Labeled Packing of Non Star Tree into its Fifth Power and Sixth Power

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Abstract

In this paper we prove that we can find a labeled packing of a non star tree $T$ into $T^6$ with $m_T + \lceil \frac{n-m_T}{5} \rceil$ labels, where $n$ is the number of vertices of $T$ and $m_T$ is the maximum number of leaves that can be removed from $T$ in such a way that the obtained graph is a non star tree. Also, we prove that we can find a labeled packing of a non star tree $T$ into $T^5$ with $m_T + 1$ labels and a labeled packing of a path $P_n$, $n \geq 4$, into $P_n^4$ with $\lceil \frac{n}{4} \rceil$ labels.

1 Introduction

All graphs considered in this paper are finite and undirected. For a graph $G$, $V(G)$ and $E(G)$ will denote its vertex set and edge set, respectively. We denote by $N_G(x)$ the set of the neighbors of the vertex $x$ in $G$. The degree $d_G(x)$ of the vertex $x$ in $G$ is the cardinality of the set $N_G(x)$. For short, we use $d(x)$ instead of $d_G(x)$ and $N(x)$ instead of $N_G(x)$. The distance between two vertices of $G$, say $x$ and $y$, is denoted by $dist_G(x,y)$, and for short we usually use $dist(x,y)$. For a subset $U$ of $V$, we denote by $G - U$ the graph obtained from $G$ by deleting all the vertices in $U \cap V$ and their incident edges. For a subset $F$ of $E$, we write $G - F := (V, E \setminus F)$.

A vertex of degree one in a tree $T$ is called a leaf and the neighbor of a leaf is its father. For a non star tree $T$, we denote by $m_T$ the maximum number of leaves that can be removed from $T$ in such a way that the obtained tree is a non star one. The number of edges of a path $P$ is its length $l(P)$. A path on $n$ vertices is denoted by $P_n$. The middle vertex of $P_5$ will be called a bad vertex.

Let $G$ be a graph of order $n$. Consider a permutation $\sigma : V(G) \rightarrow V(K_n)$, the map $\sigma^* : E(G) \rightarrow E(K_n)$ such that $\sigma^*(xy) = \sigma(x)\sigma(y)$ is the map induced by $\sigma$. We say that there is a packing of $k$ copies of $G$ (into the complete graph $K_n$) if there exist permutations $\sigma_i : V(G) \rightarrow V(K_n)$, where $i = 1, \ldots, k$, such that $\sigma_i^*(E(G)) \cap \sigma_j^*(E(G)) = \phi$ for $i \neq j$. A packing of $k$ copies of a graph $G$ will be called a $k$-placement of $G$. A packing of two copies of $G$ (i.e. a 2-placement) is also called an embedding of $G$ (into its complement $\bar{G}$). That is, we say that $G$ can be embedded in its complement if there exists a permutation $\sigma$ on $V(G)$ such that if an edge $xy$ belongs to $E(G)$, then $\sigma(x)\sigma(y)$ does not belong to $E(G)$. A permutation $\sigma$ on $V(G)$ such that $\sigma(x) \neq x$ for every $x$ in $V(G)$ is called a fixed point free permutation.

The problem of embedding paths and trees in their complements has long been one of the fundamental questions in combinatorics that has been considerably investigated [2,4,5,6,7,8]. For recent results and survey on this field, we refer to the survey papers of Wozniak [9] and Yap [10].

Concerning non star trees, the following theorem was proved by Straight (unpublished, cf. [3]).

**Theorem 1.1** Let $T$ be a non star tree, then $T$ is contained in its own complement.
This result has been improved in many ways especially in considering some additional information and conditions about embedding. An example of such a result is the following theorem contained as a lemma in [9]:

**Theorem 1.2** Let $T$ be a non-star tree of order $n$ with $n > 3$. Then there exists a 2-placement $\sigma$ of $T$ such that for every $x \in V(T)$, $\text{dist}(x, \sigma(x)) \leq 3$.

This theorem immediately implies the following:

**Corollary 1.1** Let $T$ be a non-star tree of order $n$ with $n > 3$. Then there exists an embedding of $T$ such that $\sigma(T) \subseteq T^3$.

In [6], Kheddouci et al. gave a better improvement in the following theorem:

**Theorem 1.3** Let $T$ be a non star tree and let $x$ be a vertex of $T$. Then, there exists a permutation $\sigma$ on $V(T)$ satisfying the following four conditions:

1. $\sigma$ is a 2-placement of $T$.
2. $\sigma(T) \subseteq T^4$.
3. $\text{dist}(x, \sigma(x)) = 1$.
4. for every neighbor $y$ of $x$, $\text{dist}(y, \sigma(y)) \leq 2$.

Labeled graph packing is a well known field of graph theory that has been considerably investigated. It is introduced by E. Duchene and H. Kheddouci. Below is the definition of the labeled packing problem:

**Definition 1.1**. Consider a graph $G$. Let $f$ be a mapping from $V(G)$ to the set $\{1, 2, ..., p\}$. The mapping $f$ is called a $p$-labeled-packing of $k$ copies of $G$ into $K_n$ if there exists permutations $\sigma_i : V(G) \to V(K_n)$, where $i = 1, ..., k$, such that:

1. $\sigma_i^*(E(G)) \cap \sigma_j^*(E(G)) = \phi$ for all $i \neq j$.
2. For every vertex $v$ of $G$, we have $f(v) = f(\sigma_1(v)) = f(\sigma_2(v)) = ... = f(\sigma_k(v))$.

The maximum positive integer $p$ for which $G$ admits a $p$-labeled-packing of $k$ copies of $G$ is called the labeled packing number of $k$ copies of $G$ and is denoted by $\lambda^k(G)$.

E. Duchene et al. introduced the following two results which are presented as Lemmas in [1]. These results give an upper bound for the labeled packing $\lambda^2(G)$.

**Theorem 1.4** Let $G$ be a graph of order $n$ and let $I$ be a maximum independent set of $G$. If there exists an embedding of $G$ into $K_n$, then

$$\lambda^2(G) \leq |I| + \left\lfloor \frac{n-|I|}{2} \right\rfloor$$

**Theorem 1.5** Let $G$ be a graph of order $n$ with a maximum independent set $I$ of size at least $\left\lceil \frac{n}{4} \right\rceil$. If there exists a packing of $k \geq 2$ copies of $G$ into $K_n$, then

$$\lambda^k(G) \leq |I| + \left\lfloor \frac{n-|I|}{k} \right\rfloor$$
E. Duchene et al. [1] introduced and studied the labeled graph packing problem for some vertex labeled graphs. In this paper, we are concerned with finding a p-labeled-packing of $G$ into $G^k$.

We give below the definition of the new labeled packing problem:

**Definition 1.2** Let $f$ be a mapping from $V(G)$ to the set $\{1, 2, ..., p\}$. The mapping $f$ is called a p-labeled-packing of $G$ into $G^k$ if there exists a permutation $\sigma : V(G) \rightarrow V(K_n)$, such that:
1. $\sigma$ is a 2-placement of $G$.
2. $\sigma(G) \subseteq G^k$.
3. For every vertex $v$ of $G$, we have $f(v) = f(\sigma(v))$.

The maximum positive integer $p$ for which $G$ admits a $p$-labeled-packing of $G$ into $G^k$ is called the labeled packing $k$-power number and denoted by $w^k(G)$.

Concerning the packing of a path $P_n$, $n \geq 4$, into $P^4_n$, we introduce the following result:

**Theorem 1.6** Consider a path $P_n$, $n \geq 4$, and let $u$ and $v$ be its end vertices. Then there exists a permutation $\sigma$ on $P_n$ such that $\sigma$ satisfies the following conditions:
1. $\sigma$ is a 2-placement.
2. $\sigma(P_n) \subseteq P^4_n$.
3. $\text{dist}(u, \sigma(u)) = 1$ and $\text{dist}(v, \sigma(v)) \leq 1$.
4. The length of each cycle of $\sigma$ is at most 4.

This result allows us to establish the following:

**Corollary 1.2** Consider a path $P_n$, $n \geq 4$, then $w^4(P_n) \geq \lceil \frac{n}{4} \rceil$.

To formulate our main results we need to introduce some definitions.

Let $T$ be a non star tree and let $x$ be a vertex of $T$. Then, a fixed point free permutation $\sigma$ on $V(T)$ is called a $(T, x)$-well 2-placement if it satisfies the following conditions:
1. $\sigma$ is a 2-placement of $T$.
2. $\sigma(T) \subseteq T^6$.
3. $\text{dist}(x, \sigma(x)) \leq 2$.
4. $\text{dist}(y, \sigma(y)) \leq 3$ for every neighbor $y$ of $x$.
5. $\text{dist}(y, \sigma(y)) \leq 4$ for every $y$ such that $d(y) = 1$.
6. The length of each cycle of $\sigma$ is at most 5.

We prove first:

**Theorem 1.7** Let $T$ be a non star tree and let $x$ be a vertex of $T$. Then, there exists a $(T, x)$-well 2-placement.

This implies the following:

**Corollary 1.3** Consider a non star tree $T$ with $|V(T)| = n$, then $w^6(T) \geq m_T + \lceil \frac{n-m_T}{6} \rceil$.

Let $T$ be a non star tree and let $x$ be a vertex of $T$. Then, a fixed point free permutation $\sigma$ on $V(T)$ is called a $(T, x)$-good 2-placement if it satisfies the following conditions:
1. $\sigma$ is a 2-placement of $T$.
2. \( \sigma(T) \subseteq T^5 \).
3. \( \text{dist}(x, \sigma(x)) = 1 \).
4. \( \text{dist}(y, \sigma(y)) \leq 2 \) for every neighbor \( y \) of \( x \).
5. \( \text{dist}(y, \sigma(y)) \leq 4 \) for every \( y \) such that \( d(y) = 1 \).

We prove that:

**Theorem 1.8** Let \( T \) be a non star tree and let \( x \) be a vertex of \( T \) such that \( x \) is not a bad vertex. Then, there exists a \((T, x)\)-good 2-placement.

This result allows us to establish the following:

**Corollary 1.4** Consider a non star tree \( T \), then \( w^5(T) \geq m_T + 1 \).

## 2 Labeled Packing of \( P_n \) into \( P_n^4 \)

In this section, we are going to prove Theorem 1.6, but we need first to prove the theorem for \( n = 4, \ldots, 7 \):

**Lemma 2.1** Consider a path \( P_n \) such that \( 4 \leq n \leq 7 \), and let \( u \) and \( v \) be its end vertices. Then there exists a permutation \( \sigma \) on \( V(P_n) \) satisfying the following conditions:

1. \( \sigma \) is a 2-placement.
2. \( \sigma(P_n) \subseteq P_n^1 \).
3. \( \text{dist}(u, \sigma(u)) = 1 \) and \( \text{dist}(v, \sigma(v)) \leq 1 \).
4. The length of each cycle of \( \sigma \) is at most 4.

**Proof.** For each path \( P_n \), \( n = 4, \ldots, 7 \), we will introduce below a permutation \( \sigma \) on \( V(P_n) \), satisfying the above conditions:

For \( P = x_1x_2x_3x_4 \), \( \sigma = (x_1 \ x_2 \ x_4 \ x_3) \).

For \( P = x_1x_2x_3x_4x_5 \), \( \sigma = (x_1 \ x_2 \ x_5 \ x_4)(x_3) \).

For \( P = x_1x_2x_3x_4x_5x_6 \), \( \sigma = (x_1 \ x_2 \ x_5 \ x_4)(x_3)(x_6) \).

