Gap vertex-distinguishing edge colorings of graphs

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Abstract

In this paper, we study a new coloring parameter of a graph called the Gap vertex-distinguishing edge coloring. It is a proper edge-coloring of a graph $G$ which induces a vertex distinguishing labeling of $G$ such that the label of each vertex is given by the difference between the highest and the lowest colors of its adjacent edges. The minimum number of colors required for a Gap vertex-distinguishing edge coloring of $G$ is called the gap chromatic number of $G$ and denoted by $\text{gap}(G)$. In this paper, we study the gap chromatic number for a large set of graphs $G$ of order $n$ and we even prove that $\text{gap}(G) \in \{n-1, n, n+1\}$.

Key words: graph, proper edge coloring, Gap vertex-distinguishing edge coloring.

1 Introduction and definitions

All graphs considered in this paper are finite and undirected. For a graph $G$, we use $V(G)$, $E(G)$, $\triangle(G)$ and $\delta(G)$ to denote its vertex set, edge set, maximum degree and minimum degree, respectively. For any undefined terms, we refer the reader to [5].

A vertex labeling of a graph $G$ is said to be vertex-distinguishing if distinct vertices are assigned distinct labels. Let $k$ be a non-negative integer. A $k$-edge-coloring of $G$ is a mapping $f$ from $E(G)$ to $\{1, 2, ..., k\}$. We say that an edge coloring is proper if no two adjacent edges have the same color. Many researchers investigated the question of edge coloring inducing a vertex distinguishing labeling. This is often referred to as vertex-distinguishing edge colorings. In the literature, three main different functions have been proposed to label each vertex $v$ of $G$ according to the colors of the edges incident with $v$. A vertex labeling $l$ induced by an edge-coloring $f$ is said to be:

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(1) vertex-labeling by sum if \( l(v) = \sum_{e \ni v} f(e), \forall v \in V \) (see [6])
(2) vertex-labeling by sets if \( l(v) = \bigcup_{e \ni v} f(e), \forall v \in V \) (see [1,4])
(3) vertex-labeling by multiset if \( l(v) = \biguplus_{e \ni v} f(e), \forall v \in V \) (see [2,3])

The problem of vertex-distinguishing edge colorings offers many variations and there seems to be a great interest for some years. In this paper, we define a new parameter called \textit{Gap vertex-distinguishing edge colorings} which is defined below.

\textbf{Definition 1.1} Let \( G \) be a graph and \( f \) be a mapping from \( E(G) \) to the set \( \{1, 2, ..., k\} \). For each vertex \( v \) of \( G \), the label of \( v \) is defined to be

\[
l(v) = \begin{cases} f(e)_{e \ni v} & \text{if } d(v) = 1 \\ \max_{e \ni v} f(e) - \min_{e \ni v} f(e) & \text{otherwise} \end{cases}
\]

The mapping \( f \) is called a \textit{Gap vertex-distinguishing} if distinct vertices have distinct labels. Such a coloring is called a \textit{gap-}k-coloring.

The minimum positive integer \( k \) for which \( G \) admits a gap-\( k \)-coloring is called the \textit{gap chromatic number} of \( G \) and is denoted by \( \text{gap}(G) \).

\textbf{Proposition 1.1} Every graph \( G \) has a \textit{Gap vertex-distinguishing edge coloring} if and only if it has no components which is isomorphic to \( K_1 \) or \( K_2 \).

\textbf{Proof} Since no isolated vertex of a graph \( G \) is assigned a label in an edge coloring of \( G \). Furthermore, if \( G \) contains a component \( K_2 \), then the two vertices of \( K_2 \) are assigned the same label in any edge coloring of \( G \). Hence, when considering \textit{Gap vertex-distinguishing edge coloring} of a graph \( G \), we may assume that the order of every component of \( G \) is at least 3. Let \( G \) be a graph without isolated edges and isolated vertices such that \( E(G) = \{e_1, e_2, ..., e_m\} \), if we define an edge coloring \( f \) by \( f(e_i) = 2^{i-1} \) for \( 1 \leq i \leq m \), then a \textit{Gap vertex-distinguishing edge colorings} is produced.

\[ \square \]

It easy to see that the following lemma give a lower bound for the \textit{Gap chromatic number}.

\textbf{Lemma 1} A graph \( G \) with \( n \) vertices, without components isomorphic to \( K_1 \) or \( K_2 \) has \( \text{gap}(G) \geq n \) if (i) \( \delta(G) \geq 2 \) or (ii) any vertex of degree greater than 1 has at least two adjacent vertices of degree 1; otherwise \( \text{gap}(G) \geq n - 1 \).

To illustrate these concepts, consider the graph \( G \) shown in Figure 1(a). A 6-edge coloring \( f_1 \) of \( G \) is given in Figure 1(b) and a 5-edge coloring \( f_2 \) of \( G \) is given in Figure 1(c). For example, in the Figure 1(b), the vertex \( w \) is incident
with two edges colored 2 and one edge colored 3, then $l_1(w) = 1$, while the
vertex $z$ is incident with one edge colored 6, then $l_1(z) = 6$. The resulting
vertex labelings are distinct for the both figures. By lemma 1, $\text{gap}(G) \geq 5$, we
can immediately conclude that $\text{gap}(G) = 5$.

After a strong analysis of this problem, we raised the conjecture asserting that
there is no connected graph $G$ having $\text{gap}(G) > n+1$.

**Conjecture 2** For every connected graph $G$ of order $n \geq 3$, we have

$$\text{gap}(G) \in \{n - 1, n, n + 1\}$$

In the following sections, we prove this conjecture for a large set of graphs and
we even decide the exact value of $\text{gap}(G)$. The presentation of our results is
organised in the following way: the results of section 2 confirms our conjecture
for a large part of graphs with minimum degree at least 2. In the section 3, we
prove our conjecture for some classes of graphs with minimum degree 1, such
as: path, complete binary tree and all trees with at least two leaves at a distance
2. This classification of results according to $\delta(G)$ is due to the definition of
our parameter, especially the definition of labels of vertices of degree 1.

## 2 Graphs with $\delta(G) \geq 2$

The main result of this section is the following.

