Sensor-based extraction of physical property matrices from realised state-space models

Ramin Bighamian and Hamid Reza Mirdamadi*

Department of Mechanical Engineering,
Isfahan University of Technology,
Isfahan 8415683111, Iran
E-mail: bighamia@ualberta.ca
E-mail: hrmirdamadi@cc.iut.ac.ir
*Corresponding author

Fariborz Fariborzi

Mechanical Engineering Department,
Shahrood University of Technology,
Shahrood 361995161, Iran
E-mail: bfariborzi@yahoo.com

Abstract: In some qualification such as damage detection, a more detailed representation including mass, stiffness, and damping property matrices plays a more pivotal role than an equivalent state-space model. In practice, estimating the modal parameters of the dominant modes in the signal records is superior to directly identifying the mass, stiffness, and damping matrices. In this paper, a novel technique is presented for deriving physical structural dynamics matrices from a balanced state-space realisation. In this approach, obtaining undamped modal parameters from the identified complex-valued modal parameters is not needed. Also there is no requirement for computing mass-normalised mode shapes. The only requirement of this technique is an accelerometer instrumentation of the degrees of freedom (DOFs) which are in mind for configuring the structural mode shapes. The structural state-space description is realised by eigen-system realisation algorithm with data correlation (ERA/DC). This procedure is validated with some simulations and the results are satisfactory.

Keywords: structural system identification; eigen-system realisation algorithm/data correlation; ERA/DC; structural property matrices; state-space representation; differential evolution.


Biographical notes: Ramin Bighamian received his BSc and MSc in Mechanical Engineering from Isfahan University of Technology (IUT), Iran, in 2008 and 2011, respectively. Currently, he is a PhD student in Department of Mechanical and Biomedical Engineering, University of Alberta, Canada. His main research areas include system identification, drug delivery, predictive control, signal processing, and structural health monitoring.

Hamid Reza Mirdamadi received his BSc in Civil Engineering, in 1986, MSc with honour, in Structural Engineering, in 1990, and PhD with honour, in Structural-Earthquake Engineering in 1999, all from Sharif University of Technology (SUT), Iran. Currently, he is an Assistant Professor at the Department of Mechanical Engineering, Isfahan University of Technology, Iran. His main research interests include smart structures (dynamics, vibrations, and controls), smart piezoelectric materials, micro/nano electromechanical systems (MEMS/NEMS), fluid-structure interaction, structural health monitoring (SHM), and structural system identification.

Fariborz Fariborzi received his BEng degree from Amir Kabir University of Technology, Iran, in 1984. His MSc is in Crash Dynamics in 1994 and PhD in Dynamics and Control in 1997, both from University of Waterloo, Canada. Currently, He is an Assistant Professor at Shahrood University of Technology. His main research interests include non-linear system control, system modelling and identification, human and computer wireless communications, and applications of intelligent algorithms to identification of human diseases.
1 Introduction

In the last decades, modal parameter identification has been one of the most important challenges and many researchers have reported different methods to solve this issue (Juang and Pappa, 1985; Ibrahim, 1977; Ewins and Gleeson, 1982; Kim et al., 1994). Although it is sufficient to find the modal frequencies, modal damping, and mode shapes in some qualifications such as modal analysis, but in some other fields, e.g., damage detection and finite element model updating, we need a more detailed representative model, which allows a more reliable survey. One of the most natural models for describing mechanical systems is a discrete system of second order ordinary differential equations. The mass, stiffness, and damping property matrices play a pivotal role in this formulation. Therefore, a number of efforts have been performed to overcome this model reconstruction issue. Some researchers used measurement test data directly to extract these fundamental property matrices (Koh and See, 1993; Williams, 1987; Rajaram and Junkins, 1985; Aghabian et al., 1991). Because experimental records are only available from a small number of instrumented locations in a structure, it is necessary to estimate the modal parameters of the dominant modes in the signal records, rather than identify directly the mass, stiffness, and damping property matrices of a linear time-invariant model (Beck and Jennings, 1980). So, it is desirable to find at first, the structural modal parameters and then, to obtain the structural property matrices through a predetermined model order. Peterson et al. (1993) presented a method to find the mass and stiffness matrices from the identified modal parameters, extracted by ERA/OKID (Juang et al., 1988). Their work was dependent on calculating the normal modes for extracting the mass and stiffness property matrices. As the output of ERA/OKID was complex-valued modes and the retrieval of the undamped modal parameters from the identified complex-valued ones redounded to some errors, someone prefers to solve the problem without this normalisation process. Creamer and Junkins (1988) proposed a finite element model updating method to find the physical matrices of an undamped or lightly damped structure. Using a curve fitting procedure in frequency domain for system identification, at first, they found the structural eigenvalues in a predetermined frequency range. Then, they guessed and optimised initial mass and stiffness matrices at each step, by a least-squares solution method. As the mode shapes participating in the least-squares equations were based on a trial selection, the initialising physical matrices should be designated wisely. Angelis et al. (2002) presented a new solution methodology for identification of physical parameters from an identified state-space model. They suggested an improved sensor/actuator instrumentation in which the mode shapes could be computed based on the sensed or actuated degrees of freedom (DOFs). For the computation of physical matrices, they implemented Balmes (1997) formulation obtained by orthogonality conditions, which imposed some extra constraints. The physical property matrices were obtained by ignoring the computed imaginary parts. Some other discussions for extracting the physical matrices from modal parameters are those of Potter and Richardson (1974), Kammer and Steltzer (2000), Kammer (1988), and Berman and Nagy (1983). The most challenging issues in the above-mentioned researches are:

