From Mean Field Interaction to Evolutionary Game Dynamics

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Plan

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   - Convergence in probability
   - Mean field dynamics
   - Connection to evolutionary game dynamics
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4. Population dynamics
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Mean Field Interactions (MFI)

Random selection among finite number players (Fast Simulation)
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- A player is typically a node, mobile terminal; an agent, a firm; an animal or a virus etc. Each player has its own type $\theta$ and selects an action $a \in A_\theta$. 

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- Meeting: At time $t$, with some probability, $k$ players are randomly selected from $N$ players for an encounter.
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Evolutionary games with random number of players
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**Evolutionary games with random number of players**

- Large population. At each time, there are several local interactions among random number of players.
Mean Field Interactions (MFI)

Random selection among finite number players (Fast Simulation)

- A player is typically a node, mobile terminal; an agent, a firm; an animal or a virus etc. Each player has its own type \( \theta \) and selects an action \( a \in A_\theta \).
- Meeting: At time \( t \), with some probability, \( k \) players are randomly selected from \( N \) players for an encounter.

Evolutionary games with random number of players

- Large population. At each time, there are several local interactions among random number of players.
- The population profile evolves according to some evolutionary process, learning process, adaptive process, optimization process etc.
Aims

- Evolution of the population profile of "type-action" $M^N(t)$
- Convergence to mean field when the population size grows
  - Study of the random process $M^N = \frac{1}{N} \sum_j \delta_{S_j^N}$
  - Asymptotics of $M^N_s(t) = \frac{1}{N} \sum_j \delta\{S_j^N(t) = s\}$ when $t$ goes to $+\infty$.
  - Asymptotics of $M^N(t)$ when $N$ goes to $+\infty$.
- ODE of $m(t) := \lim_{N \to \infty} M^N(t)$ (or accumulation point, $\omega$-limits etc)
- From mean field interactions to population dynamics
- Evolutionary stability and equilibria
Mean Field Interactions (description)

Let $S = \{ (\theta, a), a \in A_\theta \}$ be the set of "type-actions". Assume $S$ finite. After an encounter between $k$ players in $\mathcal{B}_N(t)$ (random set), the variation of the population profile $M^N(t) \leftrightarrow M^N(t + \Delta_N)$. The player $j \in \mathcal{B}_N(t)$ receives an instantaneous cost $C^{N,\theta_j}(S^N_j(t), S^N_{\mathcal{B}_N \setminus j}(t))$.

New states

$S^N_j(t + \Delta_N)$ is drawn according to

$L^N_{\theta_j} (\cdot | S^N_j(t), S^N_{\mathcal{B}_N \setminus j}(t), j \in \mathcal{B}_N)$.

Drift

$f^N(m) := \mathbb{E} (M^N(t + \Delta_N) - M^N(t) | M^N(t) = m, \mathcal{B}^N(t + \Delta_N))$
Non-commutative diagram?

\[
\begin{align*}
M^N(t) & \xrightarrow{t \to +\infty} \omega^N \\
N & \xrightarrow{t \to +\infty} +\infty \\
m(t) & \xrightarrow{t \to +\infty} ?
\end{align*}
\]
Let

\[ J_{k_1,\ldots,k_\Theta}(m) := \mathbb{P}(\#B_\Theta^N(t + \Delta_N) = k_\theta, \ \theta = 1, \ldots, \Theta \mid M^N(t) = m) \]

Assumption 1:

\[ \forall m, \sum_k (k_1 + \ldots + k_\Theta)^2 J_{k_1,\ldots,k_\Theta}(m) < \infty \]

\[ C^N \rightarrow C, \ L^N \rightarrow L, \ \Delta_N \rightarrow 0 \]

Result

(i) \( \frac{1}{\Delta_N} f^N \rightarrow f \). (ii) Under assumption 1, the random process \( M^N = \frac{1}{N} \sum_j \delta^N_{S_j} \) converges weakly (in Skorokhod topology) to a deterministic measure.
Sketch of Proof: Convergence of marginal measures

Extend $\tilde{M}^N$ to continuous time

$$\tilde{M}^N(t) = \tilde{M}^N \left( \frac{\lfloor Nt \rfloor}{N} \right).$$

Define the filtration $\mathcal{F}_k = \sigma(S_1^N(t), \ldots, S_N^N(t), t \leq k)$. $\phi = [\phi_1, \ldots, \phi_d]$ a bounded measurable function.