**Theorem 2.1** For every $k$-edge-connected graph $G$ of order $n$ with $k \geq 2$,

$$\text{gap}(G) = \begin{cases} n & \text{if } G \text{ is not a cycle of length } \equiv 2, 3(\text{mod } 4) \\ n + 1 & \text{otherwise} \end{cases}$$

The proof of the previous theorem consists of several results mentioned below.
Theorem 2.2 Let $C_n$ be a cycle of order $n$. Then

$$\text{gap}(C_n) = \begin{cases} n & \text{if } n \equiv 0, 1 \pmod{4} \\ n + 1 & \text{otherwise} \end{cases}$$

Proof Let $C_n = (v_1, v_2, \cdots, v_n, v_{n+1} = v_1)$. For each integer $i$ with $1 \leq i \leq n$, let $e_i = v_i v_{i+1}$. We consider two cases:

Case 1: $n \equiv 0, 1 \pmod{4}$. By Lemma 1, we have $\text{gap}(C_n) \geq n$, it then suffices to prove that $C_n$ admits a gap-$n$-coloring. Two subcases are considered:

Subcase 1.1: $n \equiv 0 \pmod{4}$. A mapping $f$ from $E(C_n)$ to $\{1, 2, \cdots, n\}$ is defined as follows (see Figure 2(a)):

$$f(e_i) = \begin{cases} n + 1 - i & \text{if } i \text{ odd} \\ 1 & \text{if } i \equiv 2 \pmod{4} \\ 2 & \text{if } i \equiv 0 \pmod{4} \end{cases}$$

This mapping induces the following Gap vertex labeling function:

$$l(v_i) = \begin{cases} n - i + 1 & \text{if } i \equiv 2 \pmod{4} \\ n - i & \text{if } i \equiv 0, 3 \pmod{4} \\ n - i - 1 & \text{if } i \equiv 1 \pmod{4} \end{cases}$$

Then it is easy to check that $l$ is a bijection from $V(C_n)$ to $\{0, 1, \cdots, n - 1\}$. Hence $\text{gap}(C_n) = n$.

Subcase 1.2: $n \equiv 1 \pmod{4}$. A mapping $f$ from $E(C_n)$ to $\{1, 2, \cdots, n\}$ is defined as follows (see Figure 2(b)):

$$f(e_i) = \begin{cases} i & \text{if } i \text{ odd} \\ n - 1 & \text{if } i \equiv 2 \pmod{4} \\ n & \text{if } i \equiv 0 \pmod{4} \end{cases}$$

This mapping induces the following Gap vertex labeling function:

$$l(v_i) = \begin{cases} n - i & \text{if } i \equiv 1, 2 \pmod{4} \\ n - i + 1 & \text{if } i \equiv 0 \pmod{4} \\ n - i - 1 & \text{if } i \equiv 3 \pmod{4} \end{cases}$$

Then it is easy to check that $l$ is a bijection from $V(C_n)$ to $\{0, 1, \cdots, n - 1\}$. Hence $\text{gap}(C_n) = n$.

Case 2: $n \equiv 2, 3 \pmod{4}$. We first prove that $\text{gap}(C_n) > n$. Let $f$ be any edge-coloring of $C_n$ which induces a Gap vertex-distinguishing $l$. We can note that:

$$\sum_{i=1}^{n} l(v_i) = |f(e_1) - f(e_n)| + \sum_{i=2}^{n} |f(e_i) - f(e_{i-1})| = \frac{n(n - 1)}{2}$$
In this formula, each term $f(e_i)$ appears two times with opposite (or same) signs, hence $\frac{n(n-1)}{2}$ is even. But this latter value is odd if $n \equiv 2, 3 (mod\ 4)$, which is a contradiction. Thus, $\text{gap}(C_n) \geq n + 1$. It then remains to show that $\text{gap}(C_n) \leq n + 1$, two subcases are considered according to whether $n \ mod\ 4 = 2$ or 3.

**Subcase 2.1: $n \equiv 3 (mod\ 4)$**. We know that $C_{n+1}$ admits a gap-$(n+1)$-coloring. Necessarily, $C_{n+1}$ must contain two successive edges of same color $j$ where $1 \leq j \leq n + 1$. By merging these two edges into a single edge colored by $j$, we obtain a gap-$(n+1)$-coloring of $C_n$ (see Figure 2(d)).

**Subcase 2.2: $n \equiv 2 (mod\ 4)$**. In this subcase, we define an edge coloring $f$ from $E(C_n)$ to $\{1, 2, \ldots, n, n + 1\}$ by (see Figure 2(c)): $f(e_n) = f(e_{n-1}) = 2$, $f(e_{n-2}) = 3$ and

$$
\begin{align*}
\text{For } 1 \leq i \leq n-3, \quad f(e_i) &= \begin{cases} 
    n + 2 - i & \text{if } i \text{ odd} \\
    1 & \text{if } i \equiv 2 (mod\ 4) \\
    2 & \text{if } i \equiv 0 (mod\ 4)
\end{cases}
\end{align*}
$$

This mapping induces the following Gap vertex distinguishing:

$$
\begin{align*}
\text{For } 1 \leq i \leq n-3, \quad l(v_i) &= \begin{cases} 
    n - i & \text{if } i \equiv 1 (mod\ 4) \\
    n + 2 - i & \text{if } i \equiv 2 (mod\ 4) \\
    n + 1 - i & \text{if } i \equiv 0, 3 (mod\ 4)
\end{cases}
\end{align*}
$$

Then it is easy to check that $l$ is a bijection from the vertex set of $E(C_n)$ to the set $\{0, 1, \ldots, n, n + 1\} \setminus \{3\}$. Hence $\text{gap}(C_n) = n + 1$.

![Figure 2](image-url)

**Figure 2.** A gap-$n$-coloring of $C_n$: (a) $n = 8$, (b) $n = 9$, (c) $n = 7$, (d) $n = 6$.

We now introduce a definition which plays a pervasive role in this section.
Definition 2.1 Let $G$ be a graph of order $n$ and let $f$ be an edge coloring of $G$. For every vertex $v$ of $G$, we define an interval vertex $I(v) = [\min_{e \ni v} f(e), \max_{e \ni v} f(e)]$. We say that $f$ is balanced if and only if: $I(v_1) \cap I(v_2) \cap \cdots \cap I(v_n) \neq \emptyset$.

The following proposition summarizes an important property of our coloring parameter.

Proposition 2.3 Let $G$ be a graph with $\delta(G) \geq 2$. If there exists a spanning subgraph $H$ of $G$ with $\delta(H) \geq 2$ and there exists a Gap vertex-distinguishing of $H$ induced by a balanced edge coloring $f$ with $k$ colors, then $\text{gap}(G) \leq k$.

Proof Under the stated hypothesis, the Gap vertex-distinguishing of $H$ is induced by a balanced edge coloring $f$ of $k$ colors. Therefore, there exists at least one integer $j$ where $1 \leq j \leq k$ such that $\forall v \in V$, we have $j \in I(v)$. By coloring the edges of $G \setminus H$ with the color $j$, we get a gap-$k$-coloring of $G$. Hence $\text{gap}(G) \leq k$.

