Correlated Evolutionarily Stable Strategies in Random Medium Access Control

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Abstract

In this paper we study a dynamic multiple access in distributed wireless networks with random number of users. We apply evolutionary game theoretic analysis to solve several problems: (a) We address the stability of Aloha-like systems with finitely many power levels. Specifically, we consider very large number of receivers distributed in several locations. Each of them receives packets from random number of users accessing the resource using Aloha-like algorithms. We provide an explicit expression for equilibria, correlated evolutionarily stable strategies, and prove some asymptotic stability results. (b) We apply correlation mechanism and evaluate the performance of random medium access when saturated users interact through interference. We introduce the benefit of correlation (BoC) to measure the gap between the probability of success at correlated evolutionarily stable strategies and the worst probability of success of evolutionarily stable strategies. We show that if only two power levels are available, the correlation mechanism reduces considerably the interference and the number of collisions. Moreover, the correlation mechanism is stable in long-term under several classes of bio-inspired evolutionary game dynamics. (c) Surprisingly, when the number of strategies is at least three, the correlation mechanisms do not improve the probability of success.

1. Introduction

Random Medium Access Control (MAC) algorithms have played an increasingly important role in the development of wired and wireless networks and the performance and stability of these algorithms, such as slotted-Aloha, Carrier Sense Multiple Access (CSMA) is still an open problem. Distributed Medium Access Control, starting from the first version of Abramson’s Aloha to the most recent algorithms used in IEEE802.11, have enabled a rapid growth of both wired and wireless networks. They aim at efficiently and fairly sharing a resource among users even though each user must decide independently (eventually after receiving some messages or listening) when and how to attempt to use the resource. MAC algorithms have generated a lot of research interest, especially recently in attempts to use multi-hop wireless networks to provide high-speed access to the internet with low-cost and low-energy consumption.

In this paper, we restrict our attention to wireless networks, where the resources are receivers, base station or access points and where users interact because of interference, i.e., interfering users cannot transmit simultaneously. There is a collision if another user (mobile) transmits with a greater power level at the same range of the receiver. Motivated by the interest of evolving dense networks, evolutionary game theory was found to be an appropriate framework to apply in networks. We provide a general evolutionary game theoretic framework to analyze networks where interfering users share a resource using an Aloha-type access control. The framework covers access control with arbitrary number of strategies and variable number of mobiles around each receiver. We study a large population of communicating terminals using a aloha-like protocol with several levels of transmission power. We examine how to choose between these power levels in order maximize their probability of success minus the cost of energy consumption. We study several solution and stability concepts: the Nash equilibrium and the Evolutionarily Stable Strategy (ESS) and the correlated evolutionarily stable strategy. The concept of ESS were introduced in mathematical biology by Maynard Smith and Price [9] in the context of evolutionary games, which allow to describe and to predict properties of large populations whose evolution depends on many local interactions, each involving a finite number of individuals. Evolutionary game dynamics are models of strategy change commonly used in evolutionary game theory. A strategy which does better than the average or its oppo-
nent, increases in frequency at the expense of strategies that do worse than the average or the opposed action. Many evolutionary game dynamics models are used in the literature [15, 8]. We compare the performances of these notions with the global cooperative solution. The payoffs that we consider are functions of the probability to have a successful transmission and of the cost for energy consumption. We derive the impact of the pricing for the use of the power levels on the system performance. We analyze various solutions concepts and refinement: Nash equilibria, Pareto optimality, strong equilibria, correlated evolutionarily stable strategy. In order to study the interactions between mobiles and their stationary strategy in the long run, we develop some evolutionary game dynamics (see [8, 15] and the references therein) in order to study the convergence and the stability of equilibria. In [12], interference between more than two mobiles and non-reciprocal interactions is studied. We propose an evolutionary access control game with random number of interacting players and arbitrary number of actions. Our work extends previous works on evolutionary networking games [12] where only two strategies are considered. We use the Price of Anarchy (PoA) concept in order to measure the gap between equilibrium and social optima in the population game. We show that the PoA of evolutionarily stable strategies is more than the PoA of Nash equilibria in any population games with arbitrary number of strategies. In the evolutionary access game with non-degenerate costs and more than three strategies, the PoA at ESS denoted by \( \text{PoA}_{\text{ESS}} \) coincides with \( \text{PoA}_{\text{NE}} \). Moreover, \( \text{PoA}_{\text{ESS}} = \text{PoA}_{\text{NE}} = 0 \). In contrast, if we have only two strategies and \( m \) mobiles around the receiver, the \( \text{PoA}_{\text{NE}} = 1 \geq \text{PoA}_{\text{ESS}} = 0 = \text{PoA}_{\text{ESS}} \). We introduce and analyze correlated evolutionarily stable strategies (CESS) in access control. CESS is a refinement of correlated equilibrium. We show that the evolutionary access control game with arbitrary number of opponent around each receiver has several CESS which are more robust in terms of fairness, stability and efficiency than correlated equilibria (CE). We define the benefit of correlation (BoC) that measures the improvement in term of probability of success by introducing correlation between mobiles in the population game. To the best to our knowledge, this work is one the first which study correlated evolutionarily stable strategies in networking context. For the two strategies case, we show that the BoC is 100% i.e \( \text{PoA}_{\text{CESS}} = 1 \) and \( \text{PoA}_{\text{NE}} = 0 \). As a consequence, for the model studied in [12] the global optimum is attained using CESS. We compute explicitly an equivalence class of CESS which is a social optimum and which is local asymptotically stable under the evolutionary dynamics. Among our contributions we present some situations where the three properties: equilibrium, stability, optimality (global cooperative solution) holds. For more than two strategies, we show that the game has no pure equilibrium (Theorem 3.3) and has a unique strictly mixed equilibrium which we compute explicitly (Theorem 4.1). We develop a general class of evolutionary game dynamics for CESS. Surprisingly, when the number of strategies is at least three, the coordination and the correlation mechanisms do not improve the probability of success. We also show that Stackelberg-based solutions of random access game are suboptimal (strictly dominated by Pareto optima).

