eq. (14) can be solved using recursion relations. We denote these integrals by

\[
\begin{align*}
ST_{l,m}(x_{i-1}, x_i) &= \int_{x_{i-1}}^{x_i} T_{l,m}(x) \, dx, \\
SXT_{l,m}(x_{i-1}, x_i) &= \int_{x_{i-1}}^{x_i} x T_{l,m}(x) \, dx,
\end{align*}
\]

(16)

By utilizing the two recursion relations (e.g. Wang and Guo, 2000, pp. 217–283):

\[
\begin{align*}
(l - m)P_{l,m}(x) &= (2l - 1)xP_{l-1,m}(x) - (l + m - 1)P_{l-2,m}(x), \\
(1 - x^2)P_{l,m}(x) &= (l + m)P_{l-1,m}(x) - lP_{l-2,m}(x), \\
l &= m + 2, m + 3, \ldots, l \text{ max},
\end{align*}
\]

(17)

and taking their definite integrals we obtain

\[
\begin{align*}
\int_{x_{i-1}}^{x_i} P_{l,m}(x) \, dx &= -\frac{2l-1}{(l+m)(l-m)}(1 - x^2)P_{l-1,m}(x)_{x_{i-1}}^{x_i} \\
&+ \frac{(l-m+1)}{(l+1)(l-m)} \int_{x_{i-1}}^{x_i} P_{l-2,m}(x) \, dx, \\
\int_{x_{i-1}}^{x_i} x P_{l,m}(x) \, dx &= -\frac{1}{l+1}(1 - x^2)P_{l,m}(x)_{x_{i-1}}^{x_i} \\
&+ \frac{l+m}{l+2} \int_{x_{i-1}}^{x_i} P_{l-1,m}(x) \, dx, \quad l = m + 2, m + 3, \ldots, l \text{ max},
\end{align*}
\]

(18)

Thus, from Eqs. (7), (16) and (18) we obtain

\[
\begin{align*}
ST_{l,m} &= -\frac{2l-1}{(l+1)(l+m)}(1 - x^2)T_{l-1,m}(x)_{x_{i-1}}^{x_i}, \\
+ \frac{l-2}{l+1} \sqrt{\frac{(l-m)(l+1)}{(l-m)(l+m)}} ST_{l-2,m}, \\
l &= m + 2, m + 3, \ldots, l \text{ max},
SXT_{l,m} &= -\frac{1}{l+1}(1 - x^2)T_{l,m}(x)_{x_{i-1}}^{x_i} \\
+ \sqrt{\frac{(l-m)(l+m)}{(l+2)(l+3)}} SXT_{l-1,m},
\end{align*}
\]

(19)

with the initial values

\[
\begin{align*}
ST_{m,m} &= \sqrt{\frac{(2m-1)!}{(2m)!}} \frac{1}{m+1}(1 - x^2)^{m/2}x_{x_{i-1}}^{x_i}, \\
&+ \frac{m}{m+1} \sqrt{\frac{(2m-1)(2m-3)}{(2m-2)!}} ST_{m-2,m}, \quad m \geq 2, \\
SXT_{m,m+1} &= -\frac{1}{m+1} \sqrt{\frac{(2m-1)!}{(2m)!}}(1 - x^2)^{m+2/2}x_{x_{i-1}}^{x_i}, \quad m \geq 0, \\
SXT_{m,m} &= -\frac{1}{m+1} \sqrt{\frac{(2m-1)!}{(2m)!}}(1 - x^2)^{m+2/2}, \quad m \geq 0.
\end{align*}
\]

(20)

\[
ST_{m,m} \text{ in Eq. (20) is itself a recursive expression with initial values}
\]

\[
\begin{align*}
ST_{0,0} &= x_i - x_{i-1}, \\
ST_{1,1} &= \frac{\sqrt{2}}{4} \left( \arccos x + \sqrt{1 - x^2} \right)_{x_{i-1}}^{x_i}.
\end{align*}
\]

(21)

Eqs. (14)–(16) and (19)–(21), are the formulas used in numerical evaluation of the spherical harmonic coefficients.