We illustrate the interest of Theorem 2.3 by considering the following example: Let $G$ be a Hamiltonian graph of order $n \equiv 0(\text{mod} 4)$. In the proof of Theorem 2.2 (subcase 1.1), it is easy to check that the proposed edge coloring of $C_n$ is balanced. Indeed, $\forall v \in V$, we have $2 \in I(v)$. Hence, we can augment the cycle $C_n$ to $G$ by weighting the added edges with color 2 without affecting the gap chromatic value of $C_n$. Thus, for every Hamiltonian graph $G$ of order $n \equiv 0(\text{mod} 4)$, we have $\text{gap}(G) = n$.

The following proposition is useful for proving Theorem 2.1. Furthermore, it provides both a tool for proving other results.

Proposition 2.4 If $G = (V, E)$ is a 2-edge connected graph of order $n$, different from a cycle of length $\equiv 1, 2$ or $3(\text{mod} 4)$, then for all integer $a \geq 0$, there exists an $(a + n)$-edge-coloring $f$ which induces the following Gap vertex-distinguishing function

$$l : V \rightarrow \{a, a + 1, \cdots, a + n - 1\}$$

Proof The proof of this proposition is done by giving a polynomial-time algorithm. We first start with some definitions used in the following. For every subset $S$ of $V$, let $N_S$ denote the set of neighboring vertices of $S$, not included in $S$.

$$N_S = \{u \in V \setminus S : \exists v \in S \text{ for which } (v, u) \in E\}$$

For every two adjacent vertices $u$ and $v$ of $G$ such that $v \in S$ and $u \in N_S$, let $P(v, u)$ be the function which returns a path (or cycle) from $v$ to a vertex $w \in S$ that passes through $u$, such that the set of vertices between $v$ and $w$...
does not belong to $S$.
For every subgraph $R$ of $G$, let $g(R)$ be a function defined on the set $E(R)$ like this:
\[ g(R) = \min \{ f(e_i) : \forall e_i \in E(R), f(e_i) \neq 1, 2 \} \]
We denote by $Q$ the set of all graphs that are isomorphic to a cycle of order multiple of 4 or a two cycles having at least one vertex in common.

**Observation 2.5** Every 2-edge connected graph $G$, different from a cycle of length $\equiv 1, 2$ or $3(mod\ 4)$ contains at least one subgraph $H \in Q$.

It follows from the lemma hypothesis that if $G$ is different from a cycle of length multiple of 4, then $\Delta(G) \geq 3$. Hence, the subgraph $H$ can be always obtained from $G$.

The basic idea of our algorithm is to find a balanced edge-coloring $f$ of a 2-edge connected spanning subgraph $G' = (V', E')$ of $G$. Initially, both sets $V'$ and $E'$ are empty set. During the algorithm, the updating of $V'$ and $E'$ is done gradually through specific edge coloring procedure (which is explained in more detail below). When an edge of $G$ is colored by this procedure it is inserted to $E'$. A vertex $v \in V$ is inserted in $V'$ if and only if it is incident with at least two colored-edges ($e, s \in E$). Note that when a vertex $v$ is inserted in $V'$, we set the label $l(v)$ as $l(v) = |f(e) - f(s)|$ and the interval $I(v)$ at $[\min(f(e), f(s)), \max(f(e), f(s))]$. Such an edge coloring ensures that for every interval $I(v)$, we have $2 \in I(v)$.

In more details, the proposed algorithm begins by coloring the edges of a 2-edge connected subgraph $H$ of order $k$ in $G$ which induces a Gap vertex-distinguishing of $H$, such that the vertices of $H$ are labeled by distinct numbers ranging from $n + a - k$ to $n + a - 1$. We can easily establish this labeling structure for every subgraph $H$ of $G$ which is isomorphic to a member of $Q$.

Then, we have proposed four edge-coloring functions to color the set of edges which construct a cycle that has a single and only single vertex in $V'$ or a path between two vertices of $V'$. This last step is iterated until all vertices are labeled (i.e: $|V'| = |V|$).

In order to color the subgraph $H$, we need to define several edge-coloring functions. For a proper understanding of our algorithm, we are going to present the algorithm for a graph $G$ which contains at least one cycle of length multiple of 4. Otherwise, all other edge-coloring functions of $H$ are described in detail in the Appendix of this paper. The different steps of the algorithm are illustrated in the example of Figure 3 where $a = 12$.

**Algorithm 1**

**Input:** An integer $a$ and a 2-edge-connected graph $G = (V, E)$ of order $n$, different from a cycle of length $\equiv 1, 2$ or $3(mod\ 4)$.

**Output:** A balanced $(a+n)$-edge-coloring $f$ of $G$ which induces a Gap vertex-distinguishing function $l: V \rightarrow \{a, a + 1, \cdot \cdot \cdot , a + n - 1\}$.

**Begin of Algorithm**
Step1: $V' \leftarrow \emptyset, E' \leftarrow \emptyset$. Let an index $t = 2$.

Step2: Take any subgraph $H = R_1 \in Q$ of $G$.

2.1 If $(R_1$ is a cycle of length $k \equiv 0 (mod\ 4))$ Then

Let $H = (v_1, v_2, \cdots, v_k, v_{k+1} = v_1)$. For each integer $i$ with $1 \leq i \leq k$, let $e_i = v_i v_{i+1}$. A mapping $f$ from $E(R_1)$ to $\{1, 2, \cdots, a + n\}$ is defined as follows:

$$\text{For } 1 \leq i \leq k, f(e_i) = \begin{cases} n + a - i + 1 & \text{if } i \equiv 2 (mod\ 4) \\ 1 & \text{if } i \equiv 0 (mod\ 4) \\ 2 & \text{if } i \equiv 0 (mod\ 4) \end{cases}$$

This mapping induces the following vertex labeling of $R_1$:

$$\text{For } 1 \leq i \leq k, l(v_i) = \begin{cases} n + a - i + 1 & \text{if } i \equiv 2 (mod\ 4) \\ n + a - i & \text{if } i \equiv 0, 3 (mod\ 4) \\ n + a - i - 1 & \text{if } i \equiv 1 (mod\ 4) \end{cases}$$

Then it is easy to check that $l$ is a bijection from the vertex set of $R_1$ to the set $\{n + a - 1, n + a - 2, \cdots, n + a - k\}$.

Otherwise all other edge-coloring functions of $R_1$ are described in detail in the Appendix of this paper.

2.2 $V' \leftarrow V(R_1), E' \leftarrow E(R_1)$. And set $z = g(R_1)$

Step3: While ($V' \neq V$) do

Begin while

3.1 Take any two adjacent vertices $u$ and $v$ such that $v \in V'$ and $u \in N_{V'}$.

3.2 Let $R_t = P(v, u)$, we represent the obtained subgraph $R_t$ by the walk $(v_1, v_2, \cdots, v_{k-1}, v_k)$. For each integer $i$ with $1 \leq i \leq k - 1$, let $e_i = v_i v_{i+1}$. We now define a coloring $f$ of the edges of $R_t$. We consider four cases according to the value of $k \ mod\ 4$.