The rest of the paper is structured as follows. In next section we formulate problem of access control with several actions and introduce the evolutionarily equilibrium concept, the efficiency metric "Price of Anarchy" and evolutionary game dynamics with time delays. In Section 3 we analyze equilibria: Nash, ESS, CE, CESS and price of anarchy, price of stability and benefit of correlation in Dirac distribution. We compute explicitly the equilibria and the price of anarchy in general distribution in Section 4.

2. Access control with several power levels

We consider a wireless communication network with distributed receivers in which some mobiles contend for access on a common, wireless communication channel. We characterize this distributed multiple access problem in terms of many random access games at each time. Random multiple access games introduce the problem of medium access. We assume that mobiles are randomly placed over an area and distributed receivers in the corresponding area (see Fig. 1). The channels are ideal for transmission and all errors are due to collisions. A mobile decides to transmit a packet with some power level or not to transmit (null power level)
to a receiver when they are within transmission range of each other. Interference occurs as in the Aloha protocol where the power control is introduced: if more than one neighbors of the receiver transmit a packet with a power level which is greater than the corresponding power of the mobile at the same time slot there is a collision. The evolutionary random multiple access game is a non-zero-sum dynamic game, at each time, the mobiles in the same neighborhood have to share a common resource, the wireless medium. We denote by $\mu$ the probability that a mobile has its receiver within its range. We assume that $\mu > 0$ if $\mu = 0$ there is no receiver, and hence no successful transmission.

**Strategies** There are $n + 1$ pure strategies. We assume that for each packet, its source can choose the transmitted power level among several power levels $\mathcal{P} = \{p_0, p_1, p_2, \ldots, p_n\}$ with $p_0 < p_1 < \ldots < p_n$ i.e a strategy for a mobile corresponds to the choice of a power level in $\mathcal{P}$. $p_0 = 0$ means that the mobile does not transmit, $p_n$ is the maximum power level available to the mobiles.

**Aloha-type payoff with pricing** If mobile $j$ transmits a packet using a power $p_j$, it incurs a transmission cost of $c(p_j) \geq 0$. The packet transmission is successful if the other users in the range of its receiver use some power levels strictly lower than $p_j$ in that given time slot, otherwise there is a collision. If there is no collision, user $i$ gets a reward of $V$ from the successful packet transmission. If $c(a) > V$ for some power $a$ then $a$ is dominated by 0 (not transmit). For the remainder, suppose that the reward $V$ is greater than the cost of transmission $\max_{p_j} c(p_j) < V$. All packets of a lower power level involved in a collision are assumed to be lost and will have to be retransmitted later. In addition, if more than one packet of a higher power level is involved in a collision then all packets are lost. The power differentiation thus allows one packet of a higher power level to be successfully transmitted in collisions that do not involve other packets of the higher power level. Then, a transmission of mobile $j$ is successful if its transmission power is strictly greater than the power levels used by the others mobiles at the same slot. When the number of mobiles transmitting at the receiver is $m + 1$, the payoff is given by $u_{m+1}^j : \mathcal{P}^{m+1} \rightarrow \mathbb{R}$

$$u_{m+1}^j(a^1, a^{-j}) = -c(a^j) + V \begin{cases} 1 & \text{if } a^j > \max_{k \neq j} a^k \\ 0 & \text{otherwise} \end{cases}$$

where $a^{-j}$ denotes $(a^1, \ldots, a^{j-1}, a^{j+1}, \ldots, a^{m+1})$, $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a pricing function. We assume that

$$c(p_0) = c(0) = 0 \leq c(p_1) \leq c(p_2) \ldots \leq c(p_n).$$

Define a population profile as a proportion of the use of the various pure strategies in the population. A population profile can be represented by an element of the $n$–simplex $\Delta_n$ of the $(n + 1)$–dimensional Euclidean space $\mathbb{R}^{n+1}$. The expected payoff of a mobile $j$ with the power level $p_j$ when facing $m$ others mobiles is given by

$$f_{m,p_j}(x) = \mu u_{m+1}^j(x, \ldots, x, p_j, x, \ldots, x)$$

(1)

where $x_{p_j}$ is the fraction of mobiles with the power level $p_j \in \mathcal{P}$, and $x = (x_{p_0}, x_{p_1}, \ldots, x_{p_n})$ is the population profile.

**Lemma 2.0.1.** $f_{m,p_j}(x) = 0$, and for $i \geq 1$,

$$f_{m,p_i}(x) = -\mu c(p_i) + V \mu (x_{p_0} + x_{p_1} + x_{p_2} + \ldots + x_{p_{i-1}})^m$$

Denote by $M$ the discrete random variable representing the number of mobiles competing with some anonymous mobile randomly selected in the population when the population profile is $x$ and by $G_M(x) = \mathbb{E}_M(s^M)$ the generating function of $M$ taking value on the set of non-negative integers. Recall that $G_M(1) = 1$. Thus, the radius of convergence of the series is at least one. The expected payoff of a mobile using the power level $p_i$ can be expressed as

$$F_{p_i}(x) = V \mu G_M(x_{p_0} + x_{p_1} + x_{p_2} + \ldots + x_{p_{i-1}}) - \mu c(p_i)$$

for all $i > 0$ and $F_{p_0}(x) = 0$. We observe that if $x$ is stochastically dominated by $y$, i.e for all $i < n, \sum_{j=i+1}^n x_j \leq \sum_{j=i+1}^n y_j$ then $F_b(x) \geq F_b(y), \forall b \in \mathcal{P}$. The function $F$ is extended to a more general generating function $[15]$ $G$ defined in a linear space that contains $\Delta_n \times \Delta_n$. The G-function satisfies $G(v,x)_{x=p_j} = F_{p_j}(x)$.

