The Finite Element Approximation of Hamilton-Jacobi-Bellman Equations

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Abstract—This paper deals with the finite element approximation of Hamilton-Jacobi-Bellman equations. We establish a convergence and a quasi-optimal $L^\infty$-error estimate, involving a weakly coupled system of quasi-variational inequalities for the solution of which an iterative scheme of monotone kind is introduced and analyzed. © 2001 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

This paper is concerned with the finite element approximation of the Dirichlet problem for the Hamilton-Jacobi-Bellman equation (HJB)

$$\max_{1 \leq i \leq M} (A^i u - f^i) = 0, \quad \text{in } \Omega,$$

$$u = 0, \quad \text{on } \partial \Omega,$$

(1.1)

where $\Omega$ is a bounded open domain of $\mathbb{R}^N$, $N \geq 1$, with boundary $\partial\Omega$ sufficiently smooth, the $f^i$ are given smooth functions, and the $A^i$ are second-order, uniformly elliptic operators.

HJB equations are encountered in several applications; for example, in stochastic control the solution of (1.1) characterizes the infimum of the cost function associated to an optimally controlled stochastic switching process without costs for switching (cf., e.g., [1,2]).

A great deal of work has been done in the 1980s for the qualitative analysis of problem (1.1) (cf., e.g., [3-5]). These qualitative studies were then followed by few works in relation to numerical methods. More precisely, finite difference approximations of positive type have been achieved.

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A finite element analysis has also been studied in [8], where a convergence result was proved but no error estimate was given.

Concerning the numerical solution arising from the finite difference approximation, several iterative methods of both sequential and parallel types have been introduced and analyzed (cf. [6,9–11]). See also [12] for round-off errors behavior related to the computation of such a solution.

In this paper, our goal is to show that problem (1.1) can be properly approximated by a finite element method which turns out to be quasi-optimally accurate in $L^\infty$. The approximation is carried out by first approximating the HJB equation by a weakly coupled system of quasi-variational inequalities for the solution of which a monotone iterative scheme of Bensoussan-Lions type [1] is developed and its geometric convergence is proved. Similarly, using the standard finite element method and a discrete maximum principle, the resulting discrete scheme is studied and shown to be monotone and geometrically convergent.

An $L^\infty$-error estimate is then established combining the geometric convergence of both the continuous and discrete iterative schemes with known uniform error estimates for elliptic variational inequalities.

It should be mentioned that the purpose of the monotone iterations analyzed in this paper is to present a basic computational scheme rather than present the best rate of convergence of the iterations. A future paper will be devoted to the computation of the finite element solution of the HJB-equation (1.1), where efficient numerical monotone algorithms will be treated.

The paper is organized as follows. In Section 2, we approximate the HJB equation by a weakly coupled system of quasi-variational inequalities, introduce a monotone iterative scheme, and prove its geometric convergence. In Section 3, we consider the discrete HJB equation, discretize the iterative scheme by the standard finite element method, and achieve a similar study to that of the continuous problem. In Section 4, we prove a fundamental lemma and give the main results. Finally, in Section 5, we make some comments on the approach and the results presented in this paper.

## 2. THE CONTINUOUS PROBLEM

In this section, we shall first recall some results related to elliptic variational inequalities that are necessary in proving some useful qualitative properties. We start by introducing some notations and assumptions.

### 2.1. Notations, Assumptions

We are given functions

$$a^i_{jk}(x), b^i_k(x), a^0_i(x) \in C^2(\Omega), \quad x \in \Omega, \quad 1 \leq k, \quad j \leq N, \quad 1 \leq i \leq M$$

such that

$$\sum_{1 \leq j, k \leq N} a^i_{jk}(x) \xi_j \xi_k \geq \alpha |\xi|^2, \quad (x \in \Omega, \quad \xi \in R^N, \quad \alpha > 0),$$

$$a^i_{jk} = a^i_{kj}, \quad a^0_i(x) \geq c_0 > 0, \quad (x \in \Omega, \quad c_0 > 0).$$