Case 1: $k \equiv 0 (mod\ 4)$. A mapping $f$ from $E(R_t)$ to $\{1, 2, \cdots, a + n\}$ is defined as follows: $f(e_{k-1}) = z - k + 2$ and

$$\text{For } 1 \leq i \leq k - 2, f(e_i) = \begin{cases} z - i & \text{if } i \text{ odd} \\ 1 & \text{if } i \equiv 0 (mod\ 4) \\ 2 & \text{if } i \equiv 2 (mod\ 4) \end{cases}$$

This mapping induces the following Gap vertex labeling of $R_t$: $l(v_{k-1}) = z - k$. And:

$$\text{For } 2 \leq i \leq k - 2, l(v_i) = \begin{cases} z - i - 1 & \text{if } i \equiv 1, 2 (mod\ 4) \\ z - i - 2 & \text{if } i \equiv 3 (mod\ 4) \\ z - i & \text{if } i \equiv 0 (mod\ 4) \end{cases}$$

Case 2: $k \equiv 2 (mod\ 4)$. A mapping $f$ from $E(R_t)$ to $\{1, 2, \cdots, a + n\}$ is
defined as follows:

For $1 \leq i \leq k - 1, f(e_i) = \begin{cases} z - i & \text{if } i \text{ even} \\ 1 & \text{if } i \equiv 3 \pmod{4} \\ 2 & \text{if } i \equiv 1 \pmod{4} \end{cases}$

This mapping induces the following Gap vertex labeling of $R_t$

For $2 \leq i \leq k - 1, l(v_i) = \begin{cases} z - i - 1 & \text{if } i \equiv 0, 1 \pmod{4} \\ z - i - 2 & \text{if } i \equiv 2 \pmod{4} \\ z - i & \text{if } i \equiv 3 \pmod{4} \end{cases}$

**Case 3:** $k \equiv 1 \pmod{4}$. A mapping $f$ from $E(R_t)$ to $\{1, 2, \cdots, a + n\}$ is defined as follows: $f(e_1) = z - 2$ and

For $2 \leq i \leq k - 1, f(e_i) = \begin{cases} z - i & \text{if } i \text{ odd} \\ 1 & \text{if } i \equiv 2 \pmod{4} \\ 2 & \text{if } i \equiv 0 \pmod{4} \end{cases}$

This mapping induces the following Gap vertex labeling of $R_t$: $l(v_2) = z - 3$, and

For $3 \leq i \leq k - 1, l(v_i) = \begin{cases} z - i - 1 & \text{if } i \equiv 0, 3 \pmod{4} \\ z - i - 2 & \text{if } i \equiv 1 \pmod{4} \\ z - i & \text{if } i \equiv 2 \pmod{4} \end{cases}$

**Case 4:** $k \equiv 3 \pmod{4}$. A mapping $f$ from $E(R_t)$ to $\{1, 2, \cdots, a + n\}$ is defined as follows: $f(e_{k-1}) = z - k + 2$ and

For $1 \leq i \leq k - 2, f(e_i) = \begin{cases} z - i & \text{if } i \text{ even} \\ 1 & \text{if } i \equiv 3 \pmod{4} \\ 2 & \text{if } i \equiv 1 \pmod{4} \end{cases}$

This mapping induces the following Gap vertex labeling of $R_t$: $l(v_{k-1}) = z - k$, and

For $2 \leq i \leq k - 2, l(v_i) = \begin{cases} z - i - 1 & \text{if } i \equiv 0, 1 \pmod{4} \\ z - i - 2 & \text{if } i \equiv 2 \pmod{4} \\ z - i & \text{if } i \equiv 3 \pmod{4} \end{cases}$

**Observation:** In the previous four cases, it is easy to check that $l$ is a bijection from the vertex set $V(R_t) - \{v_1, v_k\}$ to $\{z - 3, z - 4, \cdots, z - k\}$.

**2.3** $V' \leftarrow V' \cup V(R_t)$, $E' \leftarrow V' \cup E(R_t)$. Set $z = g(R_t)$ and $t = t + 1$.

**End while**
Step 4: For all edges $e \in E \setminus E'$, set $f(e) = 2$.

End of algorithm

Figure 3. Illustration of Algorithm 1: (a) A 2-edge connected graph $G$. (b) Coloring of $R_1$. (c),(d),(e),(f) illustrates the coloring of $R_2, R_3, R_4, R_5$, respectively. (g) A balanced gap-30-coloring of a spanning subgraph $G'$ of $G$. (h) A gap-30-coloring of $G$ which induces a Gap vertex-distinguishing function $l : V \rightarrow \{12, 13, \cdots, 29\}$.

We now present the proof of correctness of the above algorithm. We first show that this algorithm achieves its goal without blocking, i.e., both actions in Step 3 (3.1 and 3.2) satisfy the following assertions:

\[
\text{If } |V'| < |V| \text{ then } N_{V'} \neq \emptyset.
\]  \hfill (1)
For every vertex \( u \in N_{V'} \), there exists a path from \( u \) to a vertex \( v \in V' \) of order greater than 2.

By the connectivity hypothesis on \( G \), it is clear that the assertion (1) is valid. For a vertex \( u \in N_{V'} \) there exists, at last, an edge \((u, v) \in E \) such that \( v \in V' \). The 2-edge-connectivity hypothesis of \( G \) implies that every edge of \( G \) belongs to a cycle, then the two vertices \( u \) and \( v \) belong to the same cycle. Therefore, the assertion (2) also holds.

We now prove that our coloring algorithm gives a Gap vertex-distinguishing function \( l : V' \to \{a, a + 1, \cdots, a + n - 1\} \) of \( G' \) induced by a balanced edge coloring \( f \) with \( a + n \) colors. At the end of the loop of step 3, we obtain a bijection \( l \) from the set \( V' \) to the set \( \{a, a + 1, ..., a + n - 1\} \), i.e., for any two vertices \( u, v \) of \( V' \), we have \( l(u) \neq l(v) \). It then remains to show that \( f \) is a balanced edge-coloring and for every vertex \( v \) of \( V' \), we have \( \|v\| \) equal to \( \max_{e \in v} f(e) - \min_{e \in v} f(e) \) in \( G' \). By considering the degree in \( G' \) of each vertex \( v \), we have two cases:

**Case 1.** \( d(v) = 2 \): From the algorithm, it is clear that the label of vertex \( v \) of degree 2 which is incident with two edges \( e \) and \( s \) of \( E' \) is equal to \( |f(e) - f(s)| \).