### 2.1. Evolutionarily Stable strategy

A population profile $x$ is an evolutionarily stable strategy (ESS) if for any other population profile $\text{mut} \neq x$ there exists a $\varepsilon_{\text{mut}} > 0$ such that, $\forall \varepsilon \in (0, \varepsilon_{\text{mut}})$

$$\sum_{b \in \mathcal{P}} (x_b - \text{mut}_b)G(v, (1 - \varepsilon)x + \varepsilon \text{ mut})_{x=b} > 0.$$  

(2)

When the inequality (2) is non-strict, and $\varepsilon = 0$, we obtain that the probability distribution $x$ is a symmetric Nash equilibrium (NE). When the inequality (2) is non-strict the population state $x$ is neutrally stable (NSS). A NSS is in particular a NE. Denote by $\Delta_{\text{ESS}}$ (resp.$\Delta_{\text{NE}}, \Delta_{\text{NSS}}$) the set of ESS (resp. NE, NSS). Thus, the following inclusions hold:

**Lemma 2.1.1.** $\Delta_{\text{ESS}} \subset \Delta_{\text{NSS}} \subset \Delta_{\text{NE}}$. 
Note that in any game bi-linear G-function (in particular in a bi-matrix game with payoff \((A,A')\)), x is an ESS if and only if for all mutant-strategy mut,
\[
G(x,x) \geq G(mut,x), \text{ and,}
\]
\[
[\text{mut} \neq x, G(x,x) = G(mut,x) \implies G(x,\text{mut}) > G(mut,\text{mut})].
\]

**Theorem 2.1.2.** For any distribution of the discrete random variable \(M\), the evolutionary access game with random number of interacting mobiles has a (Nash) equilibrium in mixed strategies.

**Proof.** We show that the game has a symmetric Nash equilibrium. We first remark that the generating function of \(M\) is continuous in \((0,1)\). Thus, \(F\) is lower semi-continuous in \(\Delta_2\) (which is a non-empty, convex and compact subset of Euclidean space \(\mathbb{R}^{2+1}\)). The existence of symmetric Nash equilibrium in mixed strategies follows from the existence of solutions of the following variational inequalities

\[
\text{find } x \in \Delta_2 \text{ s.t. } \sum_{b \in \mathcal{P}} (x_b - \text{mut})F_b(x) \geq 0, \forall \text{ mut}
\]

which is guaranteed by the Brouwer-Schauder fixed point theorem. \(\square\)

Note that an ESS may not exist (e.g in Rock-Paper-Scissor games).

### 2.2. Bio-inspired evolutionary game dynamics

We adopt a class of evolutionary game dynamics based on the generating fitness (payoff) function (G-function) and revision protocols developed respectively by Vincent [15] and Sandholm [8] in which we introduce time delays. The time delays characterize the expected delay needed to know if a transmission is successful or not. An action \(p_i\), taken at time \(t\) will have its effect at time \(t + \tau_{pi}\). Delayed evolutionary game dynamics with asymmetric time delays have been introduced in [14]. In the evolving MAC context, we construct evolutionary game dynamics from a model of individual decision making, we introduce revision of protocols, which describe how mobiles adjust their choices of strategies during the game. A revision protocol is a Lipschitz continuous map \(\beta : \mathbb{R}^{n+1} \times (\Delta_2)^{n+2} \rightarrow \mathbb{R}_+^{(n+1) \times (n+1)}\) that takes a G-function \(G\) vectors \(G = (G(b,\cdot))_{b \in \mathcal{P}}\) and population profile \(x\) as arguments, and returns nonnegative matrices with size \((n+1) \times (n+1)\) as outputs. The scalar \(\beta_{p_i,p_j}(G; x^1, \ldots, x^{n+2})\) is called the conditional switch rate from strategy \(p_i\) to strategy \(p_j\). If the agents receiving revision opportunities independently according to rate \(\kappa\) Poisson processes, then

\[
\frac{\beta_{p_i,p_j}(G; x^1, \ldots, x^{n+2})}{\kappa}
\]

represents the probability that a mobile with the power level \(p_i\) who receives a revision opportunity switches to transmit with the power level \(p_j\). The two maps \(\beta\) and \(G\) together define a delayed non-linear differential equation

\[
\frac{d}{dt} x_{p_i}(t) = \sum_{b \in \mathcal{P}} x_b(t)\beta_{b,p_i}(G; x(t), x(t - \tau_{p_i}), \ldots, x(t - \tau_{p_j})) - x_{p_i}(t) \sum_{b \in \mathcal{P}} \beta_{p_i,b}(G; x(t), x(t - \tau_{p_i}), \ldots, x(t - \tau_{p_j})) \tag{3}
\]

with initial condition

\[
x(t) = \phi(t), \forall t \in (-\max_{b \in \mathcal{P}} \tau_b, 0]. \tag{4}
\]

Note that the delayed differential equation (3) combined with the condition \(x(0) = x_0\) can define infinitely many solutions (this class of differential equation is not covered by Cauchy-Lipschitz’s theorem). To guarantee uniqueness of solutions under locally Lipschitz property of \(\beta\) we need to impose a initial condition known in an interval with length at least \(\max_{b \in \mathcal{P}} \tau_b\).

