On the Diversity-Multiplexing Tradeoff for Multi-antenna Multi-relay Channels

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Abstract—In this paper we analyze the performance of multiple relay channels when multiple antennas are deployed only at relays. Specifically, we investigate the simple repetition-coded decode-and-forward protocol and apply two antenna combining techniques at relays, namely maximum ratio combining (MRC) on receive and transmit beamforming (TB). We assume that the total number of antennas at all relays is fixed to $N$. With a reasonable power constraint at the relays, we show that the antenna combining techniques can exploit the full spatial diversity of the relay channels and can achieve the same diversity multiplexing tradeoff as achieved by more complex space-time distributed coding techniques, such as those proposed by Laneman and Wornell (2003).

I. INTRODUCTION

It is widely believed that ad hoc networking [1] or multi-hop cellular networks [2] are important new concepts for future generation wireless systems, where either mobile or fixed nodes (often referred to as relays) are used to help forward the information to the desired user. One advantage of these structures is that it is possible to unite multiple relays in the network as a “virtual antenna array” to forward the information cooperatively, while appropriate combining at the destination realizes diversity gain. The diversity achieved in this way is often named as user cooperation diversity or cooperative diversity [4], as it mimics the performance advantages of multiple-input multiple-output (MIMO) systems [3] by exploiting the spatial diversity of the relay channels. The performance limits of distributed space-time codes, which can exploit cooperative diversity, are discussed in [4]–[7] for single-antenna relay networks. However, the design and implementation of practical codes that approach these limits is difficult and a challenging open area of research.

In this paper we exploit the spatial diversity of the relay channels in a way different from the space-time codes-based approach. We apply two kinds of antenna combining techniques at the relay, namely maximum ratio combining (MRC) [8] for reception and transmit beamforming (TB) [9] for transmission. Those techniques were often used in point-to-point single-input multiple-output (SIMO) or multiple-input single-output (MISO) wireless links, where either the transmitter or receiver is equipped with multiple antennas. It has been shown that MRC (TB) is able to achieve the information theoretic upper bound and optimal diversity multiplexing tradeoff of SIMO (MISO) systems [10]. In a relay context, we move the multiple antennas to the relays, while the source and the destination are only equipped with a single antenna. Our investigation is based on the repetition-coded decode-and-forward transmission [4], where each relay simply fully decodes the source message, re-encodes it with the same codebook as the source and forwards it to the destination. This method avoids any form of space-time coding or network coding and is easy to implement in practice. We analyze the performance of this system based on a slow fading scenario. More specifically, we examine the outage probability and the diversity multiplexing tradeoff of the network.

Note that one different assumption between our approach and the space-time coded approach is that we allow multiple antennas to be deployed at the relays. However, it will be shown later that the diversity gain that can be achieved by our approach is the same as that of the space-time coded approach proposed by Laneman and Wornell in [5], as long as the total number of antennas at all relays is fixed, regardless of the number of antennas at each relay and the number of relays. Thus, the same diversity gain can be achieved even when each relay is deployed with a single antenna. Therefore the application of our scheme is quite general.

The rest of this paper is organized as follows. In Section II, the basic system model and assumptions are introduced. Section III introduces the antenna combining techniques. The outage analysis is presented in section IV. Finally, conclusions are drawn in Section V.

II. SYSTEM MODEL

We consider a two hop network model with one source, one destination and $K$ relays. For simplicity we ignore the direct link between source and destination. The extension of all the results to include the direct link is straightforward. We assume that the source and destination are deployed with single antennas, while relay $k$ is deployed with $m_k$ antennas; the total number of antennas at all relays is fixed to $N$. This
can be expressed as
\[ \sum_{k=1}^{K} m_k = N. \]  
(1)