$$w^N(t) = \tilde{M}^N(t)) - \tilde{M}^N(0) - \sum_{k=0}^{Nt-1} f^N(\tilde{M}^N(\frac{k}{N})).$$

Then, $w^N$ is a martingale.
Sketch of Proof: Convergence of marginal measures

\[
\tilde{M}_x^N(t) - \tilde{m}_x(t) = \tilde{M}_x^N(0) - \tilde{m}_x(0) + \sum_{k=0}^{Nt-1} f_x^N(\tilde{M}_x^N\left(\frac{k}{N}\right)) - \int_0^t f_x(\tilde{m}(\tau)) d\tau
\]

By the convergence of Darboux approximation of the Riemann integral term, \(\frac{1}{N} \sum_{k=0}^{Nt-1} f_x(u, \tilde{M}_x^N(\frac{k}{N})) - \int_0^t f_x(\tilde{m}(\tau)) d\tau\) is bounded by \(C' \frac{1}{N}\) for some \(C'\). By, Lipschitz continuity:

\[
\| \tilde{M}_x^N(t) - \tilde{m}(t) \| \leq \| \tilde{M}_x^N(0) - \tilde{m}(0) + w^N(t) \| + K \int_0^t \| \tilde{M}_x^N(\tau) - \tilde{m}(\tau) \|
\]
Sketch of Proof: Convergence of marginal measures

By Gronwall’s inequality,

\[ \| \tilde{M}^N(t) - \tilde{m}(t) \| \leq [ \| \tilde{M}^N(0) - \tilde{m}(0) + w^N(t) \| + \frac{Kt}{N}] e^{Kt} \]

\[ \sup_{0 \leq t \leq T} \| \tilde{M}^N(t) - \tilde{m}(t) \| \leq [ \| \tilde{M}^N(0) - \tilde{m}(0) \| + \sup_{0 \leq t \leq T} \| w^N(t) \| + \frac{KT}{N} \]

By Doob’s inequality one has

\[ \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} \| w^N(t) \| \right)^2 \right] \leq 4 \mathbb{E}( [w^N]_T) \]

\([w^N]_T : \text{total variation of the martingale } w^N \]

\[ [w^N]_T = \sum_{t=0}^{Nt-1} \| \tilde{M}^N \left( \frac{k+1}{N} \right) - \tilde{M}^N \left( \frac{k}{N} \right) - f^N(\tilde{M}^N \left( \frac{k}{N} \right)) \| \leq \frac{Ct}{N} . \]
\[
D^{T,N}[m_0] := \sup_{t \in [0,T]} \max_{\theta, a} |\tilde{M}^N_{\theta, a}(t) - \tilde{m}^N[m_0]_{\theta, a}(t)|
\]

the maximal deviation in any population profile, from the flow induced by \(\bar{f}^N(\tilde{m})\) through \(m_0\), during \([0, T]\) where \(\tilde{m}^N[m_0]\) is the solution of the ODE

\[
\begin{cases}
\frac{d}{dt} \tilde{m}^N(t) = f^N(\tilde{m}^N(t)) \\
\tilde{m}^N(0) = m_0
\end{cases}
\]

(existence and uniqueness of \(\tilde{m}^N[m_0]\) follows from Picard-Lindelöf).

\[
\bar{D}^{T,N}[m_0] := \sup_{t \in [0,T]} \max_{\theta, a} |\tilde{M}^N_{\theta, a}(t) - m[m_0]_{\theta, a}(t)|
\]

maximal deviation from the flow induced by \(\bar{f}(m)\) through \(m_0\), during \([0, T]\).
Sketch of Proof: Convergence of marginal measures

\[ M^N(t) := \frac{1}{N} \sum_{j=1}^{N} \delta_{S_j^N(t)} \]

Martingale + Legendre’s transformation + Gronwall’s inequality

Convergence to deterministic distribution

For every \( \tau > 0 \) there exists a constant \( C \) such that for every \( \epsilon > 0 \) and \( N \) large enough one has

\[
P \left( \sup_{0 \leq \tau \leq T} \| M^N(\tau) - m(\tau) \| > \epsilon \mid M^N(0) = m_0 \right) \leq 2de^{-\epsilon^2 CN}
\]

for all \( m_0 \in \Delta_d \),
Convergence of random measure in càdlàg function spaces

The random measure $\frac{1}{N} \sum_{j=1}^{N} \delta_{S_j^N}$ with marginal $\frac{1}{N} \sum_{j=1}^{N} \delta_{S_j^N(t)}$ converges (when $N \to \infty$) to a deterministic measure (solution of ODE) under mild assumptions on the expected number of interacting players that changes action at the same time and asymptotic indistinguishability $^a$.