**Lemma 2.2.1.** If \(\beta\) is locally Lipschitz, \(\phi\) is continuous on \((-\max_{b \in \mathcal{P}} \tau_b, 0]\) and \(F\) is generated by regular \(G\)-function \(G : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}\) satisfying \(G(v,x)|_{v=b} = F_b(x)\). Then the delayed evolutionary game dynamics defined by (3) and (4) has a unique solution.

The dynamic is said positively correlated (PC) if \(\dot{x} \neq 0 \implies \sum_b |\frac{\partial \phi}{\partial b}| |G(v,x)|_{v=b} > 0\). In [8] Sandholm showed that in absence of time delays, replicator dynamics [11] (or general imitation dynamics), Smith dynamics [10] (pairwise comparison dynamics), projection dynamics [8], Brown-von Neumann-Nash dynamics [4] (excess payoff dynamics) satisfy the positive correlation (PC) property. Moreover Brown-von Neumann-Nash dynamics, projection dynamics and generalized Smith dynamics satisfy the property that every rest point of the dynamics is an equilibrium of the game. It is easy to see that the parameter \(\mu > 0\) and the times does not change the equilibria set. Hence, the following holds:

**Corollary 2.2.2.** Delayed imitation dynamics, delayed pairwise comparison dynamics, delayed projection dynamics, excess payoff dynamics and projection dynamics satisfy the positive correlation (PC) property. Moreover Brown-von Neumann-Nash dynamics, projection dynamics and Smith dynamics satisfy property that every rest point of the dynamics is an equilibrium of the game.

**Examples** If \(\beta_{p_i,p_j} = x_{p_i}(t)\max\left(0, G(w; x(t - \tau_{pi}))|_{w=p_j} - G(v; x(t - \tau_{pi}))|_{v=p_j}\right)\)
we obtain the delayed replicator dynamics
\[
\frac{d}{dt} p_i(t) = p_i(t) \left[ G(w, x(t - \tau_w)) |\tau = p_i \right] - \sum_{k=0}^{n} p_k(t) G(w', x(t - \tau_w)) |\tau = p_k \]
\]
\[ j = 0, \ldots, n, \]

Analogously, the delayed Brown-von Neumann-Nash dynamics is obtained for \( p_i, p_j = \max (0, gg) \) where
\[
gg = G(w, x(t - \tau_w)) |\tau = p_j - \sum_{k=0}^{n} p_k(t) G(w', x(t - \tau_w)) |\tau = p_k
\]
and the delayed \( \theta \)-Smith dynamics for
\[
\beta_{p_i, p_j} = \max \left( 0, G(w, x(t - \tau_w)) |\tau = p_j - G(v, x(t - \tau_v)) |\tau = p_j \right) ^\theta
\]
with \( \theta \geq 1 \). Note that under delayed evolutionary game dynamics, the ESS can be unstable (see Fig.8). If the dynamics is regular, a sufficient condition of stability of an ESS is obtained for \( \beta_{p_i, p_j} = \max (0, gg) \) where

3. Dirac distribution

In this subsection, \( M \) is the Dirac distribution \( \delta_{m-1} \) (the number of opponents in the same neighborhood is \( m - 1 \)). If there is no pricing for the energy consumption then, the high power level weakly dominates the others power levels. Using the results of [13], we obtain that the one-shot random access game between \( m \) mobiles and without cost \( c(.) \equiv 0 \) (degenerate case) has an infinite number of equilibria. Only \( (p_n, \ldots, p_n) \) is a symmetric equilibrium.

**Proposition 3.1.** If \( c(.) \equiv 0 \), then the one-shot random access game between \( m \) mobiles has an infinite number of Nash equilibria and a unique symmetric Nash equilibrium \( (p_n, \ldots, p_n) \) which is an evolutionarily stable strategy. Moreover, the game has many Pareto optimal1.

A multi-strategy is a strong equilibrium (or coalition proof) if is a configuration from which no coalition (of any size) can deviate and improve their payoff (probability minus cost) of every member of the coalition (group of the simultaneous moves), while possibly lowering the payoff of mobiles outside the coalition group. A strong equilibrium is in particular a Nash equilibrium (by taking coalition of size one) but also Pareto optimal (by considering coalition of full size). The theorem 3.2 below describes Nash equilibria, Pareto optimality and coalition proof of access game between \( m \) mobiles with two strategies \( \{p_0, p_1\} \).

**Theorem 3.2.** Suppose that \( n = 1 \) and \( 0 = c(p_0) < c(p_1) < V \). Then the one-shot random access game has (i) \( 2^m - 1 \) number of Nash equilibria, (ii) \( m \) of them are Nash-Pareto equilibria, and strong equilibria, (iii) a unique fully mixed Nash equilibrium which is not Pareto optimal, (iv) a unique ESS given by \( \left(1 - \left(\frac{c(p_1)}{V - c(p_1)}\right)^{m+1}, \left(\frac{c(p_1)}{V - c(p_1)}\right)^{m+1}\right) \). (v) the price of anarchy is zero for both ESS and Nash equilibria.