For \( P = x_1x_2x_3x_4x_5x_6x_7 \), \( \sigma = (x_1 \ x_2 \ x_5)(x_3 \ x_7 \ x_6)(x_4) \).

\[ \blacksquare \]

**Proof of Theorem 1.4.**

The proof is by induction. By the previous Lemma, \( \sigma \) exists for \( n = 4, \ldots, 7 \). Suppose now that \( n \geq 8 \) and the theorem holds for all \( n' < n \). Then \( P_n \) can be partitioned into two paths \( P' \) and \( P'' \) such that \( l(P') \), \( l(P'') \geq 3 \). Let \( x \) be the end vertex of \( P' \) and \( y \) that of \( P'' \) such that \( d_{P_n}(x) = d_{P_n}(y) = 2 \) and let \( \sigma' \) and \( \sigma'' \) be two permutations defined on \( V(P') \) and \( V(P'') \) respectively such that \( \sigma' \) and \( \sigma'' \) satisfy the four conditions mentioned in the theorem with \( \text{dist}(x, \sigma'(x)) = 1 \) and \( \text{dist}(y, \sigma''(y)) \leq 1 \). Let \( \sigma \) be a permutation defined on \( V(P_n) \), such that:

\[
\sigma(v) = \begin{cases} 
\sigma'(v) & \text{if } v \in V(P') \\
\sigma''(v) & \text{if } v \in V(P'')
\end{cases}
\]

It can be easily shown that \( \sigma \) satisfies the four conditions. \( \blacksquare \)
Proof of Corollary 1.2.
Consider a path \( P_n, n \geq 4 \), and let \( u \) and \( v \) be its end vertices. Then, by the previous theorem there exists a permutation \( \sigma \) on \( V(P_n) \) satisfying the following conditions:
1. \( \sigma \) is a 2-placement.
2. \( \sigma(P_n) \subseteq P_n^4 \).
3. \( \text{dist}(u, \sigma(u)) = 1 \) and \( \text{dist}(v, \sigma(v)) \leq 1 \).

4. The length of each cycle of \( \sigma \) is at most 4.

Let \( r \) be the number of cycles of \( \sigma \) and let \( \sigma_1, \ldots, \sigma_r \) be these cycles. Note that \( r \geq \lceil \frac{n}{4} \rceil \).
Label the vertices of \( \sigma \) by \( i \) for \( i = 1, \ldots, r \). Hence, we obtain a labeled packing of \( P_n \) into \( P_n^4 \) with \( r \) labels and so \( w^4(P_n) \geq \lceil \frac{n}{4} \rceil \). \( \square \)

3 Labeled Packing of a Non Star Tree \( T \) into \( T^5 \) and \( T^6 \)

In this section, we are going to prove the main results of this paper, but we still need to introduce some definitions and results on paths followed by a sequence of lemmas.

Consider a path \( P_n, n \geq 4 \), and let \( x \) be a vertex of \( P_n \). A fixed point free permutation \( \sigma \) on \( V(P_n) \) is called a \((P_n, x)\)-well path 2-placement, if it satisfies the following conditions:
1. \( \sigma \) is a 2-placement of \( P_n \).
2. \( \sigma(P_n) \subseteq P_n^6 \).
3. \( \text{dist}(x, \sigma(x)) \leq 2 \).
4. \( \text{dist}(y, \sigma(y)) \leq 3 \) for every \( y \in N(x) \) and for every \( y \) such that \( d(y) = 1 \).
5. The length of each cycle of \( \sigma \) is at most 5.

We will prove the following theorem:

**Theorem 3.1** Consider a path \( P_n \) and let \( x \) be a vertex of \( P_n, n \geq 4 \). Then there exists a \((P_n, x)\)-well path 2-placement.

**Lemma 3.1** Consider a path \( P_n, 4 \leq n \leq 7 \), and let \( x \) be a vertex of \( P_n \). Then there exists a \((P_n, x)\)-well path 2-placement, say \( \sigma \), such that \( \text{dist}(v, \sigma(v)) \leq 3 \) for every \( v \in V(P_n) \).

**Proof.** For each path \( P_n, n = 4, \ldots, 7 \), and for every vertex \( x \) of \( P_n \) we will introduce below a \((P_n, x)\)-well path 2-placement \( \sigma \) such that \( \text{dist}(v, \sigma(v)) \leq 3 \) for every \( v \in V(P_n) \):

For \( P = x_1 x_2 x_3 x_4 \), \( \sigma = (x_1 x_2 x_4 x_3) \) is a \((P, x)\)-well path 2-placement for every \( x \in V(P) \).
For \( P = x_1 x_2 x_3 x_4 x_5 \), \( \sigma = (x_1 x_2 x_4 x_5 x_3) \) is a \((P, x)\)-well path 2-placement for every \( x \in V(P) \).
For \( P = x_1 x_2 x_3 x_4 x_5 x_6 \), there are three choices for choosing \( x \), either \( x_1, x_2 \) or \( x_3 \). \( \sigma = (x_1 x_2 x_4)(x_3 x_6 x_5) \) is a \((P, x)\)-well path 2-placement for every \( x \in \{ x_1, x_2 \} \).
For \( P = x_1 x_2 x_3 x_4 x_5 x_6 x_7 \), there are four choices for choosing \( x \), either \( x_1, x_6, x_3 \) or \( x_4 \). \( \sigma = (x_1 x_2 x_4 x_3)(x_4 x_6 x_7) \) is a \((P, x)\)-well path 2-placement for every \( x \) of the previous choices. \( \blacksquare \)

**Proof of Theorem 3.1.**
The proof is by induction. By the previous Lemma, there exists a \((P_n, x)\)-well path 2-placement for every vertex \(x\) of \(P_n\), where \(4 \leq n \leq 7\). Suppose now \(n \geq 8\) and the theorem holds for all \(n' < n\). Let \(x\) be a vertex of \(P_n\). Since \(n \geq 8\), then \(P_n\) can be partitioned into two paths \(P'\) and \(P''\) such that \(l(P'), l(P'') \geq 3\). Without loss of generality, suppose that \(x \in V(P')\). Let \(x_1\) be the end vertex of \(P''\) such that \(d_{P_n}(x_1) = 2\). By induction, there exists a \((P', x)\)-well path 2-placement, say \(\sigma_x\), and there exists a \((P'', x_1)\)-well path 2-placement, say \(\sigma_{x_1}\). Let \(\sigma\) be a permutation defined on \(V(P_n)\) such that:

\[
\sigma(v) = \begin{cases} 
\sigma_x(v) & \text{if } v \in V(P') \\
\sigma_{x_1}(v) & \text{if } v \in V(P'')
\end{cases}
\]

It can be easily shown that \(\sigma\) is a \((P_n, x)\)-well path 2-placement. \(\square\)

Consider a path \(P_n, n \geq 4\), and let \(x\) be a vertex of \(P_n\). We say that a fixed point free permutation \(\sigma\) on \(V(P_n)\) is a \((P_n, x)\)-good path 2-placement if \(\sigma\) satisfies the following conditions:
1. \(\sigma\) is a 2-placement of \(P_n\).
2. \(\sigma(P_n) \subseteq P_n^5\).
3. \(\text{dist}(x, \sigma(x)) = 1\).
4. \(\text{dist}(y, \sigma(y)) \leq 2\) for every \(y \in N(x)\) and for every \(y\) such that \(d(y) = 1\).

We prove:

**Theorem 3.2** Let \(x\) be a vertex of \(P_n, n \geq 4\), such that \(x\) is not a bad vertex, then there exists a \((P_n, x)\)-good path 2-placement.

**Lemma 3.2** For every \(x\) in \(V(P_n), 4 \leq n \leq 7\), such that \(x\) is not a bad vertex, there exists a \((P_n, x)\)-good path 2-placement.

**Proof.** For every \(x\) in \(V(P_n), n = 4, ..., 7\), we will introduce below a \((P_n, x)\)-good path 2-placement \(\sigma_x\):

For \(P = x_1x_2x_3x_4\), there are two choices for choosing \(x\), either \(x_1\) or \(x_2\). Then:

\[\sigma_{x_1} = (x_1\ x_2\ x_4\ x_3); \ \sigma_{x_2} = (x_1\ x_3\ x_4\ x_2)\]

For \(P = x_1x_2x_3x_4x_5\), there are two choices for choosing \(x\), either \(x_1\) or \(x_4\). Then:

\[\sigma_{x_1} = \sigma_{x_4} = (x_1\ x_2\ x_4\ x_5\ x_3)\]

For \(P = x_1x_2x_3x_4x_5x_6\), there are three choices for choosing \(x\), either \(x_1, x_2\) or \(x_3\). Then:

\[\sigma_{x_1} = (x_1\ x_2\ x_4\ x_3\ x_6\ x_5); \ \sigma_{x_2} = \sigma_{x_3} = (x_1\ x_3\ x_4\ x_6\ x_5\ x_2)\]

For \(P = x_1x_2x_3x_4x_5x_6x_7\), there are four choices for choosing \(x\), either \(x_1, x_2, x_3\) or \(x_4\). Then:

\[\sigma_{x_1} = \sigma_{x_4} = (x_1\ x_2\ x_4\ x_5\ x_7\ x_6\ x_3); \ \sigma_{x_2} = \sigma_{x_3} = (x_1\ x_3\ x_4\ x_6\ x_7\ x_5\ x_2)\]
Proof of Theorem 3.2.

The proof is by induction. By the previous Lemma, there exists a \((P_n, x)\)-good path 2-placement for every \(x \in V(P_n)\) such that \(x\) is not a bad vertex, where \(4 \leq n \leq 7\). Suppose now \(n \geq 8\) and the theorem holds for all \(n' < n\). Let \(x\) be a vertex of \(P_n\). Since \(n \geq 8\), then \(P_n\) can be partitioned into two paths \(P'\) and \(P''\) such that \(l(P') = l(P'') \geq 3\). Without loss of generality, suppose that \(x \in V(P')\) such that \(P'\) is chosen to be distinct from \(P_5\). Let \(x_1\) be the end vertex of \(P''\) such that \(d_{P_n}(x_1) = 2\). By induction, there exists a \((P', x)\)-good path 2-placement, say \(\sigma_x\), and there exists a \((P'', x_1)\)-good path 2-placement, say \(\sigma_{x_1}\). Let \(\sigma\) be a permutation defined on \(V(P_n)\) such that:

\[
\sigma(v) = \begin{cases} 
\sigma_x(v) & \text{if } v \in V(P') \\
\sigma_{x_1}(v) & \text{if } v \in V(P'')
\end{cases}
\]

It can be easily shown that \(\sigma\) is a \((P_n, x)\)-good path 2-placement. □

Let \(T\) be a non star tree and let \(xy\) be an edge in \(T\). We denote by \(T_{(x,y)}\) the connected component containing \(x\) in \(T - \{xy\}\) and it is called a neighbor tree of \(y\). \(T_{(x,y)}\) is said to be a neighbor \(F\)-tree of \(y\) if \(T_{(x,y)}\) is a path of length at most two such that whenever \(T_{(x,y)}\) is a path of length two, then \(x\) is an end vertex in it.