4. The algorithm and code

The FORTRAN code dragon.f is used for spherical harmonic analysis (i.e., the calculation of
spherical harmonic coefficients for a given dataset) and synthesis. The results of synthesis can be compared to the original grid data to check the accuracy of the analysis. Fig. 1 shows a flowchart for our implementation of the procedure described in Section 3. The most computationally intensive part of the program is the recursive calculation of $ST_{l,m}$ and $SXT_{l,m}$ (see Eqs. (19) and (20)). In order to reduce the computation time, the four loops are arranged to avoid any duplication in the computations of $ST_{l,m}$ and $SXT_{l,m}$. In the case of a fixed mesh the computer time is further reduced by calculating the results of $ST_{l,m}$ and $SXT_{l,m}$ only once and storing them in a lookup table for later steps. Memory usage is thus traded for speed—but memory these days is relatively cheap. It should be noted that in our approach the faster algorithm is not used, and the code is not as fast as software.
based on the FFT technique (e.g., Spherepack3.1). As will be shown in Section 6, our approach is more advantageous for non-smooth grid data and can also be used for the approximate analysis of smooth data. The code dragon.f has been successfully tested on various FORTRAN compilers, including the Powerstation 4.0 and Intel FORTRAN compilers, and is compatible with both FORTRAN 90 and FORTRAN 95. The code can also calculate the coefficients of two other important spherical harmonic expansions, which are discussed in Appendix A.

5. Criteria for program validation

Two criteria will be used to test the validity of the program for spherical harmonic analysis. A straightforward and common sense criterion is to see whether synthesis of the decomposed spectrum returns the original gridded data (this will be called a “synthesis check” below). Another important approach is to compare the numerically computed spherical harmonic coefficients with those that can be computed analytically (this will be called a “coefficient check/comparison”).

In the next section, we will use the code dragon.f to perform a coefficient check for data whose coefficients can all be determined analytically. Analytical expressions for the harmonic coefficients of some axisymmetric cases are well known in the literature, but they only give coefficients for order $m = 0$. In order to test our code for harmonics with $m > 0$, we can consider functions with an axis of rotational symmetry displaced from the polar axis. The coefficients of such a model can be determined analytically.

Suppose that the symmetry axis of a continuous function $f(\theta, \phi)$ is located at $(\theta', \phi')$ ($\theta' \neq 0$), so that the angular distance between an arbitrary point $(\theta, \phi)$ and the symmetry axis is $\gamma$ and $f(\theta, \phi) = 0$ when $\gamma > \alpha$. We can expand $f(\theta, \phi)$ into a series of Legendre polynomials:

$$f(\theta, \phi) = \sum_{l=0}^{\infty} f_l P_l(\cos \gamma).$$

where

$$f_l = \frac{2l+1}{2} \int_0^{2\pi} f(\theta, \phi) P_l(\cos \gamma) \sin \gamma \, d\gamma$$

$$= \frac{2l+1}{2} \int_{\cos \gamma}^{1} f(x') P_l(x') \, dx'.$$

In Eq. (23), $x' = \cos \gamma$. Using the Addition Theorem of spherical harmonics (e.g. Wang and Guo, 2000, pp. 217–283)

$$P_l(\cos \gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi).$$

Eq. (22) can be rewritten as

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} f_l Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi).$$

Comparing Eq. (25) with Eq. (1), we see that the coefficients are given by

$$f_{l,m} = \frac{4\pi}{2l+1} f_l Y_{lm}^*(\theta', \phi'),$$

where $f_l$ and $Y_{lm}$ can be easily calculated from Eqs. (23) and (8)–(10). Because the center of the axisymmetric model is not at the poles ($\theta' \neq 0$), coefficients of order $m \neq 0$ also exist.

Since the coefficient check involves many harmonics when the degree is large, the comparison can be accomplished more efficiently by combining the amplitudes of all positive orders ($0 \leq m \leq l$) for each degree $l$ into:

$$\text{Amplitude} = \sqrt{\sum_{m=0}^{l} |f_{l,m}|^2}.$$  (27)

It should be mentioned that the axisymmetric model is not ideal for the validation of our code. The reason is that such a model is in fact tangentially smooth, whereas this cannot be precisely described by the bilinear function. Despite this, a cone model is chosen and proved to be effective in the next section.

6. Numerical validation

The purpose of this section is to validate the methodology described above by showing that the code dragon.f meets the two criteria mentioned in the last subsection.