**Case 2.** \( d(v) > 2 \): Let \( R(v) = \{R_d, R_{d+1}, R_{d+2} \cdots R_{d+p}\} \) denote the set of all subgraphs having a common vertex \( v \). From the algorithm, we can remark that: (see Figure 3(g))

- For any two subgraphs \( R_i \) and \( R_j \) of \( R(v) \), we have \( E(R_i) \cap E(R_j) = \emptyset \).
- \( v \) is incident with exactly two edges \( e_d \) and \( s_d \) of \( E(R_d) \). Let \( f(e_d) \geq f(s_d) \), then the label of \( v \) is fixed as \( l(v) = f(e_d) - f(s_d) \).
- For every subgraph \( R_i \) of \( R(v) \) where \( d + 1 \leq i \leq d + p \), \( v \) is incident with one or two edges of \( E(R_i) \).

Furthermore, according to the edge coloring \( f \), we can easily see that:

- For every vertex \( v \) of \( G' \), we have \( 2 \in I(v) \).
- \( 1 \leq f(s_d) \leq 2 \) and \( f(e_d) \geq g(R_d) \geq 2 \).
- For every subgraph \( R_i \) of \( R(v) \) where \( d + 1 \leq i \leq d + p \), we have \( \forall e \in E(R_i) \)
- For every edge \( e \) of \( E(R_i) \) where \( d + 1 \leq i \leq d + p \) with \( v \in e \), we have \( 2 \leq f(e) \leq g(R_d) \).

From these observations we can conclude the following:

- The edge-coloring \( f \) is balanced.
- For every vertex \( v \) of \( V' \), \( \max_{e \in v} f(e) = f(e_d) \) and \( \min_{e \in v} f(e) = f(s_d) \).

At the step 4 of the algorithm, we know that the obtained edge coloring \( f \) of \( G' \) is balanced. Hence, we can augment \( G' \) to \( G \) by coloring the added edges with color 2 without affecting the vertex labeling function \( l : V' \to \{a, a + 1, \cdots, a + n - 1\} \).

\[ \square \]
Now, we can state the proof of Theorem 2.1. To proceed, we introduce the following theorem.

**Theorem 2.6** For every 2-edge-connected graph $G$ of order $n$, different from a cycle of length $n \equiv 2$ or $3(\text{mod } 4)$, we have

$$\text{gap}(G) = n$$

**Proof** By Lemma 1, we have $\text{gap}(G) \geq n$. It then suffices to prove that $G$ admits a gap-$n$-coloring. We know by Theorem 2.2 that if $G$ is a cycle of length $n \equiv 0, 1(\text{mod } 4)$, then $\text{gap}(G) = n$. For otherwise, it is clear by Proposition 2.4 that if we set the integer parameter $a$ at 0, we obtain a gap-$n$-coloring of $G$ induced by a balanced edge coloring. Hence $\text{gap}(G) = n$.

\[ \square \]

Since for all integer $k > 2$, every $k$-edge connected graph $G$ has a 2-edge connected spanning subgraph $G'$ different from a cycle. From Theorem 2.5, $G'$ admits a gap-$n$-coloring induced by a balanced edge coloring, then we have the following corollary.

**Corollary 2.7** Let $G$ be a $k$-edge-connected graph ($k > 2$) of order $n$. Then

$$\text{gap}(G) = n$$

We can now conclude that the result of Theorem 2.1 is a direct consequence of Theorem 2.2, Theorem 2.5, and Corollary 2.6.

Here we generalize the previous results to a special case of disconnected graphs as follows.

**Theorem 2.8** If $G$ is a graph of order $n$ with components $G_1, G_2, \cdots, G_t$ such that each component of $G$ is a $k$-edge connected graph (with $k \geq 2$) different from a cycle of length $\equiv 1, 2, 3(\text{mod } 4)$, then

$$\text{gap}(G) = n$$

**Proof** Let $n_i$ be the order of $G_i$ ($1 \leq i \leq t$). The proof is essentially due to Theorem 2.4 and its algorithm. The idea is to provide a Gap vertex distinguishing edge colorings for each component of $G$ according to the parameter $a$ of Theorem 2.4 as follows.

By the application of Theorem 2.4 for each component $G_i$ of $G$, we can obtain the labeling function $l : V(G_i) \rightarrow \{a, a + 1, \cdots, a + n_i - 1\}$ induced by an edge coloring $f$ with $a + n_i$ colors such that $a = n - \sum_{j=1}^{i} n_j$. From this, it is easy to check that $l$ is a bijection from the vertex set of $G$ to the set $\{0, 1, 2, \cdots, n - 1\}$. Thus $\text{gap}(G) = n$.

\[ \square \]
3 Graphs with $\delta(G) = 1$

In this section we give the value of $\text{gap}(G)$ for some classes of graphs with $\delta(G) = 1$.

**Theorem 3.1** Let $P_n$ be a path of order $n$. Then

$$\text{gap}(P_n) = \begin{cases} 
  n - 1 & \text{if } n \equiv 0, 1 \pmod{4} \\
  n & \text{otherwise}
\end{cases}$$

**Proof** The proof of this theorem is similar to the one of Theorem 2.2. Let $P_n = v_1, v_2, \ldots, v_n$. For each integer $i$ with $1 \leq i \leq n - 1$, let $e_i = v_i v_{i+1}$. We consider two cases:

**Case 1**: $n \equiv 0, 1 \pmod{4}$. By Lemma 1, we have $\text{gap}(P_n) \geq n - 1$, it then suffices to prove that $P_n$ admits a gap-$(n - 1)$-coloring. Two subcases are considered:

**Subcase 1.1**: $n \equiv 0 \pmod{4}$. A mapping $f$ from $E(P_n)$ to $\{1, 2, \ldots, n - 1\}$ is defined as follows (see Figure 4(a)):

$$\text{For } 1 \leq i \leq n - 1, f(e_i) = \begin{cases} 
  i \cdot \frac{1}{2} & \text{if } i \text{ even} \\
  \frac{n-2}{2} & \text{if } i \equiv 3 \pmod{4} \\
  n - 1 & \text{if } i \equiv 1 \pmod{4}
\end{cases}$$

This mapping induces the following vertex labeling function: $l(v_n) = \frac{n-2}{2}$

And for $2 \leq i \leq n - 1$, $l(v_i) = \begin{cases} 
  \frac{n-i-2}{2} & \text{if } i \equiv 0 \pmod{4} \\
  n - 1 - \frac{i-1}{2} & \text{if } i \equiv 1 \pmod{4} \\
  n - 1 - \frac{i}{2} & \text{if } i \equiv 2 \pmod{4} \\
  \frac{n-i-1}{2} & \text{if } i \equiv 3 \pmod{4}
\end{cases}$

Then it is easy to check that $l$ is a bijection from $V(P_n)$ to $\{0, 1, \ldots, n - 1\}$. Hence $\text{gap}(P_n) = n - 1$.