Lemma 3.3 Consider a non star tree \(T\) containing a vertex \(x\) such that \(d(x) > 2\). Let \(\{x_1, ..., x_n\}, n > 2\), be the neighbors of \(x\). Suppose that \(T_{(x_1, x)}\) is a neighbor \(F\)-tree of \(x\) for \(i = 1, ..., m\), where \(2 \leq m < n\). Let \(T'\) be the connected component containing \(x\) in \(T - \{x_i; i = 1, ..., m\}\) and let \(G\) be the graph obtained by the union of the remaining components. Suppose that there exists a \((T', z)\)-well 2-placement \(\sigma\) such that \(\text{dist}(x, \sigma(x)) \leq 3\), where \(z\) is a vertex in \(T'\) not necessarily distinct from \(x\), then there exists a \((T, z)\)-well 2-placement, say \(\sigma_z\), such that \(\sigma_z(v) = \sigma(v)\) for every \(v \in V(T')\), \(\text{dist}(x_i, \sigma_z(x_i)) \leq 2\) for \(i = 1, ..., m\) and \(\text{dist}_T(\sigma_z(u), \sigma_z(v)) \leq 5\) whenever \(uv\) is an edge in \(G\).

Proof. Let \(r, p, q\) be the number of neighbor trees of \(x\) that are paths of length zero, one and two, respectively, in the set \(\{T_{(x_i, x)}; i = 1, ..., m\}\). In what follows we need to rename some neighbors of \(x\) for the sake of the proof. Let \(T_i = T_{(x_i, x)}\) for \(i = 1, ..., m\) such that if \(r > 0\), then \(T_i\) is the vertex \(a_i\) for \(i = 1, ..., r\), \(T_i = b_{i-r}c_{i-r}\) for \(i = r + 1, ..., p + r\) if \(p > 0\) and \(T_i = d_{i-(p+r)}e_{i-(p+r)}f_{i-(p+r)}\) for \(i = r + p + 1, ..., r + p + q\) if \(q > 0\). We will define a \((T, z)\)-well 2-placement \(\sigma_z\) according to the different values of \(r, p, q\) and \(q\). To construct \(\sigma_z\), we need first to introduce the permutations \(\Theta, \Upsilon, \Delta\) on \(V(G)\) in each case below such that:

\[
\Theta = \begin{cases} 
\theta & \text{if } r \text{ is even} \\
\theta' & \text{if } r \text{ is odd}
\end{cases}, \quad \Upsilon = \begin{cases} 
\epsilon & \text{if } p \text{ is even} \\
\epsilon' & \text{if } p \text{ is odd}
\end{cases} \quad \text{and} \quad \Delta = \begin{cases} 
\delta & \text{if } q \text{ is even} \\
\delta' & \text{if } q \text{ is odd}
\end{cases}
\]

where \(\theta, \theta', \epsilon, \epsilon', \delta, \delta'\) are permutations defined in each case below.

Case 1. \(p, q\) and \(r\) are strictly greater than one.

If \(r = 2n'\) for some \(n' \in \mathbb{Z}\), then let \(\theta = \prod_{j=1}^{j=n'} (a_{2j-1}a_{2j})\). If \(r = 2n' + 1\), then if \(n' = 1\) let \(\theta' = (a_1 a_2 a_3)\), else let \(\theta' = (a_1 a_2 a_3) \prod_{j=2}^{j=n'} (a_{2j}a_{2j+1})\).

If \(p = 2m'\) for some \(m' \in \mathbb{Z}\), then let \(\epsilon = \prod_{j=1}^{j=m'} (b_{2j}b_{2j-1}c_{2j-1})\). If \(p = 2m' + 1\), then let
\( e' = (b_1 b_2 b_3)(c_1 c_3 c_2) \) if \( m' = 1 \), else let \( e' = (b_1 b_2 b_3)(c_1 c_3 c_2) \prod_{j=2}^{j=m'} (b_{2j+1} c_{2j+1} b_{2j} c_{2j}) \).

If \( q = 2s' \) for some \( s' \in \mathbb{Z} \), then let \( \delta = \prod_{j=1}^{j=s'} (e_{2j-1} f_{2j} d_{2j-1})(e_{2j} f_{2j-1} d_{2j}) \). If \( q = 2s' + 1 \), then let \( \delta' = (d_1 e_1 f_2 e_2 f_1)(d_2 d_3 f_3 e_3) \prod_{j=1}^{j=s'} (e_{2j} f_{2j+1} d_{2j})(e_{2j+1} f_{2j} d_{2j+1}) \).

Finally, let \( \sigma_z = \Theta \sum \Delta \). 

- **Case 2.** \( r = 1 \).
  
  We will study the following subcases:

  1. \( p > 1 \) and \( q > 1 \).
     
     In this case, if \( p = 2m' \) then let \( \epsilon = (a_1 b_1)(b_2 c_2 c_1) \) if \( m' = 1 \), and if \( m' > 1 \) then let \( \epsilon = (a_1 b_1)(b_2 c_2 c_1) \prod_{j=2}^{j=m'} (b_{2j} c_{2j-1} c_{2j}). \) On the other hand, if \( p = 2m' + 1 \), then let \( \epsilon' = (a_1 b_1 c_1) \prod_{j=2}^{j=m'} (b_{2j+1} c_{2j+1} b_{2j} c_{2j}). \) Finally, let \( \sigma_z = \Delta \sum \), where \( \delta \) and \( \delta' \) are the same as in Case 1.

  2. \( p > 1 \) and \( q = 1 \).
     
     Let \( \sigma_z = \sum (a_1 d_1 f_1 e_1) \), where \( \epsilon \) and \( \epsilon' \) are the same as in Case 1.

  3. \( p > 1 \) and \( q = 0 \).
     
     Then let \( \sigma_z = \sum \), where \( \epsilon \) and \( \epsilon' \) are the same as in (1).

  4. \( p = 1 \) and \( q > 1 \).
     
     Then \( \sigma_z = (a_1 b_1 c_1) \Delta \sum \), where \( \delta \) and \( \delta' \) are the same as in Case 1.

  5. \( p = 1 \) and \( q = 1 \).
     
     Then let \( \sigma_z = (a_1 b_1 c_1 e_1)(f_1 d_1) \).

  6. \( p = 1 \) and \( q = 0 \).
     
     Then \( \sigma_z = (a_1 b_1 c_1) \).

  7. \( p = 0 \) and \( q > 1 \).
     
     If \( q = 2s' \) then let \( \delta = (d_1 e_1 f_2)(f_1 a_1 d_2 e_2) \) if \( s' = 1 \) and if \( s' > 1 \) then let \( \delta = (d_1 e_1 f_2)(f_1 a_1 d_2 e_2) \prod_{j=2}^{j=s'} (e_{2j-1} f_{2j} d_{2j-1})(e_{2j} f_{2j-1} d_{2j}). \) On the other hand, if \( q = 2s' + 1 \) then let \( \delta' = (a_1 d_1 f_1 e_1) \prod_{j=1}^{j=s'} (e_{2j} f_{2j+1} d_{2j})(e_{2j+1} f_{2j} d_{2j+1}) \). Finally, let \( \sigma_z = \Delta \sum \).

  8. \( p = 0 \) and \( q = 1 \).
     
     Then let \( \sigma_z = (a_1 d_1 f_1 e_1) \).

- **Case 3.** \( p = 1 \).

  1. \( r > 1 \) and \( q > 1 \).
     
     If \( r = 2n' \) then let \( \theta = (a_1 b_1 c_1 a_2) \) if \( n' = 1 \) and if \( n' > 1 \) then let \( \theta = (a_1 b_1 c_1 a_2) \prod_{j=2}^{j=n'} (a_{2j-1} a_{2j}). \) On the other hand, if \( r = 2n' + 1 \) then let \( \theta' = (b_1 c_1 a_1) \prod_{j=1}^{j=n'} (a_{2j} a_{2j+1}). \) Let \( \sigma_z = \Theta \sum \), where \( \delta \) and \( \delta' \) are defined as in Case 1.

  2. \( r > 1 \) and \( q = 1 \).
     
     Then let \( \sigma_z = (c_1 e_1 b_1)(f_1 d_1) \Theta \sum \), where \( \theta \) and \( \theta' \) are the same as the ones defined in Case 1.

  3. \( r > 1 \) and \( q = 0 \).
     
     Then let \( \sigma_z = \Theta \sum \), where \( \theta \) and \( \theta' \) are just like the ones in (1).
4. $r = 0$ and $q > 1$.
   If $q = 2s'$ then let $\delta = (d_1 f_1 b_1 c_1)(f_2 d_2)(e_1 e_2)$ if $s' = 1$ and if $s' > 1$ then let $\delta = (d_1 f_1 b_1 c_1)(f_2 d_2)(e_1 e_2)\prod_{j=2}^{n'}(e_{2j-1} f_{2j} d_{2j-1})(e_{2j} f_{2j-1} d_{2j})$. Otherwise, if $q = 2s' + 1$ then $\delta' = (e_1 d_1 f_1 b_1)\prod_{j=1}^{n'/2}(e_{2j} f_{2j+1} d_{2j})(e_{2j+1} f_{2j} d_{2j+1})$. Finally, let $\sigma_z = \Delta \sigma$.

5. $r = 0$ and $q = 1$.
   Let $\sigma_z = (c_1 e_1 f_1 d_1 b_1) \sigma$.

- **Case 4. $q = 1$.**

1. $r > 1$ and $p > 1$.
   If $r = 2n'$ then let $\theta = (d_1 f_1 e_1 a_1 a_2)$ if $n' = 1$ and if $n' > 1$ then let $\theta = (d_1 f_1 e_1 a_1 a_2)\prod_{j=2}^{n'}(a_{2j-1} a_{2j})$. If $r = 2n'+1$, then let $\theta' = (a_1 d_1 f_1 e_1)\prod_{j=1}^{n'/2}(a_{2j} a_{2j+1})$. Finally, let $\sigma_z = \Theta \tilde{\sigma}$, where $\epsilon$ and $\epsilon'$ are the same as the ones defined in Case 1.

2. $r > 1$ and $p = 0$.
   Then let $\sigma_z = \Theta \sigma$, where $\theta$ and $\theta'$ are the same as in (1).

3. $r = 0$ and $p > 1$.
   If $p = 2m'$ then let $\epsilon = (e_1 b_2 b_1 c_1 c_2)(d_1 f_1)$ and if $m' > 1$ then let $\epsilon = (e_1 b_2 b_1 c_1 c_2)(d_1 f_1)\prod_{j=2}^{m'}(b_{2j} c_{2j} b_{2j-1} c_{2j-1})$. On the other hand, if $q = 2m' + 1$, then let $\epsilon' = (e_1 f_1 d_1 b_1 c_1)\prod_{j=1}^{m'/2}(b_{2j} c_{2j} b_{2j+1} c_{2j+1})$. Finally, let $\sigma_z = \tilde{\sigma}$.

- **Case 5. $r = 0$.**

1. $p > 1$ and $q > 1$.
   Let $\sigma_z = \tilde{\sigma}$, where $\delta$, $\delta'$, $\epsilon$ and $\epsilon'$ are the same as the ones defined in Case 1.

2. $p > 1$ and $q = 0$.
   Then let $\sigma_z = \tilde{\sigma}$, where $\epsilon$ and $\epsilon'$ are the same as the ones defined in Case 1.

3. $p = 0$ and $q > 1$.
   Then let $\sigma_z = \tilde{\sigma}$, where $\delta$ and $\delta'$ are the same as the ones defined in Case 1.

- **Case 6. $p = 0$.**

1. $r > 1$ and $q > 1$.
   Then let $\sigma_z = \Theta \tilde{\sigma}$, where $\delta$, $\delta'$, $\theta$ and $\theta'$ are the same as the ones defined in Case 1.

2. $r > 1$ and $q = 0$.
   Then let $\sigma_z = \Theta \sigma$, where $\theta$ and $\theta'$ are the same as the ones defined in Case 1.