We restrict our discussion to the case where the channels are slow, frequency-flat fading. The data transmission is over two time slots using two hops. In the first transmission time slot, the source broadcasts the signal to all the relay terminals. The input/output relation for the source to the \( k \)th relay is given by
\[ r_k = \sqrt{\eta} h_k s + n_k, \]
(2)
where \( r_k \) is the \( m_k \times 1 \) receive signal vector, and \( \eta \) denotes the transmit power at the source. The scalar \( s \) is the unit mean power transmit signal and \( n_k \) is the \( m_k \times 1 \) complex circular additive white Gaussian noise vector at relay \( k \) with identity covariance matrix \( I_{m_k} \). The vector \( h_k \) is the \( m_k \times 1 \) channel transfer matrix from source to the \( k \)th relay. The entries of \( h_k \) are identically independent distributed (i.i.d) complex Gaussian random variables with zero mean and unit variance. In the second hop, each relay processes its received signals and re-transmits them to the destination. The signal received at the destination can be written as:
\[ y = \sum_{i=1}^{K} g_i d_k + n_d, \]
(3)
where the vector \( g_k \) is the channel matrix from \( k \)th relay to the destination, of which each entry is an i.i.d. complex Gaussian random variable with unit variance. The scalar \( n_d \) is the complex additive white Gaussian noise at the destination with unit variance. The vector \( d_k \) is the transmit signal vector at relay \( k \), which should meet the total transmit power constraint:
\[ E \left[ \| d_k \|^2_F \right] \leq \eta m_k / N, \]
(4)
where \( \| \cdot \|^2_F \) denotes the Frobenius norm. This power constraint means that the power is allocated at each relay in proportion to its number of antennas. For presentation simplicity we assume here that the total power at all relays is fixed to be \( \eta \), i.e. the same as at the source. However, all the conclusions in the paper also hold when the total power at all relays is fixed to an arbitrary constant. We assume a coherent relay channel configuration context where the \( k \)th relay can obtain full knowledge of both the backward channel vector \( h_k \) and the forward channel vector \( g_k \). Note that the forward channel knowledge can be obtained if the relay-destination link operates in a Time-Division-Duplex (TDD) mode. For fair comparison, we also assume that for each channel realization, all the backward and forward channel coefficients for all \( N \) antennas remain the same regardless of the number of relays \( K \). Fig. 1 shows the system model.

**III. ANTENNA DIVERSITY TECHNIQUES IN RELAY CHANNELS**

In this section we apply MRC and TB techniques to the system model described in section II. We assume that each relay performs MRC of the received signals, by multiplying the received signal vector by the vector \( h_k^H / \| h_k \|_F \). The signal at the output of the relay receiver is given by
\[ \tilde{r}_k = \sqrt{\eta} \sum_{i=1}^{m_k} \frac{h_{i,k}^* n_{i,k}}{\sqrt{\sum_{i=1}^{m_k} |h_{i,k}|^2}} \]
(5)
where \( h_{i,k} \) denotes the \( i \)th antenna at relay \( k \), and \( n_{i,k} \) denotes the noise factor for \( i \)th receiver input branch. The SNR at the output of the receiver can be written as:
\[ \rho_k^{m_k} = \eta \sum_{i=1}^{m_k} |h_{i,k}|^2. \]
(6)

After the relays decode the signals, each relay then performs TB of the decoded waveform. If we denote the transmitted signals as \( t_k \) with unit variance, the transmitted signal vector \( d_k \) for relay \( k \) can be written as
\[ d_k = \sqrt{\frac{\eta m_k}{N} \frac{g_k^H}{\| g_k \|_F}} t_k. \]
(7)
The destination receiver simply detects the combined signals from all \( K \) relays. If the signals are correctly decoded at all the relays (i.e. \( t_k = s \)), the output signal at the destination can be written as:
\[ y = s \sum_{k=1}^{K} \sqrt{\frac{\eta m_k}{N} \sum_{i=1}^{m_k} |g_{i,k}|^2} + n_d = \sum_{k=1}^{K} \tilde{g}_k + n_d \]
(8)
It can be seen from (8) that by applying antenna diversity schemes at the relays, the networks can be decomposed to \( K \) diversity channels, each with channel gain \( \tilde{g}_k \). The output SNR
at the destination receiver can therefore be written as:
\[ \rho_d^{m_k} = \left( \sum_{k=1}^{K} \frac{\eta m_k}{N} \sum_{i=1}^{m_k} |g_{i,k}|^2 \right)^2. \] (9)