**Subcase 1.2**: $n \equiv 1 \pmod{4}$. A mapping $f$ from $E(P_n)$ to $\{1, 2, \ldots, n - 1\}$ is defined as follows (see Figure 4(b)):

$$\text{For } 1 \leq i \leq n - 1, f(e_i) = \begin{cases} 
  \frac{i}{2} & \text{if } i \text{ even} \\
  \frac{n-1}{2} & \text{if } i \equiv 3 \pmod{4} \\
  n - 1 & \text{if } i \equiv 1 \pmod{4}
\end{cases}$$

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This mapping induces the following vertex labeling function:

\[
\text{And for } 2 \leq i \leq n - 1, l(v_i) = \begin{cases} 
\frac{n-1-i}{2} & \text{if } i \equiv 0 \pmod{4} \\
-1 - \frac{i-1}{2} & \text{if } i \equiv 1 \pmod{4} \\
-1 - \frac{i}{2} & \text{if } i \equiv 2 \pmod{4} \\
\frac{n-i}{2} & \text{if } i \equiv 3 \pmod{4}
\end{cases}
\]

Then it is easy to check that \(l\) is a bijection from \(V(P_n)\) to \(\{0, 1, \ldots, n - 1\}\). Hence \(\text{gap}(P_n) = n - 1\).

**Case 2:** \(n \equiv 2, 3 \pmod{4}\). We first prove that \(\text{gap}(P_n) > n - 1\). Let \(f\) be any edge-coloring of \(P_n\) which induces a Gap vertex-distinguishing \(l\). We can note that:

\[
\sum_{i=1}^{n} l(v_i) = f(e_1) + f(e_{n-1}) + \sum_{i=2}^{n-1} |f(e_i) - f(e_{i-1})| = \frac{n(n-1)}{2}
\]

In this formula, each term \(f(e_i)\) appears two times with opposite (or same) signs, hence \(\frac{n(n-1)}{2}\) is even. But this latter value is odd if \(n \equiv 2, 3 \pmod{4}\), which is a contradiction. Thus, \(\text{gap}(P_n) \geq n\). It then remains to show that \(\text{gap}(P_n) \leq n\), two subcases are considered according to whether \(n \bmod{4} = 2\) or 3.

**Subcase 2.1:** \(n \equiv 3 \pmod{4}\). We know that \(P_{n+1}\) admits a gap-\(n\)-coloring. Necessarily, \(P_{n+1}\) must contain two successive edges of same color \(j\) where \(1 \leq j \leq n\). By merging these two edges into a single edge colored by \(j\), we obtain a gap-\(n\)-coloring of \(P_n\) (see Figure 4(c)).

**Subcase 2.2:** \(n \equiv 2 \pmod{4}\) In this subcase, we define an edge coloring \(f\) from \(E(P_n)\) to \(\{1, 2, \ldots, n\}\) (see Figure 4(d)) by: \(f(e_{n-1}) = n - 1\) and

\[
\text{For } 2 \leq i \leq n - 2, f(e_i) = \begin{cases} 
\frac{i+1}{2} & \text{if } i \text{ even} \\
\frac{i}{2} & \text{if } i \equiv 3 \pmod{4} \\
n & \text{if } i \equiv 1 \pmod{4}
\end{cases}
\]

This mapping induces the following Gap vertex distinguishing: \(l(v_1) = n\), \(l(v_{n-1}) = \frac{n}{2} - 1\), \(l(v_n) = n - 1\)

\[
\text{And for } 2 \leq i \leq n - 2, l(v_i) = \begin{cases} 
\frac{n-i-1}{2} & \text{if } i \equiv 0 \pmod{4} \\
-1 + \frac{i+1}{2} & \text{if } i \equiv 1 \pmod{4} \\
-1 - \frac{i}{2} & \text{if } i \equiv 2 \pmod{4} \\
\frac{n-i-1}{2} & \text{if } i \equiv 3 \pmod{4}
\end{cases}
\]

Then it is easy to check that \(l\) is a bijection from the vertex set of \(E(P_n)\) to the set \(\{0, 1, \ldots, n\} \setminus \{\frac{n}{2}\}\). Hence \(\text{gap}(P_n) = n\).

\(\square\)
The complete binary tree of height $h > 0$ will be denoted by $BT_h$, note that $BT_h$ has exactly $2^{h+1} - 1$ vertices. The following theorem gives the gap chromatic number of $BT_h$.

**Theorem 3.2** For any complete binary tree $BT_h$ of order $n$ and height $h \geq 2$, we have

$$\text{gap}(BT_h) = n - 1$$

**Proof** By Theorem 3.1 $\text{gap}(BT_1) = \text{gap}(P_3) = 3$. Hence, we may restrict our attention to $h \geq 2$. By Lemma 1, we have $\text{gap}(BT_h) \geq n - 1$, it then suffices to prove that $BT_h$ admits a gap-$(n - 1)$-coloring. We define the level $l(u)$ of vertex $u$ of $BT_h$ as the number of edges along the unique path between it and the root. Similarly, the level of an edge $e = (u, v)$ of $BT_h$ is $l(e) = \max\{l(u), l(v)\}$. We represent the vertices and the edges of $BT_h$, level by level, left to right by the sequence $v_1, v_2, \ldots, v_n$, and $e_1, e_2, \ldots, e_{n-1}$, respectively (see Figure 5(a)). We now define a mapping $f$ from $E(BT_h)$ to $\{1, 2, \ldots, n - 1\}$ as follows:

For $1 \leq i \leq n - 1$, $f(e_i) = \begin{cases} 2h & \text{if } i \leq 2 \\ i + 2(h - l(e_i)) & \text{if } i \geq 3 \end{cases}$

This mapping induces the following Gap vertex labeling function: $l(v_i) = i - 1$ for $1 \leq i \leq n$. Then it is easy to check that $l$ is a bijection from $V(BT_h)$ to $\{0, 1, \ldots, n - 1\}$. Thus $\text{gap}(BT_h) = n - 1$. 

\[\square\]

**Theorem 3.3** Let $T = (V, E)$ be a tree of order $n$ which has two leaves $u$ and $v$ at a distance equal to 2. Then

$$\text{gap}(T) \leq n$$

**Proof** The proof of this theorem is done by giving a polynomial-time algorithm. We first start with some definitions used in the following: let $P = u, w, v$
be a path of \( T \) and let \( R \) be a subtree of \( T \) rooted in \( w \) and induced by the set \( V \setminus \{u, v\} \) (see Figure 6(a)). Let \( h \) be the depth of \( R \). For every level \( i \) of \( R \), let \( L_i \) denote the set of leaves at level \( i \). Let \( S \) be a subset of \( V(R) \), for every vertex \( x \) of \( V(R) \setminus S \), let \( P(x, S) \) be the function which returns a path from \( x \) to a vertex \( y \in S \), such that the set of vertices between \( x \) and \( y \) does not belong to \( S \). For every path \( P \) of \( T \), let \( g(P) \) be a function defined as follows:

\[
g(P) = \min\{l(v) : \forall v \in V(P)\}
\]

The different steps of Algorithm 2 are illustrated in the example of Figure 6.