- Orthogonality assumption: Ibrahim (1983) showed that the orthogonality condition works exactly if all the modes could be measured, or equivalently, if the number of modes in the frequency ranges of interest is equal to the number of measurement locations. But, in practice, fewer modes are measured and the physical property matrices cannot be obtained uniquely (Friswell and Mottershead, 1995). For this problem, all of the researches that extracted the physical property matrices by orthogonality condition would be just useful for small structures where one is almost sure that all the modes could be identified.

- Computation of mass-normalised eigenvectors: To have a unique and right physical model of the discrete system, we need to compute the mass-normalised mode shapes and find a system transformation based on this normalised eigenvectors. Two ways are suggested to find the normalised eigenvectors. First, someone could find the complex-valued mass-normalised eigenvectors. This method was implemented by Angelis et al. (2002) using modal state-space matrices. By this method, Angelis scaled the eigenvectors to have a unique system. Second, someone could compute the normal mode shapes from complex-valued ones (Ibrahim, 1983; Niedbal, 1984). It was shown (Friswell and Mottershead, 1995) that significant errors could arise in this procedure. The resulting mode shapes should then be mass-normalised using the modal participation factors of the applied forces from driving point measurements or using an assumed mass matrix (Alvin et al., 1995). Both of them were accompanied by some errors. Several research groups (Li et al., 2010; Jin et al., 2010; Friswell et al., 1998) updated initially assumed stiffness/damping matrices by either considering the mass matrix, or by assuming the existence of mass-normalised eigenvectors. Indeed, these works tried to escape from normalising the optimised physical property matrices through considering either a predetermined mass matrix or mass-normalised eigenvectors. The system identification algorithms, such as ERA do not compute the mass-normalised mode shapes. Tee et al. (2005) presented three methods in order to identify the physical property matrices in a damage assessment context. In this work, just one method could identify the mass matrix in addition to those of stiffness and damping using the procedure discussed by Angelis et al. (2002). They utilised sub-FOMI algorithm in substructure level system identification. Using collocated actuators/sensors in the interface DOFs and accelerometers at all the interface DOFs, they identified
the model of the substructure systems in a state-space description. For the purpose of structural health monitoring, the Angelis method was used for extracting the mass, stiffness, and damping matrices from an identified substructure state-space model. It should be noted that the physical property extraction from state-space models is different from model updating procedures, which are based on either a predetermined mass matrix or a mass-normalised mode shape matrix.

In this paper, an inverse-optimisation problem is configured to compute the physical property matrices from the identified modal parameters. We obtain some elements of the mass matrix directly from the state-space model, and then we solve an optimisation problem to find the full mass, stiffness, and damping matrices. The major originality of this paper is in that we do make any assumptions about neither normalising eigenvectors nor orthogonality conditions. As mentioned above, the orthogonality condition could only exist in a fully identified model including all modal parameters. In addition, normalising the eigenvectors redounds to significant computational errors. Since most of the proposed procedures deal with these two conditions, the results obtained from this approach could be more reliable. For simplicity, we consider that the mode shape elements are just obtained from sensor measurements. Finally, the cost function is optimised through differential evolution (DE) algorithm.

The following sections of this paper are organised as follows: In Section 2, a forward problem is presented for finding the cost function. In Section 3, it is explained how one can extract a partial mass matrix from a realised state-space model. The scheme of DE is pointed out briefly in Section 4 and, afterwards, the steps of the algorithm are presented in Section 5. Eventually, two examples are illustrated to show the efficiency of the algorithm.