We define the second-order, uniformly elliptic operators of the form

$$A^i = \sum_{1 \leq j, k \leq N} a^i_{jk}(x) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{k=1}^N b^i_k(x) \frac{\partial}{\partial x_k} + a^0_i(x)$$

and the bilinear forms associated with $A^i$: for $u, v \in H^1_0(\Omega)$

$$a^i(u, v) = \int_{\Omega} \left( \sum_{1 \leq j, k \leq N} a^i_{jk}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_k} + \sum_{k=1}^N a^i_k(x) \frac{\partial u}{\partial x_k} v + a^0_i(x)u \right) \, dx.$$
We assume that \( a^i(\cdot, \cdot) \) are coercive on \( H^1_0(\Omega) \), i.e., there exists \( \delta > 0 \) such that
\[
a^i(v, v) \geq \delta \|v\|^2_{H^1_0(\Omega)}, \quad \forall v \in H^1_0(\Omega).
\]
Let
\[
(f, \cdot) \text{ be the scalar product in } L^2(\Omega).
\]
We are also given functions \( f^i \) such that
\[
f^i \in C^2(\Omega) \quad \text{and} \quad f^i \geq 0, \quad \forall i = 1, 2, \ldots, M.
\]
We shall also need the following norm:
\[
\forall W = (w^1, \ldots, w^M) \in \prod_{i=1}^M L^\infty(\Omega),
\]
\[
\|W\|_\infty = \max_{1 \leq i \leq M} \|w^i\|_{L^\infty(\Omega)},
\]
where \( \| \cdot \|_{L^\infty(\Omega)} \) denotes the classic \( L^\infty \) norm.

### 2.2. Elliptic Variational Inequalities

Let \( f \) be a function in \( L^\infty(\Omega) \) and \( \psi \) an obstacle in \( W^{2,\infty}(\Omega) \) such that \( \psi \geq 0 \) on \( \partial \Omega \).

Let also \( A \) be an elliptic operator and \( a(\cdot, \cdot) \) its associated coercive bilinear form of the same forms as those defined in (2.4) and (2.5), respectively. We consider the following elliptic variational inequality (VI): find \( \zeta \in H^1_0(\Omega) \) such that
\[
a(\zeta, v - \zeta) \geq \langle f, v - \zeta \rangle, \quad \forall v \in H^1_0(\Omega),
\]
\[
\zeta \leq \psi, \quad v \leq \psi.
\]
Thanks to [13,14], the VI (2.11) has one and only one solution. Moreover, \( \zeta \in W^{2,p}(\Omega), 1 \leq p < \infty \) and satisfies
\[
\|\zeta\|_{W^{2,p}(\Omega)} \leq C(\|f\|_{\infty} + \|A\psi\|_{\infty}).
\]

**Definition 1.** \( z \in H^1_0(\Omega) \) is said to be a subsolution for the VI (2.11) if
\[
a(z, v) \leq \langle f, v \rangle, \quad \forall v \in H^1_0(\Omega), \quad v \geq 0,
\]
\[
z \leq \psi.
\]

Let \( X \) be the set of such subsolutions.

**Theorem 1.** (See [14].) The solution of VI (2.11) is the maximum element of \( X \).

Now, consider the mapping
\[
\sigma : L^\infty(\Omega) \rightarrow L^\infty(\Omega),
\]
\[
\psi \rightarrow \sigma(\psi) = \zeta,
\]
where \( \zeta \) is the solution of VI (2.11).

**Proposition 1.** The mapping \( \sigma \) is increasing, concave, and Lipschitz continuous with respect to \( \psi \).

**Proof.**

1. \( \sigma \) is increasing. Let \( \psi \) and \( \tilde{\psi} \) in \( L^\infty(\Omega) \) such that \( \psi \leq \tilde{\psi} \). Then \( \sigma(\psi) \leq \sigma(\tilde{\psi}) \). So \( \sigma(\psi) \) is a subsolution for the VI with obstacle \( \tilde{\psi} \). Then, using Theorem 1, we get the desired result.

2. \( \sigma \) is concave. Let \( \psi \) and \( \tilde{\psi} \) in \( L^\infty(\Omega) \) and \( \theta \in [0; 1] \) and set \( \sigma_\theta = \sigma(\theta \psi + (1 - \theta)\tilde{\psi}) \).