The negative results (ii), (iii), (v) of Theorem 3.2 in term of performance can be improve by introducing the concept of correlated evolutionarily stable strategies (CESS). We will introduce CESS in next section and exhibit a class of CESS for which the system is stable and 100% efficient (the payoff at the CESS is a social welfare) and the system is stable in long-term. We first remark that the result of Theorem 3.2 (i),(ii),(iv) for two strategies \( \varphi = 2 \) does not holds for arbitrary number of strategies \( \varphi \geq 3 \) where pure equi-

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1 An allocation of payoffs is Pareto optimal or Pareto efficient if there is no other feasible allocation that makes every user at least as well off and at least one user strictly better off.
libraria may not exist. For example, the three strategies two player game has no equilibrium in pure strategies and game has a unique completely mixed strategy \( (\frac{c(p_2)}{V}, \frac{c(p_2)-c(p_1)}{V}, \frac{c(p_2)}{V}) \) \( \in \Delta \), obtained by solving the indifferece equations. For the remainder we assume that \( 0 = c(p_0) < c(p_1) < c(p_2) < \ldots < c(p_n) < V \). We have seen in Theorem 3.2 that with two strategies, the access game between \( m \) mobiles has several pure equilibria and Pareto optimal solutions. The following Theorem 3.3 shows that Theorem cannot be extended for three strategies and an equilibrium do not exist in pure strategies.

**Theorem 3.3.** Let \( \mathcal{P} \) be the cardinality of \( \mathcal{P} \). The evolutionary random access game with more than two strategies \( |\mathcal{P}| \geq 3 \), and \( m \geq 2 \) mobiles has no pure equilibrium.

**Proof.** Fix a strategy profile \((a^1, \ldots, a^m) \in \mathcal{P}^m\) with \( |\mathcal{P}| \geq 3 \). Let \( T_1 = \arg \max_j a_j \). We distinguish two cases: \( \mathcal{P}T_1 = 1 \) or \( \mathcal{P}T_1 \geq 2 \). (a) Suppose that \( \mathcal{P}T_1 = 1 \) and let \( T_1 = \{ j \} \). One has in particular \( p^j > p_0 \).

If \( a^j = p_k < p_n \) then any mobile \( i \in \mathcal{P} \setminus T_1 \) which deviates and uses \( a^j > p^j > p_0 \) (the existence of \( a^j \) is guaranteed because there are more than three strategies) will improve its payoff from \(-c(d^k)\) to \(-c(a^j)\)

If \( a^j = p_n \), then mobile \( j \) can improve its payoff from \(-c(p_n)\) to \(-c(b^j)\) by playing a second higher power level \( b^j \in T_2 = \arg \max_j \mathcal{P}T_1 \) or one the others mobiles can save energy. (b) suppose now that \( \mathcal{P}T_1 \geq 2 \). Then if \((a^1, a^2, \ldots, a^m) \neq (p_0, \ldots, p_0)\) there is collision for all mobiles which transmit, otherwise \((a^1, a^2, \ldots, a^m) \neq (p_0, \ldots, p_0)\) and any mobile which deviates and uses a power greater than \( p_0 \) will have a successful transmission.

In both cases, there is at least one mobile that can deviate and improve its payoff. Thus, \((a^1, \ldots, a^m)\) is not an equilibrium. Since \((a^1, \ldots, a^m)\) is an arbitrary pure strategy profile in \( \mathcal{P} \) with \( |\mathcal{P}| \geq 3 \). We conclude that the game between \( m \) mobiles has no pure equilibrium if the number of pure strategies is at least three.

**Theorem 3.4.** Then the evolutionary access game with \( m \) mobiles and more than three strategies has a unique strictly mixed Nash equilibrium given by

\[
x_{p_0} = \left( \frac{c(p_1)}{V} \right)^{\frac{1}{m-1}}, \quad \text{For } 1 \leq j < n, \quad x_{p_j} = \left( \frac{c(p_{j+1})}{V} \right)^{\frac{1}{m-1}} - \left( \frac{c(p_j)}{V} \right)^{\frac{1}{m-1}}, \quad x_{p_n} = 1 - \left( \frac{c(p_1)}{V} \right)^{\frac{1}{m-1}}.
\]

**Proposition 3.5.** The price of anarchy and the price of stability (PoS) of Nash equilibria in the random access game are both zero for \( |\mathcal{P}| \geq 3 \).

### 3.6. Correlated evolutionarily stable strategy

In this subsection we focus on the concept of correlated evolutionarily stable strategies (CESS) in access games. A correlated equilibrium is introduced by Aumann [2] in 1974 and can be interpreted as a distribution of actions or messages given to the mobiles by some referee (which can be the base station or the receiver) before to play each local game. For more details on correlated equilibrium, we refer the reader to [2, 6]. As in [6], we use the concept of correlated equilibrium from the perspective of bounded rationality. In term of medium access control, the correlated mechanism phase (the phase of messages reception from the receiver before transmission) can be seen as the analogue of the probing or listening phase in CMIA algorithm (users willing to use the channel must first probe -or listen to the channel, and when it is busy, they can decide to transmit). Let \( \mathcal{N} := \{1,2, \ldots, m\} \). We define a probability space \((\Omega, 2^\Omega, \mathbb{P})\) which generates signals on which the mobiles can condition their strategic power choices where \( \Omega = \mathcal{P}^m \). The set \( \Omega \) is partitioned as follows \( I^j(b) := \{ba^1, \ldots, a^{j-1}a^{j+1}, \ldots a^m | a^j \in \mathcal{P}, j \neq 1\} \), \( I^j := \{I^j(b), b \in \mathcal{P}\} \). then, \( I^j \) has exactly \(|\mathcal{P}| = n + 1\) elements which are information sets of mobile \( j \).

We define an assignment function (called also rule) profile \( \alpha = (\alpha^1, \ldots, \alpha^m) \) as a mapping from the set of states or signals \( \Omega \) to mixed strategies set \( \Delta(\mathcal{P}) \). For all \( j \), the assignment function of the mobile \( j \), \( \alpha^j \) must satisfy : if for some \( w \), \( \alpha^j(w) = b \), then \( \alpha^j(w') = b \), for all \( w' \in I^j(s) \).