**Algorithm 2**

**Input:** A tree \( T = (V, E) \) of order \( n \) with two leaves \( u \) and \( v \) at a distance equal to 2.

**Output:** A gap-\( n \)-coloring of \( T \).

**Begin of Algorithm**

Set a mapping \( f : E(R_1) \to \{1, n\} \) as follows: \( f(vw) = n, f(uw) = 1 \)

This mapping induces the following Gap vertex labeling of \( R_1 \). \( l(v) = n, l(w) = n - 1 \) and \( l(u) = 1 \).

Let a set \( S = \{w\} \), an integer \( z = n - 1 \) and an index \( t = 2 \).

**For** \( i = 1 \) to \( h \) **do**

**Begin For**

**For** every vertex \( x \) of \( L_i \) of the subtree \( R \) **do**

**Begin For**

Let \( R_t = P(x, S) \). We denote \( R_t \) by the sequence of vertices \( v_1 = x, v_2, \ldots, v_{k-1}, v_k \). For each integer \( i \) with \( 1 \leq i \leq k - 1 \), let \( e_i = v_iv_{i+1} \). Set a mapping \( f \) of the edges of \( R_t \) as follows:

\[
For \ 1 \leq i \leq k - 1, f(e_i) = \begin{cases} 
z - \frac{i+1}{2} & \text{if } i \text{ odd} \\
\frac{i}{2} & \text{otherwise}
\end{cases}
\]
This mapping induces the following Gap vertex labeling of $R_t$:

For $1 \leq i \leq k - 1$, $l(v_i) = z - i$

$$S \leftarrow S \cup V(R_t), \quad z \leftarrow g(R_t), \quad t \leftarrow t + 1$$

End for

End for

End of Algorithm

Now, we present the proof of correctness for the above algorithm. At the end of this algorithm, we obtain a bijection $l$ from $V$ to the set $\{1, 2, ..., n\}$. It then remains to show the property of our coloring parameter. By considering the degree of each vertex $v$ of $T$, we have three cases:
Case 1. \(d(v) = 1\): From the algorithm, it is clear that \(l(v) = f(e)\) for \(v \in V\).

Case 2. \(d(v) = 2\): From the algorithm, it is clear that the label of vertex \(v\) of degree 2 which is incident with two edges \(e\) and \(s\) of \(E\) equal to \(|f(e) - f(s)|\).

Case 3. \(d(v) > 2\): Let \(R(v) = \{R_d, R_{d+1}, R_{d+2}, \ldots, R_{d+p}\}\) denote the set of all paths having a common vertex \(v\). We represent the distance between two vertices \(x, y \in V\) by \(\text{dist}(x, y)\). From the algorithm, we can remark that:

- Every path \(R_i\) of \(R(v)\) contains a leaf \(l_i\) of \(T\) which is an endpoint of \(R(v)\).
- For any two paths \(R_i\) and \(R_j\) of \(R(v)\), \(E(R_i) \cap E(R_j) = \emptyset\).
- \(v\) is incident with exactly two edges \(e_d\) and \(s_d\) of \(E(R_d)\). Let \(f(e_d) \geq f(s_d)\), then the label of \(v\) is fixed as \(l(v) = f(e_d) - f(s_d)\).
- For every path \(R_i\) of \(R(v)\) where \(d + 1 \leq i \leq d + p\), \(v\) is incident with one edge \(e_i\) of \(E(R_i)\).

Furthermore, according to the edge-coloring \(f\), we can see that:

- \(f(s_d) = \left\lfloor \frac{\text{dist}(v, l_i)}{2} \right\rfloor\)
- For every path \(R_i\) of \(R(v)\) where \(d + 1 \leq i \leq d + p\), we consider two cases for the value of \(f(e_i)\) according to the distance between \(v\) and \(l_i\):
  - \(\text{dist}(v, l_i)\) is even. We have \(f(e_i) = \frac{\text{dist}(v, l_i)}{2}\). Hence \(f(s_d) \leq f(e_i) \leq g(R_d) \leq f(e_d)\)
  - \(\text{dist}(v, l_i)\) is odd. We have \(f(e_i) = g(R_i) + \frac{\text{dist}(v, l_i) - 1}{2}\). Hence \(f(s_d) \leq f(e_i) \leq g(R_d) \leq f(e_d)\)

From these observations we can conclude that for every vertex \(v\) of \(V\), \(f(e_d) = \max_{e \ni v} f(e)\) and \(f(s_d) = \min_{e \ni v} f(e)\). Hence, \(T\) admits a gap-\(n\)-coloring.

\[
\square
\]

4 Conjecture

According to the results obtained from the above sections, we propose some conjectures.

**Conjecture 3** For every graph \(G\) of order \(n\) with minimum degree \(\delta(G) \geq 2\), we have

\[
gap(G) = \begin{cases} 
  n + 1 & \text{if } G \text{ is a cycle of length } \equiv 2, 3(\text{mod } 4) \\
  n & \text{otherwise}
\end{cases}
\]
Conjecture 4 For every tree $T$ of order $n \geq 3$, we have

$$\text{gap}(T) = \begin{cases} n & \text{if condition (ii) of Lemma 1 is fulfilled} \\ n - 1 & \text{otherwise} \end{cases}$$

5 Appendix: Step 2 of Algorithm 1

In Step 2 of Algorithm 1, it remains to handle the case when $R_1$ is a subgraph of $G$ which is isomorphic to a two cycles having at least one vertex in common. Let us recall that the goal is to define an edge coloring of $R_1$ (of order $k$) which induces the following gap vertex-distinguishing function $l : V(R_1) \to \{n + a - 1, n + a - 2, \ldots, n + a - k\}$.

It is clear that the edge set of $R_1$ can be partitioned into two sets generating graphs: cycle $C$ and a path (cycle) $P$ such that the endpoints of $P$ belong to $C$. Let $C = (v_1, v_2, \ldots, v_q, v_{q+1} = v_1)$. For each integer $i$ with $1 \leq i \leq q$, let $e_i = v_iv_{i+1}$. Let $P = (u_1, u_2, \ldots, u_t)$. For each integer $i$ with $1 \leq i \leq t - 1$, let $s_i = u_iu_{i+1}$, we assume that $v_q = u_1$. In the following, we illustrate the coloring of $R_1$, several cases are considered according to the value of $q$ and $t$.