Since \( \sigma(\psi) \leq \psi \) and \( \sigma(\tilde{\psi}) \leq \tilde{\psi} \), it follows that \( \theta \sigma(\psi) + (1 - \theta)\sigma(\tilde{\psi}) \) is a subsolution for the VI with obstacle \( \theta \psi + (1 - \theta)\tilde{\psi} \). Then using Theorem 1, we get the concavity of \( \sigma \).

3. \( \sigma \) is Lipschitz continuous. Let \( \psi \) and \( \tilde{\psi} \) in \( L^\infty(\Omega) \) and \( \zeta = \sigma(\psi); \tilde{\zeta} = \sigma(\tilde{\psi}) \). Set \( \Phi = \|\psi - \tilde{\psi}\|_{\infty} \).

Then \( \zeta - \Phi \leq \psi - \Phi \leq \tilde{\zeta} \). So, \( \zeta - \Phi \) is a subsolution for the VI with obstacle \( \psi \). It follows that \( \zeta - \Phi \leq \tilde{\zeta} \), i.e., \( \zeta \leq \tilde{\zeta} + \Phi \).

Now, interchanging the roles of \( \psi \) and \( \tilde{\psi} \), we also get \( \tilde{\zeta} \leq \zeta + \Phi \). This completes the proof. \( \blacksquare \)
2.3. Approximation of the HJB Equation by a System of Quasi-Variational Inequalities

From [3,4], we know that (1.1) can be approximated by the following weakly coupled system of quasi-variational inequalities (QVIs): find $U = (u^1, \ldots, u^M) \in (H^1_0(\Omega))^M$ such that

\begin{align}
& a^i (u^i, v - u^i) \geq (f^i, v - u^i), \quad \forall v \in H^1_0(\Omega), \\
& u^i \leq k + u^{i+1}, \quad v \leq k + u^{i+1}, \quad i = 1, \ldots, M, \\
& \text{with } u^{M+1} = u^1,
\end{align}

(2.15)

where $k$ is a strictly positive constant.

Naturally, the structure of problem (2.15) is analogous to that of the classical "obstacle problem" where the obstacle function is replaced by an implicit one, depending upon the solution. The terminology quasi-variational inequality being chosen is a result of this remark.

The theorem to follow gives the uniform convergence of the solution of system (2.15) to the solution of the HJB equation (1.1).

**Theorem 2.** (See [3,4].) Let notations and assumptions (2.1), (2.8) hold. Then there exists a unique solution $U = (u^1, \ldots, u^M)$ with $u^i \in W^{2,p}(\Omega)$, $1 \leq p < \infty$. Moreover,

$$
(u^1, \ldots, u^M) \to (u, \ldots, u) \text{ uniformly in } \overline{\Omega} \text{ as } k \to 0 \text{ and } u \in W^{2,\infty}(\Omega).
$$

Next, we shall adapt an approach developed for the quasi-variational inequality of impulse control problems, to characterize the solution of system (2.15) as a fixed point of an increasing mapping (see [1]). We shall also prove that such a fixed-point mapping generates monotone sequences which converge geometrically to the unique solution of this system.

2.4. A Fixed-Point Mapping Associated with the System of QVIs (2.15)

Let $H^+_\infty = \prod_{i=1}^M L^\infty_+(\Omega)$, where $L^\infty_+(\Omega)$ denotes the positive cone of $L^\infty(\Omega)$. We define the mapping

$$
T : H^+_\infty \to H^+_\infty
$$

(2.16)

such that $\forall i = 1, \ldots, M$, $\zeta^i$ is the solution of the following VI:

\begin{align}
& a^i (\zeta^i, v - \zeta^i) \geq (f^i, v - \zeta^i), \quad v \in H^1_0(\Omega), \\
& \zeta^i \leq k + w^{i+1}, \quad v \leq k + w^{i+1}, \\
& \text{with } w^{M+1} = w^1.
\end{align}

(2.17)

Denoting by

$$
\zeta^i - \sigma (k + w^{i+1}),
$$

(2.18)

we clearly have

$$
TW = [\sigma (k + w^2), \sigma (k + w^3), \ldots, \sigma (k + w^4), \ldots, \sigma (k + w^1)].
$$