That is, for each element \( w \in I^j \), mobile \( j \) cannot distinguish states that are in the same information set (same equivalence class). We denote the set of all pure assignment functions by \( \mathcal{A} \mathcal{P} := \{g | g : \mathcal{P} \rightarrow \mathcal{P}\} \). Thus, when a mobile chooses an assignment function \( \alpha \) and when he receives the signal \( \omega \in \Omega \) from the referee, he will choose the mixed action \( \alpha(\omega) \). We use \( \alpha(a^j|\omega) \) to denote the probability assigned on \( a^j \) under this mixed action \( \alpha(\omega) \). Then \( \alpha(w) = \{\alpha(p_0|\omega), \ldots, \alpha(p_n|w)\} \in \Delta(\mathcal{P}) \). Given a referee \( \Omega, \mathbb{P} \), we define the identity assignment function as \( \alpha^{id,m}(a^j|\omega) = 1 \) if \( \text{proj}^j(\omega) = a^j \) where \( \text{proj}^j(\omega) \) denotes the \( j \)-th element of the signal \( \omega \) and \( \alpha^{id,m}(a^j|\omega) = 0 \) for all \( \omega \) such that \( \text{proj}^j(\omega) \neq a^j \). If each mobile use the identity assignment then, the resulting probability distribution of actions profile actually played will be \( \mathbb{P} \), the same as the probability distribution of actions recommended by the referee. But when mobiles use other assignments, a different distribution may result. Given an assignment profile \( \alpha \) and a probability distribution \( \mathbb{P} \) over \( \mathcal{P}^m \), the expected payoff
is given by
\[ f(\alpha) = \sum_{a \in \Omega} \mathbb{P}(\omega) \sum_{a \in \mathcal{A}^{m}} u_m(a) \prod_{j \in \mathcal{N}} \alpha^j(a^j|w) \]
\[ = \sum_{a \in \mathcal{A}^{m}} u_m(a) Q_\alpha(a) \]
where \( Q_\alpha(a) = \sum_{a \in \Omega} \mathbb{P}(\omega) \prod_{j \in \mathcal{N}} \alpha^j(a^j|w) \). We say that two assignment functions profile \( \alpha \) and \( \beta \) are equivalent if they induce the same value \( Q_\alpha = Q_\beta \).

**Proposition 3.7.** Suppose that for all \( a \in \mathcal{A}^{m} \mathbb{P}(a) = \mathbb{P}(\sigma(a)) \), for all permutation \( \sigma \) (the distribution \( \mathbb{P} \) is said symmetric). If an assignment profile \( \beta \) is equivalent to \( \alpha \) then \( Q_\beta(a) = Q(a) \), for all \( a \in \mathcal{A}^{m} \).

**Proof.** One has, \( Q_{\alpha|m,a}(a) = \mathbb{P}(a) \), \( \forall a \in \mathcal{A}^{m} \). The result follows immediately. \( \square \)

Now, suppose a small group of mutants appears. These mutants use a mutational assignment function \( \alpha^{m} \), which is not equivalent to the identical assignment function \( \alpha^{id, m} \), but they cannot change the referee recommendation. Let \( \varepsilon \) be the portion of the population which are mutants (who use \( \alpha^{m} \)) and \( 1 - \varepsilon \) portion of the population are non-mutants who use \( \alpha^{id, m} \). At each time, \( m \) mobiles are randomly chosen to play the strategic access game. In playing the access game, the mobiles have the same referee \( (\Omega, \mathbb{P}) \). A probability distribution \( \mathbb{P} \) over \( \mathcal{A}^{m} \) is an CESS if non-mutants with identity assignment function perform better than mutants with assignment functions that are not equivalent to the identical assignment function.

**Definition 3.8.** A CESS \( \mathbb{P} \) is a symmetric distribution probability over \( \mathcal{A}^{m} \) such that for every assignment function \( \alpha^{m} \) nonequivalent to the identical assignment function \( \alpha^{id, m} \), there exists some \( \varepsilon_{m} > 0 \) such that
\[ F^m(\alpha^{m}, \varepsilon_{m}, \varepsilon^{m} + (1 - \varepsilon)\alpha^{id, m}) > F^m(\alpha^{m}, \varepsilon^{m}, \varepsilon^{m} + (1 - \varepsilon)\alpha^{id, m}) \]
for all \( \varepsilon \in (0, \varepsilon_{m}) \) where \( F^m(\beta, \alpha) = \sum_{r \in \mathcal{A}^{m}} \mathbb{P}(r, \alpha, \ldots, \alpha) \).

When the last inequality is non-strict and \( \varepsilon = 0 \), the probability distribution \( \mathbb{P} \) is a correlated equilibrium (CE). Hence, a CESS is a CE.

**Proposition 3.9.** If \( x \) is an ESS then the product measure \( x^{(m)} \) given by \( x^{(m)}(a) = \prod_{j \in \mathcal{N}} x(a^j) \) is a CESS.