$a$. This condition is weaker than anonymity.

Sketch of Proof

\[ M^N := \frac{1}{N} \sum_{j=1}^{N} \delta_{S_j^N} \]

Continuous, bounded functions \( \phi_l \)

\[ \lim_{N \to \infty} E\left( \prod_l \phi_l(S_l^N) \right) = ? \]

Snitzman’s theorem, Pair of type-state

\[ \lim_{N \to \infty} E[\phi(S_j^N)\phi(S_i^N)] = \phi(m_{\theta_i})\phi(m_{\theta_j}), \quad \frac{1}{N} \sum_{j=1}^{N} E[\phi(S_j^N)] \to \phi(m). \]

Derivation + Holder’s inequality

\[ \lim_{N \to \infty} E \left[ \phi(M^N) - \phi(m) \right]^2 = 0. \]
The mean field interaction is asymptotically equivalent to an evolutionary game

When $N$ goes to infinity, the mean field interaction model with random set $B^N(t)$ of players is equivalent to an evolutionary game $a$ in which a local interaction at time $t$ is described by

- each player is facing a population profile $m(t)$,
- the instantaneous expected cost of a player with the type $\theta$ and action $a$ is

$$C_\theta^a(m(t)) := \lim_{N \to \infty} C_{a}^{N,\theta} (M^N(t)|S^N_j(t) = (\theta, a), M^N(t) = m(t))$$

a. Notice that players are not necessarily using the same strategies.
A class of evolutionary dynamics (homogenous population)

- revision of strategies: \( L, \)

\[
f(m) = \sum_{k \geq 1} J_k(m) \sum_{a_1',...a_k'} \sum_{a_1,...,a_k} \left( \prod_{l=1}^{k} m_{a_l} \right) \times \]

\[
L_{a;a'}(m, k) \left( \sum_{l=1}^{k} (\vec{e}_{a_l'} - \vec{e}_{a_l}) \right)
\]

- evolution of system’s state, ODE: \( \frac{d}{dt} m(t) = f(m(t)) \).

For \( B^N(t) \leftrightarrow \delta_1 \) we obtain

\[
\frac{d}{dt} m_a(t) = \sum_{a' \in A} L_{a'a}(m(t))x_a(t) - m_a(t) \sum_{a' \in A} L_{aa'}(m(t))
\]
Evolutionary game dynamics (I)

### Setting
- **BNN**: Brown and von Neumann (1950), Nash (1951)
- **Replicator**: Taylor & Jonker (1978)
- **Smith dynamics**: Smith (1984)

### Differential equation
- \[ \dot{m}_a^\theta = g_a^\theta(m) - m_a^\theta \sum_{a' \in A_\theta} g_{a'}^\theta(m) \]
- \[ g_a(m) = \max(0, -C_a^\theta(m) + \sum_{a' \in A_\theta} m_{a'}^\theta C_{a'}^\theta(m)) \]
- \[ \dot{m}_a^\theta = m_a^\theta \left[ -C_a^\theta(m) + \sum_{a' \in A_\theta} m_{a'}^\theta C_{a'}^\theta(m) \right] \]
- \[ \dot{m}_a^\theta = \sum_{a' \in A_\theta} m_{a'}^\theta \max(0, -C_a^\theta(m) + C_{a'}^\theta(m)) - m_a^\theta \sum_{a' \in A_\theta} \max(0, C_{a'}^\theta(m) - C_a^\theta(m)) \]
Evolutionary game dynamics (II)

**Origin**
- Fictitious play: Brown (1951), Gilboa & Matsui (1991),

**Dynamics**

\[
\dot{m}_\theta(t) \in BR_\theta(m(t)) - m_\theta(t)
\]

\[
\dot{y}(t) \in \frac{1}{t}BR(y(t)) - y(t),
\]

\[
y(t) = \left( \frac{1}{t} \int_0^t m_1(\tau) \, d\tau, \frac{1}{t} \int_0^t m_2(\tau) \, d\tau \right)
\]

\[
m_\alpha^\theta(t) = \frac{e^{-C_\alpha^\theta(m(t))}}{\sum_{a' \in A_\theta} e^{-C_{a'}^\theta(m(t))}} - m_\alpha^\theta(t)
\]
A population profile $m$ is an **equilibrium state** if

$$\langle m - x, C(m) \rangle \leq 0, \forall x$$

This variational inequality is equivalent to:

$$\forall \theta, \forall a \in A_\theta, \left( m_\theta^a > 0 \implies C_\theta^a(m) = \min_{a' \in A_\theta} C_\theta^{a'}(m) \right)$$

**Sketch of proof**

$\iff$: (min $\leq$ any). $\implies$: convex combination.