- **Case 7. $q = 0$.**
We still have only the case where $r > 1$ and $p > 1$. Then let $\sigma_z = \Theta \tilde{\sigma}$, where $\theta$, $\theta'$, $\epsilon$ and $\epsilon'$ are the same as the ones defined in Case 1.
Note that in each of the above cases, \( \text{dist}_G(x_i, \sigma_z(x_i)) \leq 2 \) for \( i = 1, \ldots, m \), \( \text{dist}_G(c_l, \sigma_z(c_l)) \leq 4 \) for \( l = 1, \ldots, p \), \( \text{dist}_G(f_j, \sigma_z(f_j)) \leq 4 \) for \( j = 1, \ldots, q \) and \( \text{dist}_G(\sigma_z(u), \sigma_z(v)) \leq 5 \) whenever \( uv \) is an edge in \( G \). Thus, it can be easily proved that \( \sigma_z \) is a \((T, z)\)-well 2-placement. \hfill \blacksquare

**Corollary 3.1** Consider a non star tree containing a vertex \( x \) such that \( d(x) > 2 \). Let \( \{x_1, \ldots, x_n\} \), \( n > 2 \), be the neighbors of \( x \). Suppose that \( T_{(x_i, z)} \) is a neighbor \( F \)-tree of \( x \) for \( i = 1, \ldots, m \), where \( 2 \leq m < n \). Let \( T' \) be the connected component containing \( x \) in \( T - \{xx_i; \ i = 1, \ldots, m\} \). Suppose that there exists a \((T', z)\)-good 2-placement \( \sigma \) such that \( \text{dist}(x, \sigma(x)) \leq 2 \), where \( z \) is a vertex in \( T' \) not necessarily distinct from \( x \), then there exists a \((T, z)\)-good 2-placement say \( \sigma_z \) such that \( \sigma_z(v) = \sigma(v) \) for every \( v \in V(T') \) and \( \text{dist}(x_i, \sigma_z(x_i)) \leq 2 \) for \( i = 1, \ldots, m \).

**Proof.** Let \( G \) be the graph obtained by the union of all the components that do not contain \( x \) in \( T - \{xx_i; \ i = 1, \ldots, m\} \). Since \( \sigma \) is a \((T', z)\)-good 2-placement then it is a \((T', z)\)-well 2-placement, and so, by Lemma 3.3, there exists a \((T, z)\)-well 2-placement, say \( \sigma_z \), such that \( \sigma_z(v) = \sigma(v) \) for every \( v \in V(T') \), \( \text{dist}(x_i, \sigma_z(x_i)) \leq 2 \) for \( i = 1, \ldots, m \) and \( \text{dist}(\sigma_z(u), \sigma_z(v))_T \leq 5 \) whenever \( uv \) is an edge in \( G \). Thus \( \sigma_z \) is a \((T, z)\)-good 2-placement. \hfill \blacksquare

**Lemma 3.4** Let \( T \) be a non star tree and let \( x \) be a vertex of \( T \) with \( N(x) = \{x_1, \ldots, x_n\} \), \( n \geq 2 \). Suppose that there exists a \((T_{(x_i, x)}, x_i)\)-well 2-placement for \( i = 1, \ldots, p \), where \( 1 \leq p < n \). Let \( T' \) be the connected component containing \( x \) in \( T - \{xx_i; \ i = 1, \ldots, p\} \). If there exists a \((T', z)\)-well 2-placement, say \( \sigma \), such that \( \text{dist}(x, \sigma(x)) \leq 3 \), where \( z \) is a vertex of \( T' \), then there exists a \((T, z)\)-well 2-placement, say \( \sigma_z \), such that \( \sigma_z(v) = \sigma(v) \) for every \( v \in V(T') \).

**Proof.** Let \( \sigma_i \) be a \((T_{(x_i, x)}, x_i)\)-well 2-placement for \( i = 1, \ldots, p \). Then the 2-placement \( \sigma \) defined as follows:

\[
\sigma_z(v) = \begin{cases} 
\sigma(v) & \text{if } v \in V(T') \\
\sigma_i(v) & \text{if } v \in V(T_{x_i, x}) \text{ for } i = 1, \ldots, p
\end{cases}
\]

is a \((T, z)\)-well 2-placement. \hfill \blacksquare

**Lemma 3.5** Let \( T \) be a non star tree and let \( x \) be a vertex of \( T \) with \( N(x) = \{x_1, \ldots, x_n\} \), \( n \geq 4 \). Let \( T' \) be the connected component containing \( x \) in \( T - \{xx_i; \ i = 1, \ldots, p\} \), \( 3 \leq p < n \). Suppose that at least two and at most \( p - 1 \) trees in the set \( \{T_{(x_i, x)} : i = 1, \ldots, p\} \) are neighbor \( F \)-trees of \( x \) such that for the remaining non neighbor \( F \)-trees in this set there exists a \((T_{(x_i, x)}, x_i)\)-well 2-placement. If there exists a \((T', z)\)-well 2-placement, say \( \sigma' \), such that \( \text{dist}(x, \sigma'(x)) \leq 3 \), where \( z \) is a vertex of \( T' \) not necessarily distinct from \( x \), then there exists a \((T, z)\)-well 2-placement, say \( \sigma \), such that \( \sigma(v) = \sigma'(v) \) for every \( v \in V(T') \).

**Proof.** Suppose that \( T_{(x_i, x)} \) are the neighbor \( F \)-trees of \( x \) for \( i = 1, \ldots, k \), where \( 2 \leq k \leq p - 1 \), such that there exists a \((T_{(x_i, x)}, x_i)\)-well 2-placement for \( i = k + 1, \ldots, p \). Let \( T'' \) be the connected component containing \( x \) in \( T - \{xx_i; \ i = k + 1, \ldots, p\} \). Since \( \text{dist}(x, \sigma'(x)) \leq 3 \) and since \( T_{(x_i, x)} \) are neighbor \( F \)-trees of \( x \) for \( i = 1, \ldots, k \), then, by Lemma 3.3, there exists a \((T'', z)\)-well 2-placement, say \( \sigma'' \), such that \( \sigma'(v) = \sigma''(v) \) for every \( v \in V(T') \). Again, since \( \text{dist}(x, \sigma''(x)) \leq 3 \) and there exists a \((T_{(x_i, x)}, x_i)\)-well 2-placement for \( i = k + 1, \ldots, p \), then, by the previous Lemma, there exists a \((T, z)\)-well 2-placement, say \( \sigma \), such that \( \sigma(v) = \sigma''(v) \) for every \( v \in V(T'') \). \hfill \blacksquare
Lemma 3.6 Let $T$ be one of the trees in Fig. 1 such that $n \geq 2$ and $n' \geq 3$. Then there exists a $(T, x_1)$-good 2-placement and a $(T, v)$-well 2-placement for every vertex $v$ of $T$.

Proof. For each tree $T$ in Fig. 1, we give below a $(T, x_1)$-good 2-placement and a $(T, v)$-well 2-placement for every vertex $v$ of $T$.

$\sigma = (x_1 x y_1 y_2)$ is a $(T_A, x_1)$-good 2-placement and a $(T_A, v)$-well 2-placement for every $v \in V(T_A)$.

There are four choices for choosing a vertex $v$ of $T_B$, either $x_1$, $x$, $y$ or $y_1$. If $n = 2$, then $\sigma = (x_2 y_2)(y_1 y x_1 x)$ is a $(T_B, x_1)$-good 2-placement and a $(T_B, v)$-well 2-placement, for every $v \in \{x_1, x, y, y_1\}$. Otherwise, suppose that $n \geq 3$. If $n = 2k + 1$ for some $k \in \mathbb{N}$, then $\sigma = (x_1 x y_1 y_2 y) \prod_{i=1}^{k}(x_{2i} x_{2i+1})$ is a $(T_B, x_1)$-good 2-placement and a $(T_B, v)$-well 2-placement $\forall v \in V(T_B)$. On the other hand, if $n = 2k$ for some $k \in \mathbb{N}$, then $\sigma = (x_2 y_2)(y_1 y x_1 x) \prod_{i=2}^{k}(x_{2i-1} x_{2i})$ is a $(T_B, x_1)$-good 2-placement and a $(T_B, v)$-well 2-placement, for every $v \in \{x_1, x, y, y_1\}$.

Finally, if $n' = 2k + 1$ for some $k \in \mathbb{N}$, then $\sigma = (x_1 x y_1 y) \prod_{i=1}^{k}(y_{2i} y_{2i+1})$ is a $(T_C, x_1)$-good 2-placement and a $(T_C, v)$-well 2-placement $\forall v \in V(T_C)$.

And if $n' = 2k$ for some $k \in \mathbb{N}$, $k \geq 2$, then $\sigma = (x_1 x y_1 y_2 y) \prod_{i=2}^{k}(y_{2i-1} y_{2i})$ is a $(T_C, x_1)$-good 2-placement and a $(T_C, v)$-well 2-placement for every $v \in V(T_C)$. \;

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Figure 2}
\end{figure}
Lemma 3.7 Let $T$ be one of the trees in Fig. 2. Then there exists a $(T,x)$-well 2-placement and a $(T,x_1)$-well 2-placement.

Proof. For each pair $(T,x)$ and $(T,x_1)$ in Fig. 2, we give below a $(T,x)$-well 2-placement $\sigma_x$ and a $(T,x_1)$-well 2-placement $\sigma_{x_1}$.

For $T_A$, $\sigma_x = \sigma_{x_1} = (x_1 x y_2)(x_2 y_1 z_2)$.
For $T_B$, $\sigma_x = \sigma_{x_1} = (x_1 x y_2)(y_1 z_2)(x_2 z_1)$.
For $T_C$, $\sigma_x = \sigma_{x_1} = (x_1 x y_2)(y_1 z_2)(x_2 w_1 z_1)$.
For $T_D$, $\sigma_x = \sigma_{x_1} = (x_1 x y_2)(x_2 y_1 z_1)$.
For $T_E$, $\sigma_x = \sigma_{x_1} = (x_1 x y_2)(z_2 w_2 y_1)(z_1 x_2 w_1)$.

Lemma 3.8 Consider a non star tree $T$ containing an edge $xy$ such that $d_T(x) = 1$ and $d_T(y) \geq 3$. Suppose that all the neighbor trees of $y$ distinct from $T_{(x,y)}$ are neighbor $F$-trees not isomorphic to $P_1$, then there exists a $(T,x)$-well 2-placement.

Proof. Let $\{y_1, \ldots, y_n\}$, where $n \geq 2$, be the neighbors of $y$ distinct from $x$. We need to consider the following two cases concerning the degree of $y$:

1. $d(y) > 3$.
   Let $T_0$ be the connected component containing $x$ in $T - \{yy_i; i = 2, \ldots, n\}$, then $T_0$ is isomorphic to $P_4$ or $P_5$, and so, by lemma 3.1, there exists a $(T_0,x)$-well path 2-placement, say $\sigma_0$, such that $dist(v,\sigma_0(v)) \leq 3$ for every vertex $v$ of $T_0$. Since $T_{(y_i,y)}$ are neighbor $F$-trees of $y$ for $i = 2, \ldots, n$, then, by Lemma 3.3, there exists a $(T,x)$-well 2-placement.

2. $d(y) = 3$
   Then $T$ is isomorphic to one of the following trees in Fig. 3. For each pair $(T,x)$ in Fig.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{tree.png}
\caption{Figure 3}
\end{figure}

3, we give below a $(T,x)$-well 2-placement $\sigma$:
$\sigma = (x y z_2)(y_1 z_1 y_2)$ is a $(T_A,x)$-well 2-placement.
$\sigma = (x y z_1)(y_1 z_2)(y_2 w_2)$ is a $(T_B,x)$-well 2-placement.
$\sigma = (x y z_1)(y_1 z_2 w_1)(w_2 y_2)$ is a $(T_C,x)$-well 2-placement.

\begin{itemize}
\item
\end{itemize}
Lemma 3.9 Consider a non star tree containing a vertex \( x \) with \( d(x) \geq 3 \). Suppose that all the neighbor trees of \( x \) are isomorphic either to \( P_1 \) or \( P_2 \) such that \( x \) has at most one neighbor tree isomorphic to \( P_1 \), then there exists a \((T,x)\)-well 2-placement.