**Proof.** It is easy to see that \( x^{(m)} \) is a probability measure on \( \mathcal{A}^{m} \). Since \( x \) is an ESS, \( x(x, \ldots, x) \in \Delta(\mathcal{A}^{m}) \) is a Nash equilibrium of the \( m \)-player game without correlated device (Gal = 0 or mobiles ignore the signals) i.e \( \sum_{a \in \mathcal{A}^{m}} u_m(a) \prod_{j \in \mathcal{N}} x(a^j) \geq \sum_{a \in \mathcal{A}^{m}} u_m(b^k, a^k) \prod_{j \in \mathcal{N} \setminus \{k\}} x(a^j) \geq 0, \forall b^k \in \mathcal{A}^{m} \). This means that the product probability measure \( x^{(m)} \) is a CE. Moreover, \( \forall y \)
\[ u_m(x_1, \ldots, x_{m-1}, y, x_m, \ldots, x_e) > u_m(x_1, \ldots, x_{m-1}, y, x_m, \ldots, x_e) \]
implies that
\[ \sum_{a \in \Omega} \mathbb{P}(\omega) \sum_{a \in \mathcal{A}^{m}} u_m(a) \alpha^j(a^j|w) \prod_{j \in \mathcal{N} \setminus \{i\}} x_e(a^j|w) > \sum_{a \in \Omega} \mathbb{P}(\omega) \sum_{a \in \mathcal{A}^{m}} u_m(a) \alpha^j(a^j|w) \prod_{j \in \mathcal{N} \setminus \{i\}} x_e(a^j|w) \]
where \( \mathbb{P}(a) = \prod_{j \in \mathcal{N}} x(a^j), \ x_e := (1 - \varepsilon)x + \varepsilon y \)

More generally all convex combination of Nash equilibria (NE) are correlated equilibria and the set of CE is convex. This convexity properties of the CE’s set is used in some algorithms based on linear or convex programming and generically, the complexity is less than in Nash equilibria in finite action games. We denote by \( \Delta_{CESS} \) the set of CESS. Then, \( \square \)

**Lemma 3.9.1.** The following inclusions holds: \( \Delta_{ESS} \subset \Delta_{NE} \subset \Delta_{CE} \), \( \Delta_{CESS} \subset \Delta_{CE} \subset \Delta_{CE} \).

One of the principal interest to study CESS is that, in general CESS remains stable for small delays in various game dynamics and the expected payoff obtained at ESS can be improved by CESS distribution. Non-ESS can be CESS as shown below.

We now construct of a class CESS for arbitrary number of users with two actions.

**Number of mobiles which transmit with power \( p \).** We define \( numb_{p_1} : \mathcal{A}^{m} \rightarrow \mathbb{N} \) as the number of transmitters of the power \( p \). For example, \( numb_{p_0}(p_0, \ldots, p_n) = m \) and \( numb_{p_1}(p_0, a^2, \ldots, a^m) = 1 \) if \( a^j \neq p_n \), \( \forall j \geq 2 \). The set \( \{ b \in \mathcal{A}^{m} | numb_{p_1}(b) = 1, numb_{p_0}(b) = m - 1 \} \) is exactly the situations where only one of the mobiles transmits with the power \( p_n \) and the others mobiles use the power level \( p_0 \) (do not transmit). \( numb_{p_1}(b) = 1 \) if and only if there exists a unique \( j \) such that \( a^j = p_n \) i.e all permutations of the actions profile \( (p_n, a^2, \ldots, a^m) \) and permutations satisfying \( p^j < p_n \forall j \). There are \( mn^{m-1} \) possibilities. Denote by \( \Theta^m := \{ a \in \mathcal{A}^{m} | numb_{p_1}(a) = 1, numb_{p_0}(a) = m - 1 \} \).

**Proposition 3.10.** For two strategies, the probability distribution \( \mathcal{P}^{m} \) over \( \mathcal{A}^{m} \) defined as \( \mathcal{P}^{m}(a) = \begin{cases} \frac{1}{m} & \text{if } a \in \Theta^m \text{ and is a CESS with the payoff } \frac{V - c(p_1)}{m} > 0 \text{ at each slot.} \\ 0 & \text{otherwise} \end{cases} \)
class of assignment functions with the distribution $\mathbb{P}^m$ is one (i.e. the proposed method is 100% efficient). This class of CESS is also energy efficient (the energy consumption is minimized).

**Proof.** $\mathcal{G}^m$ has exactly $m$ actions. Thus, $\mathbb{P}^m$ is a probability measure. We show that every strategy $\alpha \in \mathcal{G}$ used in $\mathbb{P}^m$, $\sum_{a_j \in \mathcal{G}^m} \mathbb{P}^m(a, a^{-j}) > 0$ and any alternative strategy $b^j \in \mathcal{G} \setminus \{\alpha\}$, it holds that $\sum_{a_j \in \mathcal{G}^m} \mathbb{P}^m(a, a^{-j}) \left[ u(a, a^{-j}) - u(b^j, a^{-j}) \right] > 0$. Since $n = 1$, the inequality becomes: for $b^j \in \mathcal{G} \setminus \{p_1\}$, $\left[ u_m(p_1, a^{-j}) - u_m(b^j, a^{-j}) \right] \mathbb{P}^m(p_1, a^{-j}) = V - c(p_1) + c(b^j) > 0$. By summing over $a^{-j} \in \mathcal{G}^{m-1}$,

$$\sum_{a_j \in \mathcal{G}^{m-1}} \left[ u_m(p_1, a^{-j}) - u_m(b^j, a^{-j}) \right] \mathbb{P}^m(p_1, a^{-j}) > 0$$

for all permutation on the position of $j$ and $b^j \neq p_0$. Similarly, for $b^j \in \mathcal{G} \setminus \{p_0\}$, $a^{-j} \in \mathcal{G}^{m-1}$, one has, $\left[ u_m(p_0, a^{-j}) - u_m(b^j, a^{-j}) \right] \mathbb{P}^m(p_1, a^{-j}) = \frac{c(p_1)}{m} > 0$. Hence $\mathbb{P}^m$ is a strict CE. Thus, $\mathbb{P}^m$ is a CESS and the system has a successful transmission at each slot with probability one. The total payoff obtained at the CESS is exactly $V - c(p_1)$ which is strictly greater than the payoff obtained at the ESS. The social optimum is $V - c(p_1)$ which guarantees a successful transmission with the minimum power consumption. Thus, the CESS is efficient in term of energy consumption (by considering par example the ratio between the probability of success and cost of energy consumption as the energy-efficient metric).