2 Forward problem

If we denote the mass matrix of a linear time-invariant n-DOF structure as \( M \), the damping matrix as \( E \), the stiffness matrix as \( K \), the input vector as \( u(t) \), the input influence matrix as \( b \), and the displacement vector as \( w(t) \), the system of ODE for forced vibrations with \( r \) inputs is described by

\[
M \ddot{w}(t) + E \dot{w}(t) + K w(t) = b u(t) \quad (1)
\]

The free vibration solution to this matrix equation is

\[
\{ w(t) \} = \{ \phi \} e^{\lambda t} 
\]

(2)

This form of solution, when substituted into equation (1) will transform the ODEs into a simple algebraic matrix equation

\[
([M]\dot{\lambda}^2 + [E]\lambda + [K])\{ \phi \} e^{\lambda t} = \{0\} 
\]

(3)

For a non-trivial solution \( \{ \phi \} \), the matrix polynomial \(( [M]\dot{\lambda}^2 + [E]\lambda + [K]) \) has to be singular

\[
\|M\|\dot{\lambda}^2 + [E]\lambda + [K] = 0 
\]

(4)

This is the characteristic equation of the system and \( \| \| \) denotes the determinant of a matrix. Upon solving equation (4), one gets the diagonal spectral matrix \( \Psi \) as follows:

\[
\Psi = \begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_n
\end{bmatrix}
\]

(5)

where \( n \) is the order of the model. By substituting \( \lambda_i \) into equation (3) one obtains the corresponding model shape vector and for \( i = 1, 2, ..., n \), the modal matrix, as

\[
\Phi = \begin{bmatrix}
\phi_1 \\
\phi_2 \\
\vdots \\
\phi_n
\end{bmatrix}
\]

(6)

From equation (1) to equation (6), it is observed that by given mass, stiffness, and damping matrices, one can find a unique set of eigenvalues and mode shapes (Lixia et al., 2010). For extracting the physical property matrices from the existed modal parameters, an optimisation approach is generally needed. A novel approach for formulating on objective function is detailed below for simultaneous realisation of eigenvalues and mode shapes. Here, instead of using the differences between the targeted eigenvalues/mode shapes and their corresponding current values for constructing the objective function, the following measure is proposed

\[
CF = \min \max \left\| [M] \dot{\lambda}_i^2 + [E] \lambda_i + [K] \right\| \{ \phi_i \} 
\]

(7)

\( \| \| \) indicates the Euclidean norm of a vector and \( \lambda_i \) and \( \{ \phi_i \} \) are the desired eigenvalue and mode shape vector for the \( i^{th} \) mode. It follows that to find the mass, stiffness, and damping matrices for realising the target eigenvalues/mode shapes, one needs only to minimise \( CF \) under whatever constraints that exist on the elements of the property matrices.

3 Mass identification in collocated sensor/actuator DOFs

The second order ODEs (1) can be transformed into a system of first order ODEs by introducing the velocities as time derivatives of the displacements. By definition of a state vector \( x \), the state equation of state-space model could be found as follows:

\[
\dot{x}(t) = Ax(t) + Bu(t) 
\]

(8)

where

\[
x(t) = \begin{bmatrix}
w(t) \\
\dot{w}(t)
\end{bmatrix}_n 
\]

\[
A = \begin{bmatrix}
0_{n \times n} & I_{n \times n} \\
-M^{-1}K & -M^{-1}E
\end{bmatrix}_{2n \times 2n}
\]

\[
B = \begin{bmatrix}
0_{n \times n} \\
M^{-1}b
\end{bmatrix}_{2n \times 2n}
\]
If the outputs of the system are a combination of acceleration, velocity, and displacement measurements, the output equation is in the following form

\[ y = c_a \ddot{w} + c_v \dot{w} + c_d w \]  \hspace{1cm} (9)

where \( c_a, c_v, \) and \( c_d \) are the output influence matrices for acceleration, velocity, and displacement, respectively. For obtaining the output equations in state-space formalism, we shall arrange equation (9) only by displacement and velocity output terms. Using equation (1), the output equation is (Juang, 1994)

\[ y = Cx(t) + Du(t) \]  \hspace{1cm} (10a)

\[ C = \begin{bmatrix} c_d - c_a M^{-1} K & c_v - c_a M^{-1} E \end{bmatrix}_{nx2n} \]  \hspace{1cm} (10b)

\[ D = \begin{bmatrix} c_a M^{-1} b \end{bmatrix}_{nrx} \]