The first example of non-smooth grid data used is a cone model whose center is located at $(11^\circ, 75^\circ)$, with angular radius $\alpha = 10^\circ$ and an $1^\circ \times 1^\circ$ equiangular grid. The data for this cone model are given by

$$f(\theta, \phi) = \begin{cases} 1 - \frac{\gamma}{\alpha}, & \gamma \leq \alpha, \\ 0, & \gamma > \alpha, \end{cases}$$

(28)
where \( g \) is the angular distance from a grid node \((y, f)\) to the center of the cone. The planar contours of the model are shown in Fig. 2. Fig. 3 shows the difference between the synthesis and the original cone model, including harmonics up to degree and order 180. Inspection of Fig. 3 shows that the agreement is quite good; the RMS difference is 0.009, and the average relative error is 0.8%. As can be seen in Table 1, which lists the results of the coefficient check for selected harmonics of degree up to 180, the differences between the synthesis result and analytic solution are small except for values very close to zero and harmonics of degree larger than 80. Fig. 4 gives the results of the degree–amplitude spectrum analysis. The result shown in Fig. 4a is based on the coefficients given by Eq. (26). Fig. 4b shows the difference of the degree–amplitude spectrum based on the coefficients by Eqs. (14) and that given by Fig. 4a. The RMS difference between the two methods is 2.6E-5, and average relative difference is 1.7%. These differences comes from the high-degree harmonics, and result from the tangential smoothness of the model. By using finer resolutions, the differences can be narrowed considerably. Figs. 3, 4 and Table 1 also display results for data on a 0.25\(^\circ\) \times 0.25\(^\circ\) resolution grid. At this resolution, the RMS difference between the model and the synthesis results is 0.004, and the average relative difference is 0.4% (Fig. 3). The coefficient comparison yields an RMS difference of 2.1E-6 and an average relative difference of 0.8% (Fig. 4).

A second numerical case study is based on gridded data with many sharp changes. The purpose of this test is to validate our method for non-smooth datasets, and discuss the relationship between the maximum degree used in the spherical harmonic analysis and the quality of the synthesis. The data represent a surface that is the sum of eight overlapping cones (Fig. 5). Each cone is defined by

\[
 f(y, f) = \begin{cases} 
 h(1 - \gamma / a), & \gamma \leq a, \\
 0, & \gamma > a, 
\end{cases}
\]

(29)

where \( \gamma \) is the angular distance from a grid node to the cone’s axis, \( a \) is the radius of the cone, and \( h \) is the height of the cone. The centers of the eight cones are located on the 60\(^\circ\) longitude meridian (Fig. 5a). The cone centers are at co-latitudes 20\(^\circ\), 40\(^\circ\), 60\(^\circ\), 80\(^\circ\), 100\(^\circ\), 120\(^\circ\), 140\(^\circ\) and 160\(^\circ\); the cone radii are 12\(^\circ\), 20\(^\circ\), 32\(^\circ\), 8\(^\circ\), 12\(^\circ\), 12\(^\circ\), 20\(^\circ\) and 16\(^\circ\); the heights are 1, 1, 1, 1.4, 1, 1.2, 1.4 and 0.8, respectively. A profile of the gridded data along longitude 60\(^\circ\) is plotted in Fig. 5b; several discontinuities in the slope are evident.

First we utilize SPHEREPACK 3.1 (John and Swarztrauber, 2003) to analyze the spherical harmonics of this function on a 4\(^\circ\) \times 4\(^\circ\) grid up to degree and order 45 (the highest values compatible with this grid spacing). After these coefficients are calculated we plot the synthesis on a finer grid of size 0.5\(^\circ\) \times 0.5\(^\circ\); the result is displayed in the

Fig. 2. Cone model. Center is at (75\(^\circ\)E, 79\(^\circ\)N), bottom radius is 10\(^\circ\), and height is 1.0.

Fig. 3. Synthesis check for section of cone model (with longitude 75\(^\circ\)E). \( f \) is by cone model, and \( f^s \) is synthesis of harmonics.

where \( \gamma \) is the angular distance from a grid node \((\theta, \phi)\) to the center of the cone. The planar contours of the model are shown in Fig. 2. Fig. 3 shows the difference between the synthesis and the original cone model, including harmonics up to degree and order 180. Inspection of Fig. 3 shows that the agreement is quite good; the RMS difference is 0.009, and the average relative error is 0.8%. As can be seen in Table 1, which lists the results of the coefficient check for selected harmonics of degree up
Table 1
Coefficient check for cone model

<table>
<thead>
<tr>
<th>$l$</th>
<th>$m$</th>
<th>$f_{lm}^{'\text{in}}$</th>
<th>$f_{lm}$</th>
<th>$f_{lm}$</th>
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<tr>
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<tr>
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<td>180</td>
<td>0</td>
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</table>

Note: $f_{lm}^{'\text{in}}$ and $f_{lm}$ are evaluated using Eqs. (14) and (26). The two columns of the results are the real and imaginary components, in that order.