When all the relays are deployed with a single antenna, there is no traditional maximum ratio combining gain at the relays and the destination. However, the destination still observes a set of equal gain combined [12] amplitude signals from all relays.1 Since we assume that the backward and forward channel coefficients for each antenna are kept the same for different values of \( K \) and \( m_i \), the output SNR at the destination can be rewritten as
\[ \rho_d^N = \frac{\eta}{N} \sum_{k=1}^{K} \sum_{i=1}^{m_i} |g_{i,k}|^2. \] (10)

When all the antennas are deployed in one relay (i.e. \( K = 1 \) and \( m_1 = N \)), full diversity gain is achieved among all the \( N \) antennas at the relay and also at the destination. The SNR for this case can be rewritten as
\[ \rho_d^1 = \sum_{i=1}^{m_1} |g_{i,1}|^2. \]

IV. OUTAGE ANALYSIS

When the channel fading is slow, i.e. codewords span less than one channel block, the Shannon capacity for the Rayleigh fading channel is zero. Therefore a certain outage probability must be allowed for communicating at any finite data rate. The outage probability can be defined as
\[ P_{out} \triangleq P[C < R]. \] (12)

This further allow us to exploit and investigate the diversity-multiplexing tradeoff of the systems, which is defined as follows [13]:

**Definition 1:** (Diversity-Multiplexing Tradeoff) Consider a family of codes \( C_\eta \) operating at SNR \( \eta \) and having rates \( R \) bits per channel use. The multiplexing gain and diversity order are defined as\(^2\)
\[ r \triangleq \lim_{\eta \to \infty} \frac{R}{\log_2 \eta}, \quad d \triangleq - \lim_{\eta \to \infty} \frac{\log_2 P_{out}(R)}{\log_2 \eta}. \] (13)

We first study a simple protocol, in which all the relays participate in the decoding and forwarding process. We refer to this protocol as multi-cast decoding. An outage occurs whenever any relay or the destination fails to decode the signals. Before starting the outage analysis, we firstly introduce a lemma on the bounds of the value of \( \rho_d^{m_k} \), i.e. the output SNR at the destination given that the signal is correctly decoded at all relays. This lemma has been shown in our earlier work [14]:

**Lemma 1:** For any \( m_k \), \( \rho_d^1 \leq \rho_d^{m_k} \leq \rho_d^N \).

We omit the proof, which can be found in Appendix of [14]. This Lemma is important throughout the analysis in the paper, as it implies that the increased “equal gain combining” gain at the destination can not compensate for the loss of maximum ratio combining gain at the relay and the destination when the number of relays \( K \) is increased and the numbers of antennas at each relay are reduced. Based on Lemma 1, we now begin our outage analysis with the following lemma:

**Lemma 2:** Conditioned on all the relays correctly decoding the messages, the outage probability for the relay channels is bounded by:
\[ \frac{1}{N!} \left( \frac{N (2^{2R} - 1)}{\eta} \right)^N \geq P_{out} \geq \frac{1}{N!} \left( \frac{2^{2R} - 1}{\eta} \right)^N \] (14)

**Proof:** See Appendix I.

**Lemma 2** indicates that the full diversity of \( N \) can be achieved regardless of the number of relays \( K \), provided that the signals are correctly decoded at the relays. However, the diversity of the network might decrease if certain detection error happens at the relays. This is especially true for the multi-cast decoding protocol, for which we have the following theorem:

**Theorem 1:** For large \( \eta \), the outage probability for the multi-cast decoding is bounded by
\[ N \left( \frac{2^{2R} - 1}{\eta} \right)^N \geq P_{out} \geq \frac{2}{N!} \left( \frac{2^{2R} - 1}{\eta} \right)^N \] (15)

with equality to the right-hand side if \( K = 1 \), to the left-hand side if \( K = N \).