The physical space output matrices, \( [c_a, c_v, c_d] \), and physical space input one, \( [b] \), are known values, which contain binary information depending on how the DOFs are sensed or actuated. In the case of the physical space input matrix, by definition, \( b_{ij} = 1, 2, \ldots, n \), \( j = 1, 2, \ldots, r \) has a value of one, if the \( j \) DOF is instrumented by the \( j \)th actuator, else, it is zero. In the same way, for the physical space output matrices, \( c_{aij} = 1, 2, \ldots, m, j = 1, 2, \ldots, n \), \( a = a, v, d \), is one, if the \( j \) DOF is instrumented by the \( j \)th sensor, else, it is zero. Assume all the selected DOFs in the system are instrumented by sensor instrumentation, i.e., \( m = n \), from equation (10b) we have

\[ \bar{C} = \begin{bmatrix} -M^{-1} K & -M^{-1} E \end{bmatrix}_{n2n} \] \hspace{1cm} (11)

\[ \bar{D} = \begin{bmatrix} M^{-1} b \end{bmatrix}_{nrx} \]

Because, \( r \leq n \), some zero rows appear in \( D \) matrix for non-collocated DOFs. Therefore, by rearranging the rows of the realised \( D \) matrix in equation (11) and partitioning it into two subdivisions, a 'collocated' partition and an 'un-collocated' one, we have

\[ \bar{D} = \begin{bmatrix} D_{Colloc} \\ D_{UnColloc} \end{bmatrix} = \begin{bmatrix} (M^{-1})_{r \times r} \\ 0_{n-r \times r} \end{bmatrix} \]  \hspace{1cm} (12)

Indeed, after a state-space realisation, someone can extract a partial \( D \) matrix from the identified full \( D \) matrix, by ignoring those rows related to DOFs which are not actuator instrumented. Therefore, we can extract a partial mass matrix from this partitioned-collocated \( D \) matrix as follows

\[ M_p = D_{Colloc}^{-1} \]  \hspace{1cm} (13)

where \( p \) stands for partial mass matrix that is realised. It can be observed that the collocated partition of the direct transmission matrix \( D \) is completely correlated to the inverse-mass matrix, related to those actuator-instrumented DOFs. In the next sections, this relation is more examined.

### 4 Differential evolution (DE) algorithm

DE algorithm (Storn and Price, 1996; Wang et al., 2010; Chung and Lau, 2009) is a population-based and derivative-free approach which is efficient for optimising real-valued, multi-modal, and continuous-value cost functions. It is very similar to conventional genetic algorithms. The differences are in the way the mechanisms of mutation and crossover are performed using real floating point numbers instead of long strings of zeros and ones (binary digits). Besides, the good convergence properties, suitability for parallelisation, conceptual simplicity, and ease of use make it as an efficient optimiser which often approaches the global minimal instead of trapping in local ones (Chang et al., 2009). The scheme of DE is listed in the following:

1. Create the initial population: \( NP \) random vector populations are generated such that the feasible parameter space \([X_{min}, X_{max}]\) is entirely spanned. Each vector includes \( D_m \) members which must be optimised. If \( NP \) is very small, the population does not make a good sampling in search space, on the other hand, if \( NP \) is very large, it does not help convergence. It is suggested \( NP = 10D_m \). For the \( i^{th} \) population vector we have:

\[ pop^{i} = X_{min} + rand(1, D_m) (X_{max} - X_{min}) \]

2. Generate the perturbation: Two randomly selected vectors are subtracted from each other at \( G^{th} \) trial solution:

\[ pop^{G} = pop^{i} - pop^{G-2} \]

3. Mutation: Add a third randomly selected vector to the resulting perturbation with a differential weight \( F \). The third randomly selected vector can be replaced by the best vector at previous trial solution \( (bm) \) with the lowest value of the cost function. \( F \) should be selected such that not only do the differentials explore the tight valleys, but also they can explore new areas of the search space \((0 < F < 2)\):

\[ pop^{G+1} = bm^{G} + F (pop^{G} - pop^{G-2}) \]

4. Crossover: Perform a crossover operation to increase the offspring diversity by a selected user-input \( CR \) known as crossover ratio. Each member of a vector can be replaced by mutated ones obtained from the previous step if the random number \( \delta \) is lower than \( CR \) \((0 < CR < 1)\), \( \delta \) is randomly selected for that member. Decreasing \( CR \) redounds to increase the search robustness, but the time of solution might get longer:

\[ pop^{G+1} = \begin{cases} pop^{G+1} & \text{CR} > \delta \\ pop^{G} & \text{CR} \leq \delta \end{cases} \]

5. Selection: If the cost value \( (CF) \) of each vector constructed at the step 4 is lower than the cost value in the previous trial solution, the computed vector in the step 4 is selected for the next generation. Also the vector with the minimum cost is named \( bm \):

\[ pop^{G+1} = pop^{G+1} \quad \text{if} \quad CF^{G+1} < CF^{G} \]
Upon reaching either the maximum generation number or the desired cost function value, stop the operation, else go back to step 2.