Proof. Let \( N(x) = \{x_1, \ldots, x_n\} \), \( n \geq 3 \). If \( d(x) > 3 \), then let \( T_0 \) be the connected component containing \( x \) in \( T - \{xx_i: i = 3, \ldots, n\} \), then \( T_0 \) is isomorphic either to \( P_4 \) or \( P_5 \). Hence, by Lemma 3.1, there exists a \((T_0,x)\)-well path 2-placement, say \( \sigma_0 \). Since \( \text{dist}(x,\sigma_0(x)) \leq 2 \) and \( T_{(x,i)} \) are neighbor -trees for \( i = 3, \ldots, n \), then, by Lemma 3.3 ,there exists a \((T,x)\)-well 2-placement. On the other hand, if \( d(x) = 3 \), then \( T \) is isomorphic to one of the trees in Fig. 4.

For each pair \((T,x)\) in Fig. 4, we give below a \((T,x)\)-well 2-placement \( \sigma \):

- For \( T_A \), \( \sigma = (xy_3)(y_2 x_2 x_1)(x_3 y_1) \)
- For \( T_B \), \( \sigma = (x y_1)(x_1 y_2)(x_3 x_2) \)

Lemma 3.10 Consider a non star tree \( T \) containing an edge \( xy \) such that \( d_T(x) = 1 \) and \( d_T(y) \geq 3 \). Suppose that all the neighbor trees of \( y \) other than \( T_{(x,y)} \) are neighbor -trees not isomorphic to \( P_1 \). Then there exists a \((T,x)\)-good 2-placement.

Proof. Let \( \{y_1, \ldots, y_n\} \) be the set of neighbors of \( y \) distinct from \( x \), where \( n \geq 2 \). We are going to study two cases:

- \( y \) has a neighbor tree isomorphic to \( P_2 \).
  Without loss of generality, suppose that \( T_{(y_1,y)} = y_1 z_1 \). If \( n > 2 \), then let \( T_0 = T - \{yy_i : i = 2, \ldots, n\} \) \( \sigma_0 = (x y z_1 y_1) \) be a \((T_0,x)\)-good 2-placement. Since \( \text{dist}(y,\sigma_0(y)) = 2 \), then, by Corollary 3.1, there exists a \((T,x)\)-good 2-placement. Otherwise, that is, \( n = 2 \), then \( T \) is either isomorphic to \( T_A \) or \( T_B \) in Fig. 3. For each case, we give below a \((T,x)\)-good 2-placement \( \sigma \):
  - For \( T_A \), \( \sigma = (x y z_1)(y_1 y_2 z_2) \)
  - For \( T_B \), \( \sigma = (x y z_1 y_2)(y_3 w_2) \).

- \( y \) has a neighbor tree isomorphic to \( P_3 \).
  Without loss of generality, suppose that \( T_{(y_1,y)} = y_1 z_1 w_1 \). If \( n > 2 \), then let \( T_0 = T - \{yy_i : i = 2, \ldots, n\} \) and \( \sigma_0 = (x y z_1 w_1 y_1) \) be a \((T_0,x)\)-good 2-placement. Since \( \text{dist}(y,\sigma_0(y)) = 2 \), then, by Corollary 3.1, there exists a \((T,x)\)-good 2-placement.
Otherwise, that is, \( n = 2 \), then \( T \) is isomorphic either to \( T_B \) or \( T_C \) in Fig. 3. We showed above that there exists a \((T_B, x)\)-good 2-placement. For \( T_C \), \( \sigma = (x \ y \ z_1 \ w_1 \ y_1 \ y_2 \ w_2 \ z_2) \) is a \((T_C, x)\)-good 2-placement.

\[ \]

**Lemma 3.11** Let \( T \) be one of the trees in Fig. 5 such that \( n \geq 2 \). Then there exists a \((T, x)\)-good 2-placement \( \sigma_x \) such that \( \text{dist}(a_1, \sigma_x(a_1)) \leq 2 \) whenever \( T \) is isomorphic to \( T_E \), \( T_F \) or \( T_G \).

**Proof.** For each pair \((T, x)\) in Fig. 5, we give below a \((T, x)\)-good 2-placement \( \sigma_x \).

![Figure 5](image1)

For \( T_A \), \( \sigma_x = (x \ z_1 \ z_2 \ w \ y \ z) \)

For \( T_B \), if \( k = 2p \) for some \( p \in \mathbb{N} \), then \( \sigma_x = (x \ w \ l \ y_1 \ y_2) \) if \( k = 2 \), and if \( k > 2 \), then \( \sigma_x = (x \ w \ l \ y_1 \ y_2) \prod_{i=2}^{p}(y_{2i-1} \ y_{2i}) \). If \( k = 2p + 1 \), then \( \sigma_x = (x \ w \ y_1 \ y_2) \prod_{i=2}^{p}(y_{2i-1} \ y_{2i}) \).

For \( T_C \), \( \sigma_x = (x \ y \ w \ z_3) \).

For \( T_D \), if \( k = 2p \) for some \( p \in \mathbb{N} \), then \( \sigma_x = (x \ y \ l_1 \ l_2 \ l \ w \ z) \) if \( k = 2 \), and if \( k > 2 \), \( \sigma_x = (x \ y \ l_1 \ l_2 \ l \ w \ z) \prod_{i=2}^{p}(l_{2i-1} \ l_{2i}) \). If \( n = 2p + 1 \), then \( \sigma_x = (x \ y \ l_1 \ l \ w \ z) \prod_{i=2}^{p}(l_{2i-1} \ l_{2i}) \).

For \( T_E \), \( \sigma_x = (x \ y_2 \ y_1 \ x_1)(a_1 \ b_1 \ a_2) \).

For \( T_F \), \( \sigma_x = (x \ y_2 \ y_1 \ x_1)(a_1 \ c_1 \ b_1 \ a_2) \).

For \( T_G \), \( \sigma_x = (x \ y_2 \ y_1 \ x_1)(a_1 \ c_1 \ d_1)(b_1 \ a_2) \).

\[ \]

![Figure 6](image2)
Lemma 3.12 For the trees shown in Fig. 6, there exists a \((T_A, x)\)-good 2-placement, a \((T_A, x')\)-good 2-placement and a \((T_B, x)\)-good 2-placement.

Proof. \(\sigma = (x \ y \ x' \ w \ l)\) is a \((T_A, x)\)-good 2-placement and a \((T_A, x')\)-good 2-placement. 
\(\sigma = (x \ y \ z_1 \ w_1 \ z_2)(w_2 \ z)\) is a \((T_B, x)\)-good 2-placement.

Proof of Theorem 1.7.
The proof is by induction on the order of \(T\). For \(n = 4\), there is only one non star tree, which is \(P_4\), then, by Lemma 3.1, there exists a \((T, v)\)-well 2-placement for every \(v \in V(P_4)\). Suppose that the theorem holds for \(n' < n, n \geq 5\), and let \(T\) be a non star tree of order \(n\). Let \(x\) be a vertex of \(T\). If \(T\) is a path, then, by Theorem 3.1, there exists a \((T, x)\)-well path 2-placement. Otherwise, let \(v\) be a vertex of \(T\) and let \(\{v_1, \ldots, v_m\}, m \geq 2\), be the neighbors of \(v\). Suppose that \(\{v_1, \ldots, v_p\}, 1 \leq p < m\), are enumerated in such a way that \(x\) and \(v\) are in the same connected component in \(T - \{vv_i : i = 1, \ldots, p\}\) and let \(T'\) be this connected component. Note that \(x\) and \(v\) may be the same vertex.

Claim 1. If there exists a \((T', x)\)-well 2-placement \(\sigma\) such that \(\text{dist}(v, \sigma(v)) \leq 3\) and \(T_{(v,v)}\) are non star trees for \(i = 1, \ldots, p\), then there exists a \((T, x)\)-well 2-placement.

Proof. Since \(T_{(v,v)}\) is a non star tree for \(i = 1, \ldots, p\), then, by induction, there exists a \((T_{(v,v)}, v_i)\)-well 2-placement. Thus, by Lemma 3.4, there exists a \((T, x)\)-well 2-placement.

Claim 2. If \(2 \leq p < m\), at least two neighbor trees in the set \(\{T_{(v,v)} : i = 1, \ldots, p\}\) are neighbor \(F\)-trees of \(v\) such that the remaining neighbor trees in the set are non star trees and if there exists a \((T', x)\)-well 2-placement \(\sigma\) such that \(\text{dist}(v, \sigma(v)) \leq 3\), then there exists a \((T, x)\)-well 2-placement.

Proof. If \(T_{(v,v)}\) is a non star tree for some \(i, 1 \leq i \leq p\), then, by induction, there exists a \((T_{(v,v)}, v_i)\)-well 2-placement. Thus, since at least two neighbor trees in the set \(\{T_{(v,v)} : i = 1, \ldots, p\}\), are neighbor \(F\)-trees of \(v\), there exists, by Lemma 3.3 and Lemma 3.5, a \((T, x)\)-well 2-placement.

From now on, we shall assume that we can’t apply neither Claim 1 nor Claim 2 on any vertex \(v\) of \(T\). We will study two cases concerning the degree of \(x\).

Case 1. \(d(x) = 1\).
Let \(y\) be the father of \(x\). If \(y\) is the father of another leaf, say \(\alpha\), then let \(T' = T - \{x\}\). Note that \(T'\) is a non star tree since otherwise \(T\) is a star. So, there exists a \((T', \alpha)\)-well 2-placement, say \(\sigma'\). The 2-placement \(\sigma\) defined as follows:

\[
\sigma(v) = \begin{cases} 
\sigma'(v) & \text{if } v \in V(T) - \{\alpha, x\} \\
x & \text{if } v = \alpha \\
\sigma'(\alpha) & \text{if } v = x 
\end{cases}
\]