**Proof:** See Appendix II.

This theorem implies that for multi-cast decoding, having more relays and less antennas per relay actually loses diversity. Since requiring all the relays to fully decode the source information limits the performance of the decode and forward to that of the poorest source to relay link. Specifically, it can be seen that for \( K = N \) no diversity gain is offered by relaying i.e. the SNR exponent is \(-1\), as no diversity gain can be obtained from the source to relay links in this case. However, for \( K = 1 \) the full diversity of \( N \) can be achieved, as the diversity gain for the source to relay link is also \( N \). In terms of the diversity multiplexing multiplexing tradeoff, we have the following theorem:

**Theorem 2:** The diversity-multiplexing tradeoff curve for the multi-cast decoding scheme is bounded by
\[ 1 - 2r \leq d \leq N (1 - 2r), \quad 0 \leq r \leq 0.5 \] (16)

with equality to right-hand side if \( K = 1 \), to left-hand side if \( K = N \).

**Proof:** For large \( \eta \), replace \( R \) with \( r \log_2 \eta \) in (15), the proof is straightforward.

It can be seen from Theorem 3 that when \( K = N \), the diversity-multiplexing tradeoff for multi-cast decoding is strictly worse than that for direct transmission, which is \( d = 1 - r \) [13]. When \( K = 1 \), however, the diversity-multiplexing tradeoff is the same as the space-time distributed coding schemes proposed in [5]. In fact, we can combine the antenna diversity schemes with a protocol similar to the one proposed by [5], which exploit further the diversity of source to relay channels by selecting the qualified relays.

1Unlike [12], the equal gain combining for relay channels is applied at the transmitter instead of the receiver.
2We assume that the block length of the code is large enough, so that the detection error is arbitrarily small and the main error event is due to outage
that meet the transmission rate $R$, to improve the network performance when $K > 1$. Specifically, the protocol for the antenna diversity schemes is proposed as follows:

**Protocol 1: (Selection Decoding)** Select $\hat{K}$ relays with a total number of antennas $\hat{N}$, denoted as $\mathbb{R}(\hat{N}, \hat{K})$, that could successfully decode the source message at a transmission rate $R$, to decode and forward the messages.

We can obtain the outage probability for the selection decoding by the following theorem:

**Theorem 3:** For large $\eta$, the outage probability for the selection decoding scheme is bounded by:

$$
\left( \frac{2^R - 1}{\eta} \right)^N \sum_{N=1}^{N} \left( \frac{\hat{N}}{\hat{N}} \right) \frac{1}{\hat{N}!} \hat{N} \geq P_{out}^{\text{m},k}
$$

while the upper bound is met when the selected $\hat{N}$ antennas are all within one relay.

**Proof:** See appendix III.

It can be seen from Theorem 4 that for selection decoding full diversity can always be achieved regardless the number of relays $K$. This is clearly an advantage over the multi-antenna decoding scheme. Replacing $R$ with $r \log_2 \eta$ in (17), we can directly obtain the diversity-multiplexing tradeoff for selection decoding:

**Theorem 4:** The diversity-multiplexing tradeoff curve for selection decoding is

$$
d = N \left(1 - 2r \right), 0 \leq r \leq 0.5,
$$

which is the same as that for the space-time distributed coding protocol proposed in [5].

We claim that compared with distributed space-time coding, the messages for antenna combining techniques are simply repetition coded. Therefore it is much easier to implement than space-time coding in practice, provided that each relay antenna can obtain its forward (relay to destination) CSI. Fig. 2 shows the diversity multiplexing tradeoff for different protocols discussed in the paper, when $N = 5$.

V. CONCLUSIONS

From the above analysis, we can draw several conclusions regarding the antenna combining techniques introduced in the paper: (a) provided the messages are successfully decoded at the relays, having less relays will offer better performance due to increased combining (power) gain at the destination, though the full diversity $N$ of the network can be achieved regardless the number of antennas; (b) if all the relays participate in the decoding and forwarding process, the network performance will degrade as the number of relays increases, as the performance is always restricted to the worst source to relay link. In this sense, deploying all the antennas at single relay is the optimal choice; (c) however, full diversity can be achieved if we apply the relay selection schemes to choose the potential relays. More specifically, the diversity-multiplexing tradeoff achieved by the antenna combining techniques is the same as that achieved by more complicated space-time distributed coding schemes. In this scenario, deploying more antennas at fewer relays is still a better choice due to improved combining (power) gain.