The last property is sometimes called **Wardrop first principle** of optimality.
Evolutionary stability

Denote $m_\epsilon = \epsilon x + (1 - \epsilon)m$. A population profile $m$ is a **neutrally stable state** if $\forall x \neq m$ there exists $\epsilon_x > 0$ such that

$$\langle m - x, C(m_\epsilon) \rangle \leq 0, \ \forall \epsilon \in (0, \epsilon_x)$$

A population profile $m$ is an **evolutionarily stable state** if $\forall x \neq m$ there exists $\epsilon_x > 0$ such that

$$\langle m - x, C(m_\epsilon) \rangle < 0, \ \forall \epsilon \in (0, \epsilon_x)$$

A population profile $m$ is an **unbeatable state** if $\forall x \neq m$ one has

$$\langle m - x, C(m_\epsilon) \rangle < 0, \ \forall \epsilon \in (0, 1)$$

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1. Hamilton 1967, Smith’72,82, Weibull’95, Hofbauer & Sigmund’98, Gintis 2000, Cressman’03, Samuelson’03, Vincent’05, Sandholm’09
Immediate consequences

Relation between the solution concepts

\[ ES \iff NSS \iff ESS \iff \text{unbeatable state} \]

Price of Evolutionary Stability

\[ \text{PoA}_{ESS} = \frac{\max_{\mathbf{m}^*} ESS\langle \mathbf{m}^*, C(\mathbf{m}^*) \rangle}{SO} \]

\[ 1 \leq \text{PoS}_{ES} \leq \text{PoS}_{NSS} \leq \text{PoS}_{ESS} \leq \text{PoS}_{unbeat.\, state} \leq \text{PoA}_{unbeat.\, state} \leq \text{PoA}_{ESS} \leq \text{PoA}_{NSS} \leq \text{PoA}_{ES} \leq +\infty \]
Existence of equilibria in evolving games with random number of players

Let $d = \#S$.

**Result**

For any distribution of $B^N(t)$ and any continuous function $r$ on the non-empty, convex and compact subset $\prod_{\theta} \Delta(A_{\theta})$ of the Euclidean space $\mathbb{R}^d$, the evolving game has a least one "static" equilibrium state.

**Sketch of proof**

**Connection target projection dynamics and best reply**
A sufficient condition (PC)

Result

Suppose that the drift limit \( \tilde{f} \) satisfies
\[
\tilde{f}(m) \neq 0 \implies \langle \tilde{f}(m), C(m) \rangle = \sum_{\theta,a} C_{\theta}^a(m)f_{\theta,a}(m) > 0
\]
where
\[
f_y(m) = \sum_{k \geq 1} J_k(m)f_y^k(m),
\]

\[
f_y^k(m) = \sum_{a_1, \ldots, a_k} \left( \prod_{l=1}^{k} m_{a_l} \right) \left( \sum_{j=1}^{k} \eta_{a,y}^j(m, k) \right)
\]

\[
-m_y \left( \sum_{j=1}^{k} \sum_{a_{-j}} \left( \prod_{l=1}^{k} m_{a_l} \right) \eta_{y,a_{-j}}^t(m, k) \right)
\]

Then any "stationary" equilibrium state is a rest point of ODE.
A sufficiency condition for stationarity (NS)

Result

Suppose that the polymatrix of transition $L$ satisfies

$$L_{\theta,a,a_{-j};\theta,b}(m) > 0 \iff a, b \in A_\theta, \ C^\theta_a(m) > C^\theta_b(m)$$

for each $j, a_{-j}$ and $m$. Then,

- The mean field dynamics is positively correlated.
- Any rest point of the ODE is a stationary equilibrium state.
Particular class of games

Potential multi-type games

There exists a $C^1$-function $W$

$$\frac{\partial}{\partial m^\theta_a} W(m) = C^\theta_a(m)$$

Multi-type games with monotone expected cost

$$\forall x, \langle m - x, C(m) - C(x) \rangle \geq 0$$

Smith-stability : $\forall x \in BR(m) \setminus \{x\}, \langle m - x, C(x) \rangle < 0$ is equivalent to Evolutionary Stability $^a$. Moreover, the set of equilibria is convex set ; $ES$ set $\iff NSS$ set.