(2.19)

2.5. Some Properties of the Mapping $T$

Let us first consider $\tilde{U}^0 = (\tilde{u}^{1,0}, \ldots, \tilde{u}^{M,0})$ solution to the following variational equation:

$$
a^i (\tilde{u}^{i,0}, v) = (f^i, v), \quad \forall v \in H^1_0(\Omega), \quad 1 \leq i \leq M.
$$

(2.20)

Thanks to (2.1)–(2.8), there exists a unique positive solution to problem (2.20). Moreover, $\tilde{u}^{i,0} \in W^{2,\infty}(\Omega)$. 


PROPOSITION 2. Under the preceding notations and assumptions, the mapping $T$ satisfies

$$TV \leq TW, \quad \text{whenever } V \leq W,$$

$$TW \geq 0, \quad \forall W \in \mathbb{H}^+,$$

$$TW \leq \tilde{U}^0, \quad \forall W \in \mathbb{H}^+. \quad (2.21)$$

Proof.

1. $TV \leq TW, \; \forall V \leq W$. Let $V = (v^1, \ldots, v^M), \; W = (w^1, \ldots, w^M)$ in $\mathbb{H}^+$ such that $v^i \leq w^i, \forall i = 1, \ldots, M$. Then since $\sigma$ is increasing, it follows that $\sigma(k + v^i) \leq \sigma(k + w^i)$. Thus, $TV \leq TW$.

2. $TW \geq 0, \; \forall W \in \mathbb{H}^+$. This follows directly from the fact that $f^i \geq 0$ and classic comparison results in elliptic variational inequalities.

3. $TW \leq \tilde{U}^0, \; \forall W \in \mathbb{H}^+$. For $\varphi \in H^1(\Omega)$ we denote by

$$\varphi^+ = \max(\varphi, 0), \quad \varphi^- = \max(-\varphi, 0).$$

The fact that both of the solutions $\zeta^i$ of (2.17) and $\hat{u}^{i,0}$ of (2.20) belong to $H^1_0(\Omega)$, we readily have

$$\zeta^i - (\zeta^i - \hat{u}^{i,0})^+ \in H^1_0(\Omega).$$

Moreover, as $(\zeta^i - \hat{u}^{i,0})^+ \geq 0$, it follows that

$$\zeta^i - (\zeta^i - \hat{u}^{i,0})^+ \leq \zeta^i \leq k + w^{i+1}.$$

Therefore, we can take $\nu = \zeta^i - (\zeta^i - \hat{u}^{i,0})^+$ as a trial function in (2.17). This gives

$$a^i \left( \zeta^i, (\zeta^i - \hat{u}^{i,0})^+ \right) \geq \left( f^i, (\zeta^i - \hat{u}^{i,0})^+ \right).$$

Also, for $\nu = (\zeta^i - \hat{u}^{i,0})^+$ equation (2.20) becomes

$$a \left( \hat{u}^{i,0}, (\zeta^i - \hat{u}^{i,0})^+ \right) = \left( f^i, (\zeta^i - \hat{u}^{i,0})^+ \right),$$

so, by addition, we obtain

$$-a^i \left( (\zeta^i - \hat{u}^{i,0})^+, (\zeta^i - \hat{u}^{i,0})^+ \right) \geq 0,$$

which, by (2.6), yields

$$\zeta^i = \hat{u}^{i,0}, \quad \forall i = 1, 2, \ldots, M,$$

i.e.,

$$TW \leq \tilde{U}^0.$$  \hfill (2.23)

PROPOSITION 3. The mapping $T$ is concave on $\mathbb{H}^+$.