5 Steps of extracting physical property matrices from state-space model

As stressed before, in this paper the modal eigenvectors are only identified in a signal-based context. Therefore, the physical property model order is equal to the number of sensors installed on the structure. Therefore, for optimising the cost function in equation (7), the number of identified modes should be equal to the number of sensors; and this issue is consolidated by optimising the number and location of actuators. If the number of identified modes is more than the number of sensors, one can reduce them by removing the modes with the least contributions using some methods such as MAC and MSV (Juang, 1994). Note that the identification of mode shapes using actuator signal information, used by Angelis et al. (2002), can decrease the number of required sensors.

To better describe the computational procedure used in the current research, the computational steps are reported as follows:

1. Extract the eigenvalues and eigenvectors of the structure. In this paper, ERA/DC (Juang et al., 1988) is used to identify the state-space model including the modal parameters absolutely based on sensor/actuator signal information contents. Since this type of system identification algorithm works based on a correlation between input and output signals, the effects of noise and accelerometer errors can be neglected as much as possible.

2. Be sure the number of identified modes is equal to or more than the number of sensor-instrumented DOFs. One can remove the unnecessary eigenvalues using MAC or MSV.

3. Identify the mass matrix parameters correlated to the collocated sensor/actuator DOFs. The other terms of the mass matrix, which are not identified, and the stiffness and damping matrices are those variables that are subject to the DE optimisation process.

4. Construct the mass, stiffness, and damping matrices in a symmetrical realisation.

5. Optimise the cost function introduced in equation (7) by DE algorithm explained in the previous section.

For defining the mass distribution property of a structure, the simplest procedure is to assume that the mass parameters are concentrated at those points whose translational DOFs are defined. We note that by inserting the partial mass matrix identified by the collocated sensor/actuator DOFs into the full mass matrix, making the other non-diagonal and non-identified mass parameters to become zero, and realising the stiffness and damping property matrices in a symmetric format, DE has to optimise the cost function uniquely. So, the desired physical property matrices are obtained. Some limitations considered in this research are listed below:

1. Only accelerometer sensors may be in use, because we need this type of output in order to find the partial mass matrix parameters [equation (10b)]. It is noted that this kind of output extraction is a common practice.

2. At least, one DOF must be instrumented in a collocated sensor/actuator configuration. This condition leads to identifying at least one of the mass parameters, and the assumption of a diagonal construction of the mass matrix imposes a confining constraint in the solution space of the optimisation algorithm in order to find a unique physical property model.

To find the partial mass matrix from the state-space model, assume that the \( D \) matrix in the state-space model is obtained through 2 collocated DOFs. Therefore, we have

\[
D_{\text{colloc}} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}
\]

To establish the symmetry properties of the partial mass matrix, after computing the partial mass matrix from equation (13), we should assimilate the non-diagonal mass terms into an averaged value of the symmetric ones. We have

\[
M_p = \begin{bmatrix}
M_{p1}^{11} & \frac{1}{2}(M_{p1}^{12} + M_{p2}^{21}) \\
\frac{1}{2}(M_{p1}^{21} + M_{p2}^{12}) & M_{p2}^{22}
\end{bmatrix}
\]

6 Illustrative examples

In this section, two structures are examined to warrant the accuracy of the proposed algorithm. The first structure is a 3-DOF mass-spring-dashpot system with proportional damping, and afterward, an 8-DOF truss finite element model is illustrated. Both structures have been used in other researches and the reader may compare the results obtained in this paper with those obtained by Koh and See (1993), Agbabian et al, (1991) and Angelis et al. (2002).