The analysis in the paper also implies that given a certain amount of available antennas in the network, the wired cooperation (i.e. all the antennas belong to one terminal) outperforms the wireless cooperation (i.e. each antenna belongs to different terminals). We further note that the recently proposed fixed relay concept [2] in mesh networks allows the possibility to deploy large number of antennas at the relay. This provide a good application for the antenna combining techniques discussed in the paper.

**APPENDIX I**

**PROOF OF Lemma 2**

Based on Lemma 1, it is clear that

$$
P_{out}^{1} \geq P_{out}^{m,k} \geq P_{out}^{N},
$$

where $P_{out}^{1}$ denotes the outage probability for $N$ relay case and $P_{out}^{N}$ for 1 relay case, given that the signals are correctly decoded at all the relays. Note that

$$
\rho_{d}^{1} \geq \frac{\eta}{N} \sum_{k=1}^{K} \sum_{i=1}^{m_{i}} |g_{i,k}|^2 = \frac{\rho_{d}^{N}}{N},
$$

inequality (19) can be extended as:

$$
P_{out}^{m,k} \left[ \frac{1}{2} \log_2 \left( 1 + \frac{\rho_{d}^{N}}{N} \right) < R \right] \geq P_{out}^{m,k} \left[ \frac{1}{2} \log_2 \left( 1 + \frac{\rho_{d}^{N}}{N} \right) < R \right]
$$

21
After some modification, (21) can be rewritten as:

\[
P \left[ \sum_{k=1}^{K} \sum_{i=1}^{m_k} |g_{i,k}|^2 \leq \frac{N (2^{2R} - 1)}{\eta} \right] \geq P_{\text{out}}^{m_k}
\]

\[
P \left[ \sum_{k=1}^{K} \sum_{i=1}^{m_k} |g_{i,k}|^2 \leq \frac{2^{2R} - 1}{\eta} \right]. \tag{22}
\]

Since \( \sum_{k=1}^{K} \sum_{i=1}^{m_k} |g_{i,k}|^2 \) is chi-square distributed with dimension \( 2N \), for small \( \varepsilon \) it is easy to show that

\[
P \left( \sum_{k=1}^{K} \sum_{i=1}^{m_k} |g_{i,k}|^2 \leq \varepsilon \right) \approx \frac{1}{N!} \varepsilon^N. \tag{23}
\]

Put (23) to (22) finishes the proof of Lemma 2.

**APPENDIX II**

**PROOF OF THEOREM 1**

If we denote \( C_r^{k,m_k} = 0.5 \log_2 (1 + P_{\text{out}}^{m_k}) \) as the Shannon capacity from source to relay \( k \) channel for each channel realization. The outage probability is given by:

\[
P_{\text{out}} = P \left[ \min \left( C_r^{k,m_k} < R \right) \right] + P \left[ \min \left( C_r^{k,m_k} > R \right) \right] P_{\text{out}}^{m_k} = 1 - \prod_{k=1}^{K} \left( 1 - P \left[ \sum_{i=1}^{m_k} |h_{i,k}|^2 < \frac{2^{2R} - 1}{\eta} \right] \right)
\]

\[
+ P_{\text{out}}^{m_k} \prod_{k=1}^{K} \left( 1 - P \left[ \sum_{i=1}^{m_k} |h_{i,k}|^2 < \frac{2^{2R} - 1}{\eta} \right] \right)
\]

\[
\eta^{-\infty} \approx 1 - \prod_{k=1}^{K} \left( 1 - \frac{1}{m_k} \left( \frac{2^{2R} - 1}{\eta} \right)^{m_k} \right) + P_{\text{out}}^{m_k}. \tag{24}
\]

where \( P_{\text{out}}^{m_k} \) is bounded by (14). For large \( \eta \), retaining only the term containing the lowest exponent of \( 1/\eta \) in the first term. (24) can be further modified as

\[
P_{\text{out}} \approx K \sum_{k=1}^{K} \left( \frac{2^{2R} - 1}{\eta} \right)^{m_k} + P_{\text{out}}^{m_k}. \tag{25}
\]