Fig. 4. Degree-amplitude spectrum analysis of cone model: (a) uses spherical harmonic coefficients given by Eq. (26); (b) is difference between amplitude spectrum given by Eqs. (14) and (a), for $1^\circ \times 1^\circ$ (solid line) and $0.25^\circ \times 0.25^\circ$ (dashed line) grids.
Close inspection shows that the synthesized data are smoother than the original data in Fig. 5b. The difference between the two is shown in the right-hand plot of Fig. 6. The RMS difference is 0.034, and the average relative difference is 3.6%.

Next we perform the same analysis with our code for maximum degrees and orders 45, 90, 135, 180, 270, and 360. The results are shown in Fig. 7. As the maximum degree and order increases, the synthesis results ($f_s$ in the left-hand plots) achieve sharper transitions. The differences ($f_s - f$) between the synthesis and the original data diminish gradually with increasing degree; the RMS differences are 0.038, 0.020, 0.013, 0.010, 0.007, and 0.006 and the average relative differences are 3.4%, 2.0%, 1.4%, 1.2%, 0.9% and 0.8%, respectively. This demonstrates the effectiveness of our methodology on non-smooth data. The bilinear approximation incorporates additional information on scales below the grid spacing, describing function behavior within the grid cells and any discontinuities in the slope at the grid edges. Note that, if 135 is chosen for the maximum degree and order, the given $0.5^\circ \times 0.5^\circ$ grid data begin to preferably get back by synthesis (left-hand plots of Fig. 7).

What should the maximum degree be for a spherical harmonic expansion and synthesis of...
Fig. 7. Same as Fig. 6 except that synthesis checks are done based on dragon. $f$ coefficients for maximum degrees of 45, 90, 135, 180, 270 and 360.
non-smooth data? According to Kampes (1998), the number of latitudinal grid points is a lower bound for the maximum degree. The Kampes rule may not be relevant, however, because the appropriate maximum degree also depends on the degree of non-smoothness in the dataset, the
recovery of bilinear functions within grid cells, and
the desired synthesis check precision. Because non-
smoothness can only occur along the grid edges, a
sharper dataset will require a higher maximum
degree. A higher maximum degree is also necessary
to recover the bilinear functions within grid cells,
and if we require higher precision in the synthesis
check. In the multiple cones model tested here, for
example, achieving synthesis check RMS differences
of 0.02 and 0.006 requires analysis to a maximum
degree of 90 and 360, respectively.

Our third numerical case study is the smooth
great height data shown in Fig. 8, which is sampled
on a global 1.2° × 1.2° grid. The data are computed
based on the spherical harmonic coefficients of a
model for the Earth’s gravity field that is complete
to degree and order 150 (Reigber et al., 2005). Note
that the spherical harmonic expansion used in
Reigber et al. (2005) takes the form of Eq. (A.2)
in Appendix A. Reigber’s coefficients \( c_{l,m}, s_{l,m} \)
therefore have to be transformed into the \( f_{l,m} \)
defined in this paper by using Eq. (A.5). The goal
of this analysis is to probe the degree of error
involved when our method is applied to an
extremely smooth dataset.

The results of the coefficient check can be seen in
Table 2. Spherical harmonic analysis of the data
using Eq. (14) gives the coefficients indicated by
\( f_{l,m} \), which are close to the original data
\( f_{l,m} \) given by the

![Fig. 8. Earth's geoid height on a global 1.2° × 1.2° grid (m).](image)

Table 2
Coefficient check for global geoid height model

<table>
<thead>
<tr>
<th>( l )</th>
<th>( m )</th>
<th>( f'_{l,m}(m) )</th>
<th>( f_{l,m}(m) )</th>
<th>( l )</th>
<th>( m )</th>
<th>( f'_{l,m}(m) )</th>
<th>( f_{l,m}(m) )</th>
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<td>1.99E-1</td>
<td>150 20</td>
<td>-2.93E-2</td>
<td>-1.95E-2</td>
</tr>
</tbody>
</table>
Fig. 9. Degree–amplitude spectrum analyses for geoid height data: (a) is based on spherical harmonic coefficients given for gravity model (Reigber et al., 2005), and (b) is difference between spectrum computed by Eq. (14) and that shown in (a).