Case 1 $q \equiv 2 \pmod{4}$ and $t \equiv 0, 1, 2 \pmod{4}$. A mapping $f$ of $E(C) \setminus \{e_q, e_{q-1}\}$ is defined as follows:

$$\text{For } 1 \leq i \leq q - 2, f(e_i) = \begin{cases} a + n - i + 1 & \text{if } i \text{ odd} \\ 1 & \text{if } i \equiv 2 \pmod{4} \\ 2 & \text{if } i \equiv 0 \pmod{4} \end{cases}$$

Then, the following cases define the coloring of the remaining edges of $R_1$.

Subcase 1.1 $t \equiv 0 \pmod{4}$. $f(e_q) = 2$, $f(e_{q-1}) = n + a - q + 2$ and

$$\text{For } 1 \leq i \leq t - 1, f(s_i) = \begin{cases} g(C) - i & \text{if } i \text{ even} \\ 1 & \text{if } i \equiv 1 \pmod{4} \\ 2 & \text{if } i \equiv 2 \pmod{4} \end{cases}$$

Subcase 1.2 $t \equiv 1 \pmod{4}$. $f(e_q) = 2$, $f(e_{q-1}) = a + n - q + 1$, $f(s_1) = g(C) + 1$ and

$$\text{For } 2 \leq i \leq t - 1, f(s_i) = \begin{cases} g(C) - i + 1 & \text{if } i \text{ odd} \\ 1 & \text{if } i \equiv 2 \pmod{4} \\ 2 & \text{if } i \equiv 0 \pmod{4} \end{cases}$$

Subcase 1.3 $t \equiv 2 \pmod{4}$. We use the same coloring scheme as in Subcase 1.2 except that $f(s_{t-1}) = g(C) - t + 3$.

Case 2 $q \equiv 2 \pmod{4}$ and $t \equiv 3 \pmod{4}$. A mapping $f$ of $E(R_1)$ is defined as
follows:

For $2 \leq i \leq q - 2$, $f(e_i) = \begin{cases} 
  a + n - i - 2 & \text{if } i \text{ odd} \\
  2 & \text{if } i \equiv 2(\text{mod } 4) \\
  1 & \text{if } i \equiv 0(\text{mod } 4)
\end{cases}$

$f(e_1) = a + n$, $f(e_{q-1}) = a + n - 4$, $f(e_q) = 1$, $f(s_1) = a + n - 2$ and

For $2 \leq i \leq t - 1$, $f(s_i) = \begin{cases} 
  g(C) - i + 1 & \text{if } i \text{ odd} \\
  1 & \text{if } i \equiv 0(\text{mod } 4) \\
  2 & \text{if } i \equiv 2(\text{mod } 4)
\end{cases}$

Case 3 $q \equiv 3(\text{mod } 4)$ and $t \equiv 0(\text{mod } 4)$. A mapping $f$ of $E(R_1)$ is defined as follows:

For $1 \leq i \leq q - 1$, $f(e_i) = \begin{cases} 
  a + n - i + 1 & \text{if } i \text{ odd} \\
  2 & \text{if } i \equiv 2(\text{mod } 4) \\
  1 & \text{if } i \equiv 0(\text{mod } 4)
\end{cases}$

$f(e_q) = 1$, $f(s_{t-1}) = g(C) - t + 1$ and

For $1 \leq i \leq t - 2$, $f(s_i) = \begin{cases} 
  g(C) - i - 1 & \text{if } i \text{ odd} \\
  1 & \text{if } i \equiv 0(\text{mod } 4) \\
  2 & \text{if } i \equiv 2(\text{mod } 4)
\end{cases}$

Case 4 $q \equiv 3(\text{mod } 4)$ and $t \equiv 1(\text{mod } 4)$. A mapping $f$ of $E(R_1)$ is defined as follows:

For $1 \leq i \leq q - 1$, $f(e_i) = \begin{cases} 
  a + n - i + 2 & \text{if } i \text{ even} \\
  2 & \text{if } i \equiv 1(\text{mod } 4) \\
  1 & \text{if } i \equiv 3(\text{mod } 4)
\end{cases}$

$f(e_q) = a + n - t + 2$, $f(s_{t-1}) = g(C) - t + 1$ and

For $1 \leq i \leq t - 2$, $f(s_i) = \begin{cases} 
  g(C) - i - 1 & \text{if } i \text{ even} \\
  1 & \text{if } i \equiv 1(\text{mod } 4) \\
  2 & \text{if } i \equiv 3(\text{mod } 4)
\end{cases}$

Case 5 $q \equiv 3(\text{mod } 4)$ and $t \equiv 2, 3(\text{mod } 4)$. A mapping $f$ of $E(C) \setminus \{e_q\}$ is
defined as follows:

For \( 1 \leq i \leq q - 1 \),
\[
f(e_i) = \begin{cases} 
    a + n - i + 2 & \text{if } i \text{ even} \\
    1 & \text{if } i \equiv 1 \pmod{4} \\
    2 & \text{if } i \equiv 3 \pmod{4} 
\end{cases}
\]

Then, the following cases define the coloring of the remaining edges of \( R_1 \).

**Subcase 5.1** \( t \equiv 2 \pmod{4} \). \( f(e_q) = n + a - q + 1 \) and

For \( 1 \leq i \leq t - 1 \),
\[
f(s_i) = \begin{cases} 
    g(C) - i + 1 & \text{if } i \text{ even} \\
    1 & \text{if } i \equiv 3 \pmod{4} \\
    2 & \text{if } i \equiv 1 \pmod{4} 
\end{cases}
\]

**Subcase 5.2** \( t \equiv 3 \pmod{4} \). \( f(e_q) = n + a - q + 2 \), \( f(e_{t-1}) = g(C) - t + 2 \) and

For \( 1 \leq i \leq t - 2 \),
\[
f(s_i) = \begin{cases} 
    g(C) - i & \text{if } i \text{ even} \\
    1 & \text{if } i \equiv 3 \pmod{4} \\
    2 & \text{if } i \equiv 1 \pmod{4} 
\end{cases}
\]

**Case 6** \( q \equiv 1 \pmod{4} \) and \( t \equiv 1 \pmod{4} \). A mapping \( f \) of \( E(R_1) \) is defined as follows:

For \( 1 \leq i \leq q - 1 \),
\[
f(e_i) = \begin{cases} 
    a + n - i + 1 & \text{if } i \text{ odd} \\
    1 & \text{if } i \equiv 2 \pmod{4} \\
    2 & \text{if } i \equiv 0 \pmod{4} 
\end{cases}
\]

\( f(e_q) = 2 \), \( f(s_1) = g(C) - 1 \) and

For \( 2 \leq i \leq t - 1 \),
\[
f(s_i) = \begin{cases} 
    g(C) - i - 1 & \text{if } i \text{ even} \\
    1 & \text{if } i \equiv 2 \pmod{4} \\
    2 & \text{if } i \equiv 0 \pmod{4} 
\end{cases}
\]

**References**