is a \((T, x)\)-well 2-placement. Otherwise, suppose that \(y\) is the father of the leaf \(x\) only. If there exists a set of leaves, say \(\{\alpha_1, \ldots, \alpha_k\}, k \geq 2\), such that all of these leaves have the same father, say \(\beta\), with \(d(\beta) = k + 1\), then let \(T' = T - \{\alpha_1, \ldots, \alpha_k\}\). If \(T'\) is a non star tree then
there exists a \((T', x)\)-well 2-placement, say \(\sigma_x\), such that \(\text{dist}(\beta, \sigma_x(\beta)) \leq 4\), since \(\beta\) is a leaf in \(T'\), and so \(\sigma = \sigma_x(a_1, \ldots, a_k)\) is a \((T, x)\)-well 2-placement. Else, that is \(T'\) is a star, then \(T\) is isomorphic to \(T_A\) in Fig. 1 if \(k = 2\), and if \(k > 2\) then \(T\) is isomorphic to \(T_C\) in Fig. 1 for \(n' = k\). Thus, by Lemma 3.6, there exists a \((T, x)\)-well 2-placement. Otherwise, suppose that the set \(\{a_1, \ldots, a_k\}\) doesn’t exist. Hence, we can remark that for any edge \(ab\) in \(E(T)\), \(T(\{a,b\})\) is either a neighbor \(F\)-tree of \(b\) or a non star tree. Since \(T\) is not a path, then there exists a vertex in \(T\) with degree strictly greater than two. Let \(z\) be the first vertex away from \(x\) such that \(d(z) > 2\), \(z'\) be the nearest neighbor of \(z\) to \(x\) and let \(\{z_1, \ldots, z_m\}\), \(m \geq 2\), be the neighbors of \(z\) distinct from \(z'\). Note that \(T(z', z)\) is a path and \(T(z, z')\) is a non star tree since otherwise the set \(\{a_1, \ldots, a_k\}\), which is described above, exists. \(T(z', z)\) is a path of length at most two, since otherwise there exists, by Theorem 3.1, a \((T(z', z), x)\)-well path 2-placement, say \(\sigma_x\), such that \(\text{dist}(z', \sigma_x(z')) \leq 3\), and so we can apply Claim 1 on \(z'\). Suppose first that the path \(T(z', \alpha)\) is of length zero, that is \(z\) is the neighbor of \(x\), then \(z\) is the father of the leaf \(x\) only. If \(z\) has a non neighbor \(F\)-tree, then suppose that \(T(z_1, z)\) is a non star tree. Since \(T(z_1, z)\) is a non star tree, then there exists a \((T(z_1, z), x)\)-well 2-placement \(\sigma'\) such that \(\text{dist}(z, \sigma'(z)) \leq 3\), and so we can apply Claim 1 on \(z\), a contradiction. Hence, all the neighbor trees of \(z\) are neighbor \(F\)-trees and so there exists a \((T, x)\)-well 2-placement by Lemma 3.8. Suppose now that the path \(T(z', z)\) is of length two or one. If \(z\) has a unique neighbor \(F\)-tree or at least three neighbor \(F\)-trees other that \(T(z', z)\), then suppose that \(T(z_1, z)\) is a neighbor \(F\)-tree and let \(T'\) be the connected component containing \(x\) in \(T - \{zz_i; i = 2, \ldots, m\}\). Thus, \(T'\) is a path of length at least three and at most six and so, by Lemma 3.1, there exists a \((T', x)\)-well 2-placement \(\sigma'\) such that \(\text{dist}(v, \sigma'(v)) \leq 3\) for every vertex \(v\) of \(T'\). Hence, we can apply Claim 1 or Claim 2 on \(z\), a contradiction. Thus, \(z\) has no neighbor \(F\)-tree or \(z\) has only two neighbor \(F\)-trees distinct from \(T(z', z)\). If \(T(z', z)\) is a path of length two, then let \(T'\) be the connected component containing \(x\) in \(T - \{zz_i; i = 1, \ldots, m\}\). Since \(T'\) is a path of length three, then there exists a \((T', x)\)-well path 2-placement \(\sigma'\), by Lemma 3.1, such that \(\text{dist}(z, \sigma'(z)) \leq 3\), and so we can apply Claim 1 or Claim 2 on \(z\), a contradiction. Thus, \(T(z', z)\) is a path of length one, that is \(z'\) is the father of \(x\). If \(z\) has no neighbor \(F\)-tree distinct from \(T(z', z)\), then each neighbor of \(z\) has at least two neighbors. Let \(\{a_1, \ldots, a_q\}\), \(q \geq 1\), be the neighbors of \(z_1\) distinct from \(z\). If \(z_1\) has at least two neighbor \(F\)-trees or no neighbor \(F\)-tree distinct from \(T(z_1, z)\), then let \(T'\) be the connected component containing \(x\) in \(T - \{zz_j; j = 2, \ldots, m\}\) and \(i = 1, \ldots, q\). Thus, \(T'\) is a path of length three and so, by Lemma 3.1, there exists a \((T', x)\)-well path 2-placement \(\sigma'\) such that \(\text{dist}(v, \sigma'(v)) \leq 3\) for every vertex \(v\) of \(T'\). Since \(T(z, z)\) are non neighbor \(F\)-trees for \(j = 2, \ldots, m\), then, by Lemma 3.4, there exists a \((T'', x)\)-well 2-placement \(\sigma''\), where \(T''\) is the connected component containing \(x\) in \(T - \{zz_j; j = 1, \ldots, q\}\), such that \(\sigma''(v) = \sigma'(v)\) for every vertex \(v\) of \(T''\) and so we can either apply Claim 1 or Claim 2 on \(z_1\), a contradiction. Thus, \(z_1\) has a unique neighbor \(F\)-tree distinct from \(T(z_1, z)\), say \(T(a_1, z_1)\). Let \(T'\) be the connected component containing \(x\) in \(T - \{zz_j; j = 2, \ldots, q\}\), then \(T'\) is a path of length at least four and at most six, and so, by Lemma 3.1, there exists a \((T', x)\)-well path 2-placement \(\sigma'\) such that \(\text{dist}(v, \sigma'(v)) \leq 3\) for every vertex \(v\) of \(T'\). Thus, by Lemma 3.4, there exists a \((T'', x)\)-well 2-placement \(\sigma''\), where \(T''\) is the connected component containing \(x\) in \(T - \{za_1; i = 2, \ldots, q\}\), such that \(\sigma''(v) = \sigma'(v)\) for every \(v\) of \(T''\) and so we can either apply Claim 1 on \(z_1\), a contradiction. Thus, \(z\) has only two neighbor \(F\)-trees distinct from \(T(z', z)\). If \(m > 2\), then suppose that \(T(z_1, z)\) and \(T(z_2, z)\) are the neighbor \(F\)-trees of \(z\). The tree \(T'\), which is the connected component containing \(x\) in \(T - \{zz_i; i = 3, \ldots, m\}\), is either isomorphic to one of the trees in Fig. 2 or to the tree \(T_A\) in Fig. 1, and so, by Lemma 3.6 and Lemma 3.7, there
exists a \((T', x)\)-well 2-placement \(\sigma'\) such that \(\text{dist}(z, \sigma'(z)) \leq 3\). Thus, we can apply Claim 1 on \(z\), a contradiction. Hence, \(m = 2\) and so \(T\) is isomorphic to one of the trees in Fig. 2, and so, by Lemma 3.7, there exists a \((T, x)\)-well 2-placement.

**Case 2.** \(d(x) > 1\).

If \(x\) or any of its neighbors is a father of at least two leaves, say \(\alpha_1\) and \(\alpha_2\), then let \(T' = T - \{\alpha_1, \alpha_2\}\). \(T'\) is a star, since otherwise there exists a \((T', x)\)-well 2-placement, say \(\sigma'\), such that \(\text{dist}(v, \sigma'(v)) \leq 3\) for every \(v \in \{x \cup N(x)\}\) and so we can apply Claim 2 on the father of \(\alpha_1\) and \(\alpha_2\), a contradiction. Hence, \(T\) is isomorphic to \(T_A\) or \(T_B\) in Fig. 1, and so, by Lemma 3.6, there exists a \((T, x)\)-well 2-placement. Else, suppose that neither \(x\) nor any of its neighbors is a father of at least two leaves. If there exists a set of leaves, say \(\{\alpha_1, ..., \alpha_m\}\), \(m \geq 2\), such that all of the leaves have the same father, say \(\beta\), with \(d(\beta) = m + 1\), then let \(T' = T - \{\alpha_1, ..., \alpha_m\}\). Note that \(T'\) is a non star tree since neither \(x\) nor any of its neighbors is a father of at least two leaves. Hence, there exists a \((T', x)\)-well 2-placement, say \(\sigma_x\), such that \(\text{dist}(\beta, \sigma_x(\beta)) \leq 4\) since \(\beta\) is an end vertex in \(T'\). Then \(\sigma = \sigma_x(\alpha_1, ..., \alpha_m)\) is a \((T, x)\)-well 2-placement. Otherwise, suppose that the set of leaves \(\{\alpha_1, ..., \alpha_m\}\) doesn’t exist. Since \(T\) is not a path and all the previous cases are not satisfied then there exists \(y \in N(x)\) such that \(T_{(x,y)}\) is a non star tree. There exists no neighbor \(y\) of \(x\) such that \(T_{(x,y)}\) and \(T_{(y,x)}\) are non star trees, since otherwise there exists a \((T_{(x,y)}, x)\)-well 2-placement, and so we can apply Claim 1 on \(x\). If there exists \(y \in N(x)\) such that \(T_{(x,y)}\) is a non star tree and \(T_{(y,x)}\) is a path of length two, then all the neighbor trees of \(x\) are neighbor \(F\)-trees. And since \(T\) is not a path then \(d(x) > 2\).

Let \(\{y_1, ..., y_m\}, m \geq 2\), be the neighbors of \(x\) distinct from \(y\), \(T_0\) be the connected component containing \(x\) in \(T - \{xy_i; i = 1, ..., m\}\). Then \(T_0\) is a path of length three and so there exists, by Lemma 3.1, a \((T_0, x)\)-well 2-placement. Hence, we can apply Claim 2 on \(x\), a contradiction. Thus, there exists no neighbor \(y\) of \(x\) such that \(T_{(x,y)}\) is a non star tree and \(T_{(y,x)}\) is a path of length two. If there exists \(y \in N(x)\) such that \(T_{(x,y)}\) is a non star tree and \(T_{(y,x)}\) is a path of length one, then if \(d(x) = 2\), let \(y_1\) be the neighbor of \(x\) distinct from \(y\). \(T_{(y_1,x)}\) is not a neighbor \(F\)-tree of \(x\) since \(T\) is not a path. Let \(\{b_1, ..., b_r\}\), \(r \geq 1\), be the neighbors of \(y_1\) distinct from \(x\) and let \(T_0\) be the connected component containing \(x\) in \(T - \{y_1b_i; i = 1, ..., r\}\). Then \(T_0\) is a path of length three, and so there exists, by Lemma 3.1, a \((T_0, x)\)-well path 2-placement \(\sigma_0\) such that \(\text{dist}(y, \sigma_0(y)) \leq 3\). Thus, \(y_1\) has a unique neighbor \(F\)-tree distinct from \(T_{(x,y)}\), since otherwise we can apply either Claim 1 or Claim 2 on \(y_1\). Suppose that \(T_{(b_i,y_1)}\) is that tree and let \(T'\) be the connected component containing \(x\) in \(T - \{y_1b_i; i = 2, ..., r\}\). Hence, \(T'\) is a path of length at least four and at most six, and so, by Lemma 3.1, there exists a \((T', x)\)-well path 2-placement, say \(\sigma'\), such that \(\text{dist}(y, \sigma'(y)) \leq 3\). Thus, we can apply Claim 1 on \(y_1\), a contradiction. Hence, \(d(x) > 2\), and so all the neighbor trees of \(x\) are paths of length one with at most one is a path of length zero. Hence, by Lemma 3.9, there exists a \((T, x)\)-well 2-placement. Otherwise, suppose that there exists no \(y \in N(x)\) such that \(T_{(x,y)}\) is a non star tree and \(T_{(y,x)}\) is a path of length one. Then, there exists \(y \in N(x)\) such that \(T_{(x,y)}\) is a non star tree, \(T_{(y,x)}\) is a path of length zero and \(d(x) = 2\). Let \(N(x) = \{y, y'\}\). If \(d(y') = 2\), then let \(a\) be the neighbor of \(y'\) distinct from \(x\) and let \(\{a_1, ..., a_k\}, k \geq 1\), be the neighbors of \(a\) distinct from \(y'\). Let \(T_0\) be the connected component containing \(x\) in \(T - \{aa_i; i = 1, ..., k\}\), then \(T_0\) is a path of length three, and so, by Lemma 3.1, there exists a \((T_0, x)\)-well path 2-placement \(\sigma_0\) such that \(\text{dist}(a, \sigma_0(a)) \leq 3\). Thus, \(a\) has a unique neighbor \(F\)-tree distinct from \(T_{(y', a)}\), since otherwise we can apply either Claim 1 or Claim 2 on \(a\). Suppose that \(T_{(a_1, a)}\) is that tree and let \(T'\) be the connected component containing \(x\) in \(T - \{aa_i; i = 2, ..., k\}\). Then \(T'\) is a path of length at
least four and at most six, and so, by Lemma 3.1, there exists a \((T', x)\)-well path 2-placement, say \(\sigma'\), such that \(\text{dist}(a, \sigma'(a)) \leq 3\). Thus, we can apply Claim 1 on \(a\), a contradiction. Hence, \(d(y') > 2\). Let \(\{y'_1, ... , y'_l\}\), \(l \geq 2\), be the neighbors of \(y'\) distinct from \(x\). If \(y'\) has a non neighbor \(F\)-tree then suppose that \(T_{(y'_1, y')}\) is that tree. Note that \(T_{(y'_1, y')}\) is a non star tree and so there exists a \((T_{(y'_1, y')}, x)\)-well 2-placement say \(\sigma'\) such that \(\text{dist}(y', \sigma'(y')) \leq 3\). Thus, Claim 1 is applied on \(y'\), a contradiction. Hence, all the neighbor trees of \(y'\) are \(F\)-trees. If \(d(y') > 3\), then let \(T'\) be the connected component containing \(x\) in \(T - \{y'_1, ..., y'_l\}\). Since \(T'\) is a non star tree, then there exists a \((T', x)\)-well 2-placement and so we can apply Claim 2 on \(y'\), a contradiction. Thus, \(d(y') = 3\) and \(T\) is isomorphic to one of the trees in Fig. 2, and so, by Lemma 3.7, there exists a \((T, x)\)-well 2-placement. □