Fig. 10. Synthesis check for two cross sections of global geoid height model: (a) and (c) are grid data along latitudes 0° and 45°S, respectively; (b) and (d) are differences between synthesis results and corresponding grid data.
with that calculated from the given coefficients (Fig. 9a) for degrees less than 100, since the relative differences are small up to that point (Fig. 9b). Over the whole range of degrees, the RMS difference is 0.05 m and the average relative difference is 1.7%. Fig. 10 gives synthesis check results over two profiles with latitudes 0° (Fig. 10a,b) and 45°S (Fig. 10c,d). The differences between the synthesis results and the original data (Fig. 10a,c) are very small for both sections (Fig. 10b,d). The RMS profiles with latitudes 0° and 45°S give synthesis check results over two sections (Fig. 10b,d). The differences between the synthesis and original data (Fig. 10a,c) are very small up to that point (Fig. 9b). Over the whole range of degrees, the RMS difference is 0.28 m.

This analysis shows that when our method (dragon.f) is used on a very smooth dataset, it may have significant errors in the higher degrees and orders. The error obviously arises from the bilinear approach, which is inconsistent with the smoothness of the dataset. Despite this, dragon.f can be used for the approximate analysis of smooth data. For geophysical studies of planetary body deformations, the approximate analysis of smooth data can be several orders of magnitude larger than the total number of latitudinal grid points. The appropriate maximum degree can only be empirically determined, however, by taking into account the model resolution, the non-smoothness of the dataset, and the error tolerance requirement.

The code can also be used for the approximate analysis of smooth data. The authors sincerely welcome readers to utilize our method and code in different fields of geoscience. It should be noted that the piecewise bilinear modeling approach adopted in this paper is not the only way to model non-smooth data. Its advantages and disadvantages compared with other methods remain to be investigated.

Acknowledgments

We thank two anonymous reviewers for their very helpful suggestions. Dr. Soofi helped test the code on more FORTRAN compilers. This work is supported by the National Natural Science Foundation of China grant 40574010, and by the Chinese Academy of Sciences grant KZCX3-SW-132.

Appendix A

Two other important spherical harmonic expansions and their relation to this paper.

Two other spherical harmonic expansions have also been widely used (e.g. Heiskanen and Moritz, 1967; Wahr et al., 1998):

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ a_{l,m} R_{l,m}(\theta, \phi) + b_{l,m} S_{l,m}(\theta, \phi) \right] \quad (b_{l,0} = 0)$$  \hspace{1cm} (A.1)

and

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ c_{l,m} H_{l,m,1}(\theta, \phi) + s_{l,m} H_{l,m,2}(\theta, \phi) \right] \quad (s_{l,0} = 0),$$  \hspace{1cm} (A.2)

where $R_{l,m}(\theta, \phi) = \cos m\phi P_{l,m}(\cos \theta)$ and $S_{l,m}(\theta, \phi) = \sin m\phi P_{l,m}(\cos \theta)$ are harmonic functions; $H_{l,m,1}(\theta, \phi)$ and $H_{l,m,2}(\theta, \phi)$ are geodetically normalized surface spherical harmonics. The later ones are defined by

$$H_{l,m,1}(\theta, \phi) + i H_{l,m,2}(\theta, \phi) = (A_{l,m})^{-1/2} P_{l,m}(\cos \theta) e^{im\phi},$$

$$A_{l,0} = \frac{1}{2l+1},$$

$$A_{l,m} = \frac{1}{2l+1}(l+m)\frac{(l-m)!}{(l+m)!}, \quad 0 < m \leq l.$$  \hspace{1cm} (A.3)

We have derived the relationship between the coefficients of Eqs. (A.1) and (A.2) and those of
Eq. (1):

\[
\begin{align*}
    a_{l,0} &= \sqrt{\frac{2l+1}{4\pi}} f_{l,0}, \\
    a_{l,m} &= (-1)^m \sqrt{\frac{2l+1}{\pi}} \left( \frac{1-m}{l+m} \right) \Re(f_{l,m}) \quad (m > 0), \\
    b_{l,m} &= (-1)^{m+1} \sqrt{\frac{2l+1}{\pi}} \left( \frac{m-l}{l+m} \right) \Im(f_{l,m}) \quad (m > 0),
\end{align*}
\]

\[(A.4)\]

Since the coefficients \( f_{lm} \) have been solved for using Eq. (14), the coefficients in Eqs. (A.1) and (A.2) can be easily calculated using Eqs. (A.4) and (A.5).

References