**Benefit of Correlation:** One of the advantages of CESS is that it has a potential to reduce the distance between the optimal solution and the ESS solution obtained as an outcome of users which decide independently. Using Lemma 3.9.1, the following inequalities holds: $\text{PoS}_{\text{CESS}} \geq \text{PoS}_{\text{ESS}} \geq \text{PoA}_{\text{ESS}} \geq \text{PoA}_{\text{NE}},$ $\text{PoA}_{\text{CESS}} \geq \text{PoA}_{\text{NE}} \geq \text{PoA}_{\text{ESS}} \geq \text{PoA}_{\text{NE}} \geq \text{PoA}_{\text{CE}}$

We define the benefit of correlation as $\text{PoS}_{\text{CESS}} - \text{PoA}_{\text{NE}}$

**Corollary 3.10.1.** The benefit of correlation in the random access game is 100% for $\Omega = \mathcal{G}^m$, BoC = 1

3.11. Probability of success: Comparison at NE, ESS and CESS

In the numerical examples below, we show how the pricing function can optimize the network throughput or probability of success. We first investigate the impact of the different parameters and pricing on the NE, ESS and CESS, and the convergence of the replicator dynamic with time delays. We also discuss the impact of the pricing function on the system capacity. We took the parameters: three strategies $\mathcal{P} = \{p_0, p_1, p_2\}$, $n = 2$, $V = 1$, $\mu = 0.8$ (80% of coverage). The expected probability of success of a mobile is given by $\sum_{i=1}^{n} x_i G(x_1 + \ldots + x_{i-1})$. We consider two examples of distribution: geometric distribution $Geo(p = 0.3)$ and Dirac distribution $\delta_m, m = 30$.

![Figure 2. Probability of success - Dirac distribution $m = 30$](image)

![Figure 3. Geometric probability of success with parameter $p = 0.3$](image)

![Figure 4. Probability of success at CESS](image)

These numerical examples (Fig. 2, 3, 4) confirm the optimality of CESS in random access game with two strategies: probability to have a successful transmission at the CESS is maximum and equal to the probability that a resource exists times the probability that a randomly selected mobile has a packet to transmit which we represent by $\mu = 0.8$.

**Limitation of CESS’s approach in access game** We have seen that for $c(\mathcal{P}) \leq 2$ or for $c(\cdot) = 0$, CESS can improve the performance of system and some class of CESS are social welfare. But for $c(\mathcal{P}) \geq 3$, the correlated equilibria is reduced the Nash equilibria. Hence,
Proposition 4.2. For any distribution of the random variable $M$, and $\PoD \geq 3$, $\PoS = \PoP = 0 = \PoPE = \PoPA$.

4.3. ESS: Convergence, non-convergence and instability

Consider three strategies and fix the parameters $V = 1, c(p_2) = 1/2, c(p_1) = 1/4, c(p_0) = 0$. In figure 5, we plot the trajectories of replicator dynamics without time delays using the Dynamo’s [7] diagram phase. We observe the convergence to the interior rest point starting from any point in the relative interior of the simplex. Moreover, the interior rest point is asymptotically stable.

In figures 6, 7 and 8, we describe numerical examples of our evolutionary game model with Poisson distribution of mobiles under the delayed replicator dynamics in which each pure strategy is associated with its own delay ($\tau_{p_2} = 0, \tau_{p_1} = \tau$). We took a poisson distribution with parameter $\lambda = 5$. In the figure 8 we illustrate a situation where the ESS is not stable (for large delays) and in the figures 6, 8 the dynamics converge to the ESS which is stable (for small delays). We derive that an ESS can be unstable for large time delays. In the particular case of pairwise interactions $M = \delta$, where the payoff functions $F_i$ are linear, equilibrium can be obtained from the oscillating trajectories of the delayed replicator dynamics if its remains at the relative interior of the simplex starting from any interior function $\phi(t) \in int\Delta_t$, $t \in (-\tau, 0)$, then the time average trajectories of the delayed replicator dynamics converge to the ESS i.e. $A_{j,\phi(T)} = \frac{1}{T} \int_0^T x_{p_j}(t) \, dt \longrightarrow x_{p_j}^*$. Since the set $\Delta_t$ is compact, there exists $\phi$ such that the subsequence $A_{j,\phi(T)}$ converges (to some point $x_j$). Similarly, the vector $A_{x(T)}$ converges to $x$. We show that $x$ is a rest
This implies that when $T$ goes the infinity, one has $F_{p_j}(x) = \text{constant}$. By uniqueness of the interior equilibrium, we conclude that $x = x^\ast$. Note that for the Brown-von Neumann-Nash dynamics the limit of the time average trajectories can be different than $x^\ast$ (see [3] for details on the so-called Shapley triangle and Time Average of the Shapley Polygon TASP).

5. Concluding remarks and future work

Biological models and tools have inspired a growing number of studies and of designs of decentralized wireless networks. In various ways, this paper falls in this category of biology-inspired algorithms and decision making theory. The evolutionary game theoretic framework proposed here incorporates several actions for each mobile for interference, and admission control. We have analyzed correlated evolutionarily stable strategies which preserve robustness, stability and high performance compared to ESS and Nash equilibrium. An interesting extension that we leave for future work is the evolutionary communication equilibria [5] that extends the correlated evolutionarily stable strategies by introducing more general signal (message) space.

References