Observing that \( \frac{1}{a^n} < \frac{1}{b^n} \) when \( a > b \), \( P_{\text{out}} \) is maximized when \( m_k = 1, K = N \). Therefore for large \( \eta \)

\[
P_{\text{out}} \leq N \left( \frac{2^{2R} - 1}{\eta} \right), \tag{26}
\]

where \( P_{\text{out}}^{m_k} \) is omitted due to its higher exponent. \( P_{\text{out}} \) is minimized when \( m_k = 1, K = N \) and \( P_{\text{out}}^{m_k} = P_{\text{out}}^{N_k} \). We obtain the lower bound

\[
P_{\text{out}} \geq \frac{2}{N!} \left( \frac{2^{2R} - 1}{\eta} \right)^N \tag{27}
\]

and thus complete the proof.

**APPENDIX III**

**PROOF OF THEOREM 2**

Since \( \Re \left( \tilde{N}, \tilde{K} \right) \) is a random set, we utilize the total probability law and write

\[
P_{\text{out}} = \sum_{\Re \left( \tilde{N}, \tilde{K} \right)} P \left[ \Re \left( \tilde{N}, \tilde{K} \right) \right] P_{\text{out}}^{m_k | \Re \left( \tilde{N}, \tilde{K} \right)}, \tag{28}
\]

where \( P_{\text{out}}^{m_k | \Re \left( \tilde{N}, \tilde{K} \right)} \) denotes the outage probability conditioned on \( \Re \left( \tilde{N}, \tilde{K} \right) \) is chosen, and can be bounded by (14) by replacing \( N \) with \( \tilde{N} \). The probability for any relay to be chosen can be expressed as:

\[
P \left[ r \in \Re \left( \tilde{N}, \tilde{K} \right) \right] = P \left[ \sum_{i=1}^{m_k} |h_{i,k}|^2 \leq \frac{2^{2R} - 1}{\eta} \right]
\]

\[
= 1 - P \left[ \sum_{i=1}^{m_k} |h_{i,k}|^2 \leq \frac{2^{2R} - 1}{\eta} \right]. \tag{29}
\]

Therefore any \( \Re \left( \tilde{N}, \tilde{K} \right) \) exists with a probability that can be written as:

\[
P \left[ \Re \left( \tilde{N}, \tilde{K} \right) \right] = \prod_{r \in \Re \left( \tilde{N}, \tilde{K} \right)} \left( 1 - P \left[ \sum_{i=1}^{m_k} |h_{i,k}|^2 \leq \frac{2^{2R} - 1}{\eta} \right] \right)
\]

\[
\times \prod_{r \in \Re \left( N - \tilde{N}, K - \tilde{K} \right)} P \left[ \sum_{i=1}^{m_k} |h_{i,k}|^2 \leq \frac{2^{2R} - 1}{\eta} \right] \tag{30}
\]

Based on (23), at high SNR, \( P \left[ \Re \left( \tilde{N}, \tilde{K} \right) \right] \) can be approximated as

\[
P \left[ \Re \left( \tilde{N}, \tilde{K} \right) \right] \approx \left( \frac{2^{2R} - 1}{\eta} \right)^{N - \tilde{N}} \prod_{r \in \Re \left( N - \tilde{N}, K - \tilde{K} \right)} \frac{1}{m_k!}, \tag{31}
\]

which can be bounded by:

\[
\frac{1}{(N - \tilde{N})!} \left( \frac{2^{2R} - 1}{\eta} \right)^{N - \tilde{N}} \leq P \left[ \Re \left( \tilde{N}, \tilde{K} \right) \right] \leq \left( \frac{2^{2R} - 1}{\eta} \right)^{N - \tilde{N}}. \tag{32}
\]

Note that the bounds are independent of \( K \). Putting (32) and (14) into (28), we obtain the bounds (17) and thus complete the proof.

**REFERENCES**