Proof of Corollary 1.3.
Let \(T_0 = T - \{\alpha_1, ..., \alpha_{m_T}\}\), where \(\{\alpha_1, ..., \alpha_{m_T}\}\) is the maximal set of leaves that can be removed from \(T\) in such a way that the obtained tree is a non star one. Since \(T_0\) is a non star tree then there exists a \((T_0, x)\)-well 2-placement for some \(x\) in \(T_0\). We define a packing of \(T\) into \(T^6\), say \(\sigma\), as follows:

\[
\sigma(v) = \begin{cases} 
\sigma' & \text{if } v \in V(T') \\
\alpha_i & \text{if } v = \alpha_i \text{ for } i = 1, ..., m_T
\end{cases}
\]

Label \(\alpha_i\) by \(i\), for \(i = 1, ..., m_T\). Let \(r\) be the number of cycles of \(\sigma\) and let \(\sigma_1, ..., \sigma_r\) be those cycles. Remark that \(r \geq \lceil \frac{n-m_T}{5} \rceil\). Label the vertices of each cycle \(\sigma_i\) by \(m_T + i\) for \(i = 1, ..., r\). Hence, we obtain an \((m_T + r)\)-labeled packing of \(T\) into \(T^6\) and so \(w^6(T) \geq m_T + \lceil \frac{n-m_T}{5} \rceil\). □

Proof of Theorem 1.8.
The proof is by induction on the order of \(T\). For \(n = 4\) there is only one non star tree, which is \(P_4\). By Lemma 3.2, there exists a \((T, v)\)-good 2-placement for every \(v \in V(P_4)\). Suppose that the theorem holds for \(n' < n\), \(n \geq 5\), and let \(T\) be a non star tree of order \(n\). Let \(x\) be a vertex of \(T\) such that \(x\) is not a bad vertex. If \(T\) is a path, then, by Theorem 3.2, there exists a \((T, x)\)-good path 2-placement. Else, we will study two cases concerning the degree of \(x\):

Case 1. \(d(x) = 1\).
Let \(y\) be the father of \(x\). If \(y\) is a father of another leaf \(\alpha\), then let \(T' = T - \{x\}\). Note that \(T'\) is a non star tree since otherwise \(T\) is a star. Hence, there exists a \((T', \alpha)\)-good 2-placement, say \(\sigma'\). The 2-placement \(\sigma\) defined as follows:

\[
\sigma(v) = \begin{cases} 
\sigma'(v) & \text{if } v \in V(T') - \{\alpha\} \\
\sigma'(\alpha) & \text{if } v = x \\
x & \text{if } v = \alpha
\end{cases}
\]

is a \((T, x)\)-good 2-placement. Otherwise, suppose that \(y\) is the father of the leaf \(x\) only. If there exists a set of leaves, say \(\{\alpha_1, ..., \alpha_m\}\), \(m \geq 2\), such that all of these leaves have the same father, say \(\beta\), with \(d(\beta) = m + 1\), then let \(T' = T - \{\alpha_1, ..., \alpha_m\}\). If \(T'\) is a non star tree, then there exists a \((T', x)\)-good 2-placement, say \(\sigma_x\), such that \(\text{dist}(\beta, \sigma_x(\beta)) \leq 4\) since \(\beta\) is a leaf in \(T'\). Hence, \(\sigma = \sigma_x(\alpha_1, ..., \alpha_m)\) is a \((T, x)\)-good 2-placement. And if \(T'\) is a star then \(T\) is isomorphic to \(T_A\) or \(T_C\) in Fig. 1, and so, by Lemma 3.6, there exists a \((T, x)\)-good 2-placement.
Otherwise, suppose that the set \( \{ \alpha_1, ..., \alpha_m \} \) doesn’t exist. Hence, we can remark that for any edge ab in \( E(T) \), \( T(ab) \) can be either a neighbor \( F \)-tree of \( b \) or a non star tree. Let \( z \) be the first vertex away from \( x \) such that \( d(z) > 2 \). \( z \) exists since \( T \) is not a path. Let \( z' \) be the nearest neighbor of \( z \) to \( x \) and let \( \{ z_1, ..., z_m \} \), \( m \geq 2 \), be the neighbors of \( z \) distinct from \( z' \). Note that \( T(z',z) \) is a path and \( T(z,z') \) is a non star tree, since otherwise the set of leaves described above exists. If \( T(z',z) \) is a path of length at least three, then let \( \sigma_x \) be a \((T(z',z),x)\)-good path 2-placement and let \( \sigma_z \) be a \((T(z,z'),z)\)-good 2-placement if \( z \) is not a bad vertex in \( T(z,z') \), and if it is, then let \( \sigma_z \) be a \((T(z,z'),z'')\)-good 2-placement, where \( z'' \) is a neighbor of \( z \) in \( T(z,z') \). Since \( \text{dist}(z', \sigma_x(z')) \) \( \leq 2 \), then the 2-placement \( \sigma \) defined as follows:

\[
\sigma(v) = \begin{cases} 
\sigma_x(v) & \text{if } v \in V(T(z',z)) \\
\sigma_z(v) & \text{if } v \in V(T(z,z')) 
\end{cases}
\]

is a \((T,x)\)-good 2-placement. Otherwise, \( T(z',z) \) is a path of length at most two. If \( T(z',z) \) is a path of length two, then let \( \sigma' \) be a \((T(z',y),z')\)-good 2-placement. The 2-placement \( \sigma \) defined as follows:

\[
\sigma(v) = \begin{cases} 
\sigma'(v) & \text{if } v \in V(T) - \{x, y, z'\} \\
x & \text{if } v = z' \\
y & \text{if } v = x \\
\sigma'(z') & \text{if } v = y 
\end{cases}
\]

is a \((T,x)\)-good 2-placement. Else, if \( T(z',z) \) is a path of length one, that is, \( x \) is a neighbor of \( z' \), then if \( z \) is not a bad vertex in \( T(z,z') \), let \( \sigma' \) be a \((T(z,z'),z)\)-good 2-placement. The 2-placement \( \sigma \) defined as follows:

\[
\sigma(v) = \begin{cases} 
\sigma'(v) & \text{if } v \in V(T) - \{x, z, z'\} \\
x & \text{if } v = z \\
z' & \text{if } v = x \\
\sigma'(z) & \text{if } v = z' 
\end{cases}
\]

is a \((T,x)\)-good 2-placement. And if \( z \) is a bad vertex in \( T(z,z') \) then \( T \) is isomorphic to \( T_B \) in Fig. 6, and so, by Lemma 3.12, there exists a \((T,x)\)-good 2-placement. Otherwise, \( T(z',z) \) is a path of length zero, that is, \( x \) is a neighbor of \( z \). If there exists a neighbor of \( z \), say \( z_1 \), such that \( T(z_1,z) \) is not a neighbor \( F \)-tree of \( z \), then let \( \sigma_{z_1} \) be a \((T(z_1,z),z_1)\)-good 2-placement if \( z_1 \) is not a bad vertex in \( T(z_1,z) \), and if it is, then let \( \sigma_{z_1} \) be a \((T(z_1,z),w_1)\)-good 2-placement, where \( w_1 \) is a neighbor of \( z_1 \) in \( T(z_1,z) \). \( T(z_1,z) \) is a non star tree since \( z \) is the father of the leaf \( x \) only, then there exists a \((T(z_1,z),x)\)-good 2-placement, say \( \sigma_x \), such that \( \text{dist}(z, \sigma_x(z)) \) \( \leq 2 \). We define a \((T,x)\)-good 2-placement \( \sigma \) as follows:

\[
\sigma(v) = \begin{cases} 
\sigma_{z_1}(v) & \text{if } v \in V(T(z_1,z)) \\
\sigma_x(v) & \text{if } v \in V(T(z,z_1)) 
\end{cases}
\]

Otherwise, suppose that all the neighbor trees of \( z \) are neighbor \( F \)-trees. Then, by Lemma 3.10, there exists a \((T,x)\)-good 2-placement.
Case 2. $d(x) > 1$.

If $x$ or any of its neighbors is a father of at least two leaves distinct from $x$, say $\alpha_1$ and $\alpha_2$, then let $T' = T - \{\alpha_1, \alpha_2\}$. If $T'$ is a non star tree such that $x$ is not a bad vertex in $T'$ then let $\sigma_x$ be a $(T', x)$-good 2-placement. Then $\sigma = \sigma_x(\alpha_1 \alpha_2)$ is a $(T, x)$-good 2-placement. If $x$ is a bad vertex in $T'$ then $T$ is isomorphic to $T_A$ or $T_C$ in Fig. 5, and so, by Lemma 3.11, there exists a $(T, x)$-good 2-placement. And if $T'$ is a star then $T$ is isomorphic either to $T_B$ in Fig. 5 or to $T_A$ in Fig. 6, and so, by Lemma 3.11 and Lemma 3.12, there exists a $(T, x)$-good 2-placement. 