Proof. Let $\theta \in [0, 1]$. Then from (2.19) we have

$$T(\theta V + (1 - \theta)W) = \left( \sigma(k + \theta v^2 + (1 - \theta)w^2), \ldots, \sigma(k + \theta v^1 + (1 - \theta)w^1) \right),$$

$$\sigma(k + \theta v^i + (1 - \theta)w^i), \ldots, \sigma(k + \theta v^1 + (1 - \theta)w^1).$$

Then using the concaveness of $\sigma$ (see Proposition 1), we get

$$T(\theta V + (1 - \theta)W) \geq \theta \left( \sigma(k + v^2), \ldots, \sigma(k + v^1) \right) + (1 - \theta) \left( \sigma(k + v^2), \ldots, \sigma(k + v^1) \right) \geq \theta TV + (1 - \theta)TW$$

which proves the concaveness of $T$.  \hfill (2.23)
PROPOSITION 4. \( T \) is Lipschitz continuous on \( \mathbb{H}^+ \), i.e.,

\[
\|TV - TW\|_\infty \leq \|V - W\|_\infty, \quad \forall V, W \in \mathbb{H}^+. \tag{2.24}
\]

PROOF. Let \( V = (v_1, \ldots, v_M) \); \( W = (w_1, \ldots, w_M) \) in \( \mathbb{H}^+ \). Then from (2.19), we have

\[
TV = \left[ \sigma (k + v^1), \ldots, \sigma (k + v^i), \ldots, \sigma (k + v^M) \right],
\]
\[
TW = \left[ \sigma (k + w^1), \ldots, \sigma (k + w^i), \ldots, \sigma (k + w^M) \right],
\]

and by (2.10)

\[
\|TV - TW\|_\infty = \max_{1 \leq i \leq M} \| (TV)^i - (TW)^i \|_{L^\infty(\Omega)},
\]

where \((TV)^i\) and \((TW)^i\) denote the \(i\)th components of the vectors \( V \) and \( W \), respectively.

Now, since the mapping \( \sigma \) is Lipschitz continuous (see Proposition 1), it follows that

\[
\|\sigma (k + v^i) - \sigma (k + w^i)\|_{L^\infty(\Omega)} \leq \|(k + v^i) - (k + w^i)\|_{L^\infty(\Omega)} \\
\]

Thus,

\[
\|TV - TW\|_\infty \leq \max_{1 \leq i \leq M} \|v^i - w^i\|_{L^\infty(\Omega)} = \|V - W\|_\infty.
\]

This completes the proof. \( \square \)

2.6. A Continuous Iterative Scheme

An iterative scheme for the solution of system of QVIIs (2.15) is given as follows.

Starting from \( \bar{U}^0 \), solution of (2.20), (respectively, \( \bar{U}^0 = 0 \)) we define

\[
\bar{U}^{n+1} = T\bar{U}^n, \quad n = 0, 1, \ldots, \tag{2.25}
\]

respectively,

\[
\bar{U}^{n+1} = T\bar{U}^n, \quad n = 0, 1, \ldots. \tag{2.26}
\]

NOTATION 1. We denote by \( K \) the positive cone of \( \mathbb{H}^+ \), i.e.,

\[
K = \left\{ W \in \mathbb{H}^+ \text{ such that } 0 \leq W \leq \bar{U}^0 \right\}. \tag{2.27}
\]

LEMMA 1. Let \( 0 < \lambda < \inf (k/\|\bar{U}^0\|_\infty, 1) \). Then we have

\[
T(0) \geq \lambda \bar{U}^0. \tag{2.28}
\]

PROOF. From (2.26), \( T(0) = \bar{U}^1 = (\bar{u}^{i,1}, \ldots, \bar{u}^{i,M}) \), where \( \bar{u}^{i,1}, 1 \leq i \leq M \) is the solution of the following variational inequality:

\[
a^i (\bar{u}^{i,1}, v - \bar{u}^{i,1}) \geq (f^i, v - \bar{u}^{i,1}), \quad \forall v \in H^0_0(\Omega), \quad v \leq k,
\]

\[
\bar{u}^{i,1} \leq k. \tag{2.29}
\]

Then it is clear that

\[
v = (\bar{u}^{i,1} - \lambda \bar{u}^{0}) - \bar{u}^{i,1}
\]
can be taken as a trial function in the VI (2.29). So taking
\[ v = -\left(\hat{u}^{i,1} - \lambda \hat{u}^{i,0}\right)^- \]
as a trial function in (2.20) and using the fact that \( f^i \geq 0 \), we get by addition
\[ a^i \left( \hat{u}^{i,1} - \lambda \hat{u}^{i,0} \right) \geq (f^i - \lambda f^i, (\hat{u}^{i,1} - \lambda \hat{u}^{i,0})^-) \geq (1 - \lambda) f^i, (\hat{u}^{i,1} - \lambda \hat{u}^{i,0})^- \geq 0. \]
Thus, by (2.6)
\[ (\hat{u}^{i,1} - \lambda \hat{u}^{i,0})^- = 0, \]
i.e.,
\[ \hat{u}^{i,1} \geq \lambda \hat{u}^{i,0}, \quad \forall i = 1, 2, \ldots, M, \]
which completes the proof.