$^a$ Notice that the cost function is non-linear.
Particular games (cont’d)

Games with cooperative dynamics

\[ \frac{\partial}{\partial m_{a'}} f_{\theta,a}(m) \geq 0 \]

Uniqueness of equilibrium state, strict monotonicity

\[ \forall m \neq x, \langle m^{\theta} - x^{\theta}, C^{\theta}(m) - C^{\theta}(x) \rangle > 0 \]
Evolving Games with Delayed Expected Cost

Delayed evolutionary game dynamics

\[ \frac{d}{dt} m(t) = f \left( m(t), \{ m(t - \tau^\theta_a) \}_{\theta,a} \right) \]

Two important results

- Unbeatable state, EES, SES, ESS, NSS can be **unstable**. Evolutionary stable set can be **unstable set** under time delayed game dynamics.
- Possible **survival** of **dominated** strategies

![Graph showing fraction of transmitters over time for different τ values](image)

- τ = 0.002 ms
- τ = 2 ms, τ = 4 ms
Mean Field Asymptotics of Markov Decision Evolutionary Games
Let $S = \{(\theta, s), s \in S_{\theta}\}$ be the set of "type-state". $A_{\theta,s}$ set of action of type $\theta$ in state $s$. Assume $S$ finite. $u_{\theta}(\cdot|s) \in \Delta(A_{\theta,s})$. After an encounter between $k$ players in $B^{N}(t)$, the variation of the population profile $M^{N}(t) \leftrightarrow M^{N}(t + \Delta_{N})$. The player $j \in B^{N}(t)$ receives an instantaneous cost $\nabla^{N}(X^{N}_{j}(t), X^{N}_{B^{N}\setminus j}(t))$.

**New states**

$X^{N}_{j}(t + \Delta_{N})$ is drawn according to $L^{N}_{\theta j}(.|X^{N}_{j}(t), X^{N}_{B^{N}\setminus j}(t), j \in B^{N}, \vec{u})$.

**Drift**

$f^{N}(m) := \mathbb{E} \left( M^{N}(t + \Delta_{N}) - M^{N}(t) \mid M^{N}(t) = m, B^{N}(t + \Delta_{N}), \vec{u} \right)$
Fix a Markov strategy profile $u$. Let $m[u, m_0](t)$ solution of

$$\dot{m} = f(u, m)$$

Then:

- $M^N[u, m_0](t) \xrightarrow{t \to +\infty} \omega^N[u, m_0]$

- $N \xrightarrow{t \to +\infty} +\infty$

- $m[u, m_0](t) \xrightarrow{t \to +\infty} ??$

- $N \xrightarrow{N \to +\infty} +\infty$

**Questions:**

- Convergence/nonconvergence of $m[u, m_0](t)$ as $t$ goes to infinity?

- Convergence/nonconvergence of $\omega^N[u, m_0]$ as $N$ goes to $\infty$?
Fix a policy \( u \). Let \( m[u,m_0](t) \) solution of the ODE \( \dot{m} = f(u,m) \) starting from \( m_0 \).

\[
\begin{align*}
M^N[u,m_0](t) & \xrightarrow{t \to +\infty} \omega^N[u,m_0] \\
N & \xrightarrow{\to +\infty} \\
m[u,m_0](t) & \xrightarrow{t \to +\infty} \text{?}
\end{align*}
\]

- Under which conditions, the two limits coincide (if they exist)?
- If the dynamics do not converge, is there link between the time average of orbits of the ODE \( \dot{m} = f(u,m) \) starting from \( m(0) = m_0 \), and the \( \omega \) limit of \( \omega^N[u,m_0] \)?
Extension

Mean Field Games (continuum of players)

\[ dm(t) = f(u, m)dt + \sigma(t)dW_t \]

Mixing atomic and non-atomic players

a single "big player" has a non-negligible influence in all the population.

\[ \frac{1}{N} \sum_{j=1}^{N} \gamma_j \delta x_j^N. \]

Mean field limit under more general class of strategies

Mean field dynamics with migration

extend to the case type can change (inner and outer game).

\[ \text{evolutionary game dynamics with migration}^a \]

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\[ \text{a. Tembine H., Altman E., ElAzouzi R., Sandholm W. H.,} \]

Evolutionary game dynamics with migration for hybrid power control in wireless communications, 47th IEEE CDC’2008
Some references