6.1 Example 1: A 3-DOF mass-spring-dashpot system

In this subsection, a simple but general structural model, which is used by Angelis et al. (2002), is illustrated. We use a full set of sensors, instead of sensing only two last DOFs. Here, we extract mass, stiffness, and damping property by optimising the proposed cost function without any additional assumption on either orthogonality condition or normalising the identified complex-valued mode shapes, as opposed by Angelis et al. (2002) and Alvin et al. (1995). The finite element model matrices of the system, shown in Figure 1, are presented in Table 1.
Figure 1 A 3-DOF mass-spring system with proportional damping

\[
M = \begin{bmatrix}
0.8 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1.2
\end{bmatrix}, \quad \quad K = \begin{bmatrix}
4 & -1 & -1 \\
-1 & 4 & -1 \\
-1 & -1 & 4
\end{bmatrix}, \quad \quad E = \begin{bmatrix}
0.4 & -0.1 & -0.1 \\
-0.1 & 0.4 & -0.1 \\
-0.1 & -0.1 & 0.4
\end{bmatrix}
\]

The system is excited by a random input signal of \( F(k) \) for \( k = 1, 2, \ldots, 5,000 \), and the output signals are simulated with a sampling time of 0.01. By employing the ERA/DC algorithm, the input and output signals are used to find the state-space model and modal parameters. The identified \( D \)-matrix terms related to the collocated sensor/actuator DOFs, eigenvalues, and eigenvectors are listed in Table 2. We note that the resulting modal parameters are complex conjugate.

Table 2  Identified D-matrix terms related to those collocated DOFs and modal parameters

\[
D = \begin{bmatrix}
1.25 & -0.00 \\
-0.00 & 0.50
\end{bmatrix}, \quad \lambda_1 = -0.28 \pm 2.34i, \quad \lambda_2 = -0.07 \pm 1.17i, \quad \lambda_3 = -0.17 \pm 1.84i
\]

\[
\phi = \begin{bmatrix}
0.12 \pm 0.05i \\
-0.01 \pm 0.00i \\
-0.04 \pm 0.02i
\end{bmatrix}, \quad \phi = \begin{bmatrix}
-0.04 \pm 0.04i \\
-0.07 \pm 0.06i \\
-0.04 \pm 0.04i
\end{bmatrix}, \quad \phi = \begin{bmatrix}
-0.02 \pm 0.04i \\
0.03 \pm 0.05i \\
-0.06 \pm 0.09i
\end{bmatrix}
\]

As presented before, one can identify mass parameters correlated to the collocated DOFs. Based on Figure 1, the first two DOFs are instrumented with a collocated sensor/actuator pair. It can be observed that the contribution of non-diagonal mass terms \( \left( \frac{D_{12}}{D_{11}}, \frac{D_{21}}{D_{22}} \right) \) are very small and the partially identified mass matrix is practically a diagonal one. Therefore, one can identify the first two diagonal mass terms by inverting the diagonal \( D \)-elements \( \left( \frac{1}{D_{11}}, \frac{1}{D_{22}} \right) \).

The partial mass matrix can be computed as

\[
M_p = \begin{bmatrix}
0.80 & 0.00 \\
0.00 & 2.00
\end{bmatrix}
\]

To optimise the cost function, the symmetric mass, stiffness, and damping property matrices are realised as tabulated in Table 3.

Table 3  Symmetric mass, stiffness, and damping matrices including optimisation variables

\[
M = \begin{bmatrix}
0.8 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & \nu_1
\end{bmatrix}, \quad \quad K = \begin{bmatrix}
\nu_2 & \nu_3 & \nu_4 \\
\nu_3 & \nu_5 & \nu_6 \\
\nu_4 & \nu_6 & \nu_7
\end{bmatrix}, \quad \quad E = \begin{bmatrix}
\nu_8 & \nu_9 & \nu_{10} \\
\nu_9 & \nu_{11} & \nu_{12} \\
\nu_{10} & \nu_{12} & \nu_{13}
\end{bmatrix}
\]

Using DE adjusting variables, \( CR = 0.95, F = 0.7, \) and \( NP = 150 \), for an optimisation problem with 13 variables, the mass, stiffness, and damping matrices are computed, which are the same as those in Table 1. It is noted that even for a very large range of constraint values for the above optimisation adjusting variables, this algorithm can optimise the cost function very well, however, as a more confined range of values are selected, a fewer steps are taken.

6.2 Extraction of physical property matrices in the presence of noise

In this section, we are going to extract the mass, stiffness, and damping property matrices from the identified state-space model in the presence of noise. Consider the output signals are polluted with a Gaussian, zero-mean, random white noise sequence, whose RMS value is adjusted to be 5% of the original unpolluted system responses extracted from 3-DOFs. Using the identified \( D \)-matrix terms related to the collocated DOFs, the following partial mass matrix can be obtained:

\[
M_p = \begin{bmatrix}
0.804 & 0.000 \\
0.000 & 2.011
\end{bmatrix}
\]

Using the \( DE \) adjusting variables, the same as previous section, the realised property matrices, as presented in Table 4, are optimised.
Sensor-based extraction of physical property matrices from realised state-space models