Otherwise, $x$ and each of its neighbors is the father of at most one leaf. If there exists a set of leaves, say $\{\alpha_1, ..., \alpha_m\}$, $m \geq 2$, such that all of the leaves have the same father, say $\beta$, with $d(\beta) = m + 1$, then let $T' = T - \{\alpha_1, ..., \alpha_m\}$. Note that $T'$ is a non star tree since neither $x$ nor any of its neighbors is a father of at least two leaves. If $x$ is not a bad vertex in $T'$, then there exists a $(T', x)$-good 2-placement, say $\sigma_x$, such that $\text{dist}(\beta, \sigma_x(\beta)) \leq 4$ since $\beta$ is a leaf in $T'$. Thus $\sigma = \sigma_x(\alpha_1... \alpha_m)$ is a $(T, x)$-good 2-placement. And if $x$ is a bad vertex in $T'$, then $T$ is isomorphic to $T_D$ in Fig. 5, and so, by Lemma 3.11, there exists a $(T, x)$-good 2-placement. Otherwise, suppose that the set of leaves $\{\alpha_1, ..., \alpha_m\}$ doesn’t exist in $T$. Since $T$ is not a path, neither $x$ nor any of its neighbors is the father of at least two leaves and the set of leaves having the same father doesn’t exist, then there exists $y \in N(x)$ such that $T_{(x,y)}$ is a non star tree. Whenever $x$ is a bad vertex in $T_{(x,y)}$, let $x_1$ and $x_2$ be the neighbors of $x$ in $T_{(x,y)}$ and $y_1$ and $y_2$ be that of $x_1$ and $x_2$, respectively. If there exists $y \in N(x)$ such that $T_{(x,y)}$ and $T_{(y,x)}$ are non star trees, then let $\sigma_y$ be a $(T_{(y,x)}, y)$-good 2-placement if $y$ is not a bad vertex in $T_{(y,x)}$, and if it is, then let $\sigma_y$ be a $(T_{(y,x)}, y')$-good 2-placement, where $y'$ is a neighbor of $y$ in $T_{(y,x)}$. If $x$ is not a bad vertex in $T_{(x,y)}$, then let $\sigma_x$ be a $(T_{(x,y)}, x)$-good 2-placement. The 2-placement $\sigma$ defined as follows:

$$
\sigma(v) = \begin{cases} 
\sigma_x(v) & \text{if } v \in V(T_{(x,y)}) \\
\sigma_y(v) & \text{if } v \in V(T_{(y,x)}) 
\end{cases}
$$

is a $(T, x)$-good 2-placement. And if $x$ is a bad vertex in $T_{(x,y)}$, then let $T'$ be the connected component containing $x$ in $T - \{x_1, x_2\}$ and let $\sigma_x$ be a $(T', x)$-good 2-placement. Thus $\sigma = \sigma_x(x_2 y_2 x_1 y_1)$ is a $(T, x)$-good 2-placement. Otherwise, if there exists $y \in N(x)$ such that $T_{(x,y)}$ is a non star tree and $T_{(y,x)}$ is a path of length zero, then if $x$ is not a bad vertex in $T_{(x,y)}$, let $\sigma_x$ be a $(T_{(x,y)}, x)$-good 2-placement. The 2-placement $\sigma$ defined such that:

$$
\sigma(v) = \begin{cases} 
\sigma_x(v) & \text{if } v \in V(T_{(x,y)}) - \{x\} \\
y & \text{if } v = x \\
\sigma_x(x) & \text{if } v = y 
\end{cases}
$$

is a $(T, x)$-good 2-placement. And if $x$ is a bad vertex in $T_{(x,y)}$ then $\sigma = (x y x_1 y_1 x_2 y_2)$ is a $(T, x)$-good 2-placement. Else, if there exists $y \in N(x)$ such that $T_{(x,y)}$ is a non star tree and $T_{(y,x)}$ is a path of length two, then if $x$ is not a bad vertex in $T_{(x,y)}$, let $\sigma_x$ be a $(T_{(x,y)}, x)$-good 2-placement and let $T_{(y,x)} = yzw$. The 2-placement $\sigma$ defined such that:

$$
\sigma(v) = \begin{cases} 
\sigma_x(v) & \text{if } v \in V(T_{(x,y)}) - \{x\} \\
y & \text{if } v = x \\
\sigma_x(x) & \text{if } v = z \\
w & \text{if } v = y \\
z & \text{if } v = w 
\end{cases}
$$
is a \((T, x)\)-good 2-placement. And if \(x\) is a bad vertex in \(T_{(x,y)}\), then \(\sigma = (x y w z)(x_1 y_1 x_2 y_2)\) is a \((T, x)\)-good 2-placement. Else, there exists \(y \in N(x)\) such that \(T_{(x,y)}\) is a non star tree and \(T_{(y,x)}\) is a path of length one. If \(d(x) = 2\), then let \(N(x) = \{y_1, y_2\}\). Suppose that \(T_{(y_1,x)} = y_1 x_1\) and \(T_{(x,y_1)}\) is a non star tree. Let \(\{a_1, ..., a_m\}, m \geq 1, \) be the neighbors of \(y_2\) distinct from \(x\). If \(y_2\) has a non neighbor \(F\)-tree, then suppose that \(T_{(a_1,y_2)}\) is that tree and let \(\sigma_{a_1}\) be a \((T_{(a_1,y_2)}, a_1)\)-good 2-placement if \(a_1\) is not a bad vertex in \(T_{(a_1,y_2)}\), and if it is, then let \(\sigma_{a_1}\) be a \((T_{(a_1,y_2)}, b)\)-good 2-placement, where \(b\) is a neighbor of \(a_1\) distinct from \(y_2\). If \(x\) is not a bad vertex in \(T_{(y_2,a_1)}\), then let \(\sigma_x\) be a \((T_{(y_2,a_1)}, x)\)-good 2-placement. Finally, the 2-placement \(\sigma\) defined as follows:

\[
\sigma(v) = \begin{cases} 
\sigma_x(v) & \text{if } v \in V(T_{(y_2,a_1)}) \\
\sigma_{a_1}(v) & \text{if } v \in V(T_{(a_1,y_2)}) 
\end{cases}
\]

is a \((T, x)\)-good 2-placement. If \(x\) is a bad vertex in \(T_{(y_2,a_1)}\), then \(y_2\) has only two neighbors distinct from \(x\) such that \(T_{(a_2,y_2)}\) is the vertex \(a_2\). Let \(\{b_1, ..., b_l\}, l \geq 1, \) be the neighbors of \(a_1\) distinct from \(y_2\), \(T'\) be the connected component containing \(x\) in \(T - \{a_1 b_i; \ i = 1, ..., l\}\) and let \(\sigma' = (x_1 x y_2 y_1)(a_1 a_2)\). Whenever \(T_{(b_i,a_1)}, 1 \leq i \leq m, \) is not a neighbor \(F\)-tree of \(a_1\), let \(\sigma_i\) be a \((T_{(b_i,a_1)}, b_i)\)-good 2-placement if \(b_i\) is not a bad vertex in \(T_{(b_i,a_1)}\), and if it is, then let \(\sigma_i\) be a \((T_{(b_i,a_1)}, w_i)\)-good 2-placement, where \(w_i\) is a neighbor of \(b_i\) in \(T_{(b_i,a_1)}\). If all the neighbor trees of \(a_1\) are non neighbor \(F\)-trees, then the 2-placement \(\sigma\) defined as follows:

\[
\sigma(v) = \begin{cases} 
\sigma'(v) & \text{if } v \in V(T') \\
\sigma_i(v) & \text{if } v \in V(T_{(b_i,a_1)}) \text{ for } i = 1, ..., l 
\end{cases}
\]

is a \((T, x)\)-good 2-placement. If \(a_1\) has at least two neighbor \(F\)-trees, then suppose that \(T_{(b_i,a_1)}\) is a neighbor \(F\)-tree of \(a_1\) for \(i = 1, ..., p\), where \(2 \leq p \leq l\). By Corollary 3.1, there exists a \((T'', x)\)-good 2-placement \(\sigma''\) such that \(\sigma''(v) = \sigma'(v)\) for every \(v\) of \(T'\), where \(T''\) is the connected component containing \(x\) in \(T - \{a_1 b_i; \ i = p + 1, ..., l\}\). If \(T'' = T\), then \(\sigma''\) is a \((T, x)\)-good 2-placement. Otherwise, a \((T, x)\)-good 2-placement \(\sigma\) is defined as follows:

\[
\sigma(v) = \begin{cases} 
\sigma''(v) & \text{if } v \in V(T'') \\
\sigma_i(v) & \text{if } v \in V(T_{(b_i,a_1)}) \text{ for } i = p + 1, ..., l 
\end{cases}
\]

Finally, if \(a_1\) has a unique neighbor \(F\)-tree, then suppose that \(T_{(b_1,a_1)}\) is that tree. Let \(T'\) be the connected component containing \(x\) in \(T - \{a_1 b_i; \ i = 2, ..., l\}\), then \(T'\) is isomorphic to one of the trees, \(T_E, T_F\) or \(T_G\), in Fig. 5, and so, by Lemma 3.11, there exists a \((T, x)\)-good 2-placement \(\sigma'\) such that \(\text{dist}(a_1, \sigma'(a_1)) \leq 2\). If \(l = 1\), then \(\sigma'\) is a \((T, x)\)-good 2-placement. Else, a \((T, x)\)-good 2-placement \(\sigma\) is defined as follows:

\[
\sigma(v) = \begin{cases} 
\sigma'(v) & \text{if } v \in V(T') \\
\sigma_i(v) & \text{if } v \in V(T_{(b_i,a_1)}) \text{ for } i = 2, ..., l 
\end{cases}
\]

Now, suppose that all the neighbor trees of \(y_2\) are neighbor \(F\)-trees, then \(d(y_2) \geq 3\) since \(T\) is not a path. Let \(T_0\) be the connected component containing \(x\) in \(T - \{y_2 a_i; \ i = 1, ..., m\}\) and \(\sigma_0 = (x y_2 y_1 x_1)\). Then, by Corollary 3.1, there exists a \((T, x)\)-good 2-placement since \(\text{dist}(y_2, \sigma_0(y_2)) = 2\) and all the neighbor trees of \(y_2\) are neighbor \(F\)-trees. Finally, if \(d(x) > 2\),
then each neighbor tree of \(x\) is a path of length one. Let \(N(x) = \{y_1, \ldots, y_r\}, r > 2\), and let \(T_{(y_i,x)} = y_i x_i\), for \(i = 1, \ldots, r\). If \(d(x) > 4\), then let \(T'\) be the connected component containing \(x\) in \(T - \{xy_i; i = 4, \ldots, r\}\). Since \(T'\) is a non star tree and \(x\) is not a bad vertex in \(T'\), then there exists a \((T, x)\)-good 2-placement. Since \(T_{(y_i,x)}\) are neighbor \(F\)-trees of \(x\) for \(i = 4, \ldots, r\), then, by Corollary 3.1, there exists a \((T', x)\)-good 2-placement. Otherwise, that is, \(d(x) < 5\), then \(\sigma = (x y_3 y_2 x_2 x_3 y_1 x_1)\) is a \((T, x)\)-good 2-placement if \(d(x) = 3\), and \(\sigma' = (x y_3 y_2 x_2 x_3 y_1 x_4 y_1 x_1)\) is a \((T, x)\)-good 2-placement if \(d(x) = 4\). □

**Proof of Corollary 1.4.**

Let \(T' = T - \{\alpha_1, \ldots, \alpha_{m_T}\}\), where \(\{\alpha_1, \ldots, \alpha_{m_T}\}\) is the maximal set of leaves that can be removed from \(T\) in such a way that the obtained tree is a non star one. Since \(T'\) is a non star tree, then there exists a \((T', x)\)-good 2-placement, say \(\sigma_x\), where \(x\) is any non bad vertex of \(T'\). We define a packing \(\sigma\) of \(T\) into \(T^5\) as follows:

\[
\sigma(v) = \begin{cases} 
\sigma_x(v) & \text{if } v \in V(T') \\
\alpha_i & \text{if } v = \alpha_i \text{ for } i = 1, \ldots, m_T
\end{cases}
\]

Label \(\alpha_i\) by \(i\) for \(i = 1, \ldots, m_T\) and label all the vertices in \(T'\) by \(m_T + 1\). Hence, we obtain an \((m_T + 1)\)-labeled packing of \(T\) into \(T^5\), and so \(w^5(T) \geq m_T + 1\). □

**References**