**Proposition 5.** Let \( \gamma \in [0; 1] \) such that
\[ W - \hat{W} \leq \gamma W, \quad \forall W, \hat{W} \in K. \tag{2.30} \]
Then combining Proposition 3 and Lemma 1, the following holds:
\[ TW - T\hat{W} \leq \gamma (1 - \lambda) TW. \tag{2.31} \]

**Proof.** By (2.30), we have \((1 - \gamma) W \leq \hat{W}\). Then, using the fact that \( T \) is increasing and concave, it follows that
\[ (1 - \gamma) TW + \gamma T(0) \leq T[(1 - \gamma) W + \gamma 0] \leq T\hat{W}. \]
Finally, using Lemma 1 we get (2.31).

### 2.7. Convergence of the Continuous Iterative Scheme

Making use of properties (2.21),(2.23) and Proposition 5, we have the following convergence result.

**Proposition 6.** The sequences \((\hat{U}^n)\) and \((\tilde{U}^n)\) are well defined in \( K \) and converge, respectively, from above and below, to the unique solution of system of QVIs (2.15).

**Proof.** The proof will be carried out in four steps.

**Step 1.** The sequence \((\hat{U}^n)\) stays in \( K \) and is monotone decreasing.

From (2.17),(2.25) it is easy to see that \( \forall n \geq 1, \hat{U}^n = (\hat{u}^{1,n}, \ldots, \hat{u}^{n,M}) \) is solution to
\[
\begin{align*}
a^i (\hat{u}^{i,n}, v - \hat{u}^{i,n}) &\geq (f^i, v - \hat{u}^{i,n}), \quad \forall v \in H^1_0(\Omega), \quad v \leq k + \hat{u}^{i+1,n-1}, \\
\hat{u}^{i,n} &\leq k + \hat{u}^{i+1,n-1}, \quad i = 1, 2, \ldots, M, \\
\hat{u}^{M+1,n} &\equiv \hat{u}^{1,n}. \tag{2.32}
\end{align*}
\]
Since \( f^i \geq 0 \) and \( \hat{u}^{i,0} \geq 0 \), combining comparison results in variational inequalities with a simple induction, it follows that \( \hat{u}^{i,n} \geq 0 \), i.e.,
\[ \hat{U}^n \geq 0, \quad \forall n \geq 0. \tag{2.33} \]
Furthermore, by (2.23),(2.25) we have
\[ \hat{U}^1 = T\hat{U}^0 \leq \hat{U}^0. \]
Thus, inductively
\[ 0 \leq \hat{U}^{n+1} = T\hat{U}^n \leq \hat{U}^n \leq \cdots \leq \hat{U}^0, \quad \forall n \geq 0. \] (2.34)

**STEP 2.** \((\hat{U}^n)\) converges to the solution of the system (2.15).

From (2.33),(2.34) it is clear that
\[ \lim_{n \to \infty} \hat{u}^{i,n}(x) = \bar{u}^i(x), \quad x \in \Omega \quad \text{and} \quad \bar{u}^i \in K. \] (2.35)

Moreover, from (2.33) we have \(k + \hat{u}^{i+1,n-1} \geq 0\). Then we can take \(v = 0\) as a trial function in (2.32), which yields
\[ \delta \|\hat{u}^{i,n}\|_{H^1(\Omega)}^2 \leq a^i(\hat{u}^{i,n}, \hat{u}^{i,n}) \leq \|f^i\|_{L^2(\Omega)} \|\hat{u}^{i,n}\|_{H^1(\Omega)} \]

or more simply
\[ \|\hat{u}^{i,n}\|_{H^1(\Omega)} \leq C, \]

where \(C\) is a constant independent of \(n\). Hence, \(\hat{u}^{i,n}\) stays bounded in \(H^1(\Omega)\) and consequently we can complete (2.35) by
\[ \lim_{n \to \infty} \hat{u}^{i,n} = \bar{u}^i \text{ weakly in } H^1(\Omega). \] (2.36)