Table 4

<table>
<thead>
<tr>
<th>Optimised physical property matrices in the presence of noise</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ M = \begin{bmatrix} 0.804 &amp; 0 &amp; 0 \ 0 &amp; 2.011 &amp; 0 \ 0 &amp; 0 &amp; 1.206 \end{bmatrix} ]</td>
</tr>
<tr>
<td>[ K = \begin{bmatrix} 4.019 &amp; -1.005 &amp; -1.005 \ -1.005 &amp; 4.021 &amp; -1.005 \ -1.005 &amp; -1.005 &amp; 4.020 \end{bmatrix} ]</td>
</tr>
<tr>
<td>[ E = \begin{bmatrix} 0.402 &amp; -0.100 &amp; -0.100 \ -0.100 &amp; 0.402 &amp; -0.100 \ -0.100 &amp; -0.100 &amp; 0.402 \end{bmatrix} ]</td>
</tr>
</tbody>
</table>

It can be observed that the proposed algorithm is capable of identifying the physical matrices very excellently, even in the presence of a reasonable noise. It is very important that the above results may be acquired without any extra assumptions on either the orthogonality conditions or the mode shape normalisation. While, in large structures, only some modes can be identified, the orthogonality may not be satisfied and extracting the physical properties is merged with erroneous results. Also, note that the extracted physical properties are completely real-valued, without any imaginary parts.

Table 5

<table>
<thead>
<tr>
<th>Truss finite element model matrices</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ M = \begin{bmatrix} 100 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 100 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 100 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 100 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 100 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 100 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 100 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 100 \end{bmatrix} ]</td>
</tr>
<tr>
<td>[ K = {10^4} \times \begin{bmatrix} 2.707 &amp; 0 &amp; 0 &amp; 0 &amp; -1 &amp; 0 &amp; -0.354 &amp; -0.354 \ 0 &amp; 1.707 &amp; 0 &amp; -1 &amp; 0 &amp; 0 &amp; -0.354 &amp; -0.354 \ 0 &amp; 0 &amp; 2.707 &amp; 0 &amp; -0.354 &amp; 0.354 &amp; -1 &amp; 0 \ 0 &amp; -1 &amp; 0 &amp; 1.707 &amp; 0.354 &amp; -0.354 &amp; 0 &amp; 0 \ -1 &amp; 0 &amp; -0.354 &amp; 0.354 &amp; 2.707 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0.354 &amp; -0.354 &amp; 0 &amp; 1.707 &amp; 0 &amp; -1 \ -0.354 &amp; -0.354 &amp; -1 &amp; 0 &amp; 0 &amp; 0 &amp; 2.707 &amp; 0 \ -0.354 &amp; -0.354 &amp; 0 &amp; 0 &amp; 0 &amp; -1 &amp; 0 &amp; 1.707 \end{bmatrix} ]</td>
</tr>
<tr>
<td>[ E = \begin{bmatrix} 136.4 &amp; 0 &amp; 0 &amp; 0 &amp; -50 &amp; 0 &amp; -17.7 &amp; -17.7 \ 0 &amp; 86.4 &amp; 0 &amp; -50 &amp; 0 &amp; 0 &amp; -17.7 &amp; -17.7 \ 0 &amp; 0 &amp; 136.4 &amp; 0 &amp; -17.7 &amp; 17.7 &amp; -50 &amp; 0 \ 0 &amp; -50 &amp; 0 &amp; 86.4 &amp; 17.7 &amp; -17.7 &amp; 0 &amp; 0 \ -50 &amp; 0 &amp; -17.7 &amp; 17.7 &amp; 136.4 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 17.7 &amp; -17.7 &amp; 0 &amp; 86.4 &amp; 0 &amp; -50 \ -17.7 &amp; -17.7 &amp; -50 &amp; 0 &amp; 0 &amp; 0 &amp; 136.4 &amp; 0 \ -17.7 &amp; -17.7 &amp; 0 &amp; 0 &amp; 0 &amp; -50 &amp; 0 &amp; 86.4 \end{bmatrix} ]</td>
</tr>
</tbody>
</table>

6.3 Example 2: an 8-DOF truss structure

In this section, an 8-DOF truss structure, shown in Figure 2, is case studied. The finite element model matrices are presented in Table 5 (Angelis et al., 2002). The damping property matrix of the truss structure is constructed so as to result in a classical damping.

Figure 2

An 8-DOF truss structure instrumented by 4 actuators and 8 accelerometer sensors

Source: Angelis et al. (2002)
The DOFs of the model are sorted such that the displacement vector can be written as $q(t)^T = [u_1(t) \ v_1(t) \ u_2(t) \ v_2(t) \ u_3(t) \ v_3(t) \ u_4(t) \ v_4(t)]$, where $u_i$ stand for the horizontal DOFs and $v_i$ for the vertical ones. The DOFs 2, 3, 5, and 8 are the actuator-instrumented DOFs while all of DOFs are accelerometer-instrumented. By simulating the model and identifying the state-space model by using signal sequences and ERA/DC algorithm for 5,000 time steps and sample time of 0.0004, one can identify the partial mass related to collocated DOFs as follows:

$M_p =$ 

$$
\begin{bmatrix}
100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 100 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 100 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 100 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 100 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 100 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 100 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 100
\end{bmatrix}
$$

The above partial mass matrix is obtained by inverting the partial $D$ matrix, which is attained through ignoring the 1st, 4st, 6th, and 7th rows. By constructing the mass, stiffness, and damping property matrices symmetrically, setting to zero the other non-diagonal mass parameters, and using DE adjusting variables, $CR = 0.9$, $F = 0.7$, and $NP = 400$, after 15,000 search steps, the physical matrices are computed as listed in Table 6. It can be observed that the physical property system matrices for a structure with a classical damping are extracted accurately, and one can use this model for further purposes such as damage detection and so on.

## 7 Conclusions

In this paper, a new approach was presented for extracting the physical property matrices from the identified state-space model. Selected structural DOFs were subject to sensor-instrumentation in order to find the system eigenvectors from the signal sequence data. The position and number of actuators should be optimised in such a way that the number of identified modes is more than the number of installed sensors. The mass, stiffness, and damping property matrices were identified using an inverse-optimisation problem without any extra assumptions on either orthogonality conditions, or the retrieval of normal modes, or any scaling procedure for the mode shapes. The superiority of this algorithm compared to those of other researches was delineated by either large-scale structures or incompletely instrumented small-scale structures, in which not all of the modes may be identified. In addition, all of the physical properties were acquired as real-valued quantities, without any imaginary parts. DE algorithm was used as the optimiser of the proposed cost function. Two general examples with consistent mass matrices were described and reasonable results were obtained. In a distributed mass case, the contributions of diagonal and non-diagonal mass
elements are affected by each other and same results can be obtained. Since the optimisation approach proposed in this research is useful for systems with a small number of DOFs, the number of parameters will increase and the optimisation algorithm may become inefficient. The productivity of this novel procedure is impressive in a substructure-level system identification context.

References


Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>State matrix</td>
</tr>
<tr>
<td>B</td>
<td>Input influence matrix (state-space model)</td>
</tr>
<tr>
<td>b</td>
<td>Input influence matrix (physical model)</td>
</tr>
<tr>
<td>C</td>
<td>Output influence matrix (state-space model)</td>
</tr>
<tr>
<td>C_d</td>
<td>Output influence matrix with full sensor instrumentation (state-space model)</td>
</tr>
<tr>
<td>e_a, e_v, e_d</td>
<td>Output acceleration, velocity, and displacement influence matrices</td>
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<tr>
<td>CF</td>
<td>Optimisation cost function</td>
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<tr>
<td>D</td>
<td>Direct transmission matrix</td>
</tr>
<tr>
<td>D_n</td>
<td>Number of optimisation members</td>
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<tr>
<td>D</td>
<td>Direct transmission matrix with full sensor instrumentation (state-space model)</td>
</tr>
<tr>
<td>D_Colloc</td>
<td>Collocated partition of direct transmission matrix</td>
</tr>
<tr>
<td>D_Uncolloc</td>
<td>Un-collocated partition of direct transmission matrix</td>
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<tr>
<td>E</td>
<td>Structural damping matrix</td>
</tr>
<tr>
<td>K</td>
<td>Structural stiffness matrix</td>
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<td>M</td>
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<td>M_p</td>
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<td>m</td>
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<tr>
<td>n</td>
<td>Number of structural DOFs</td>
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<td>r</td>
<td>Number of actuators</td>
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<td>u(t)</td>
<td>Applied input vector</td>
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<td>v(t)</td>
<td>Displacement vector</td>
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<tr>
<td>w(t)</td>
<td>Velocity vector</td>
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<td>y</td>
<td>Output vector</td>
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<td>\lambda_i</td>
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<tr>
<td>\phi_i</td>
<td>i^{th} structural mode shape</td>
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<tr>
<td>\Phi</td>
<td>Modal matrix</td>
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<tr>
<td>\Psi</td>
<td>Diagonal spectral matrix</td>
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