**STEP 3.** \(\bar{U} = (\bar{u}^1, \ldots, \bar{u}^M)\) coincides with the solution of system (2.15). Indeed, since
\[ \hat{u}^{i,n}(x) \leq k + \hat{u}^{i,n-1}(x), \]

then (2.35) implies
\[ \hat{u}^i(x) \leq k + \hat{u}^{i+1}(x). \]

Now let \(v \leq k + \hat{u}^{i+1}\), then \(v \leq k + \hat{u}^{i,n-1}, \forall n \geq 0\). We can, therefore, take \(v\) as a trial function for the VI (2.32). Consequently, combining (2.35),(2.36) with the weak lower semicontinuity of \(a^i(v, v)\) and passing to the limit in problem (2.32), we obtain:
\[ a^i(\bar{u}^i, v - \bar{u}^i) \geq (f^i, v - \bar{u}^i), \quad \forall v \in H^1_0(\Omega), \quad v \leq k + \hat{u}^{i+1}. \]

Finally, since \(U = (u^1, \ldots, u^M)\) is the unique solution of (2.15), we clearly have
\[ U = \bar{U} = (\bar{u}^1, \ldots, \bar{u}^M). \]

**STEP 4.** The monotone property of the sequence \((\hat{U}^n)\) can be shown similarly to that of sequence \((\hat{U}^n)\). Let us prove its convergence to the solution of system (2.15). Indeed, apply (2.30),(2.31) with
\[ W = \hat{U}^0, \quad \tilde{W} = \hat{U}^0, \quad \gamma = 1, \]

therefore,
\[ T\hat{U}^0 - T\hat{U}^0 \leq (1 - \lambda)T\hat{U}^0 \]

so
\[ 0 \leq \hat{U}^1 - \hat{U}^1 \leq (1 - \lambda)\hat{U}^1 \]

and applying (2.31) again, this yields
\[ 0 \leq \hat{U}^2 - \hat{U}^2 \leq (1 - \lambda)^2\hat{U}^2 \]

and quite generally
\[ 0 \leq \hat{U}^n - \hat{U}^n \leq (1 - \lambda)^n\hat{U}^n \leq (1 - \lambda)^n \|
\]
Thus,
\[ \tilde{U}^n - \tilde{U}^n \to 0, \text{ a.e.,} \]
from which it follows that
\[ \tilde{U}^n \to U = U, \]
the unique solution of the system of QVIs (2.15).

**Remark 1.** From the above proposition, one can observe that the solution of system QVI (2.15) is a fixed point of \( T \), i.e.,
\[ U = TU. \]

The following estimations provide a rate of convergence for sequences (2.25),(2.26). We shall see that this result will play a major role in the finite element error analysis section.

### 2.8. Geometric Convergence of the Iterative Scheme

**Proposition 7.** There exists a positive constant \( 0 < \mu < 1 \) such that
\[
\begin{align*}
\| \tilde{U}^n - U \|_\infty & \leq \mu^n \| \tilde{U}^0 \|_\infty, \\
\| \tilde{U}^n - U \|_\infty & \leq \mu^n \| \tilde{U}^0 \|_\infty.
\end{align*}
\]

**Proof.** By (2.34),(2.35), we have
\[ 0 \leq U \leq \tilde{U}^0 \]
so
\[ 0 \leq \tilde{U}^0 - U \leq \tilde{U}^0. \]

Then, applying (2.31) with \( \gamma = 1 \), we get
\[ 0 \leq T\tilde{U}^0 - TU \leq (1 - \lambda)T\tilde{U}^0, \]
and by (2.25),(2.37)
\[ 0 \leq \tilde{U}^1 - U \leq (1 - \lambda)\tilde{U}^1. \]

Now, using (2.31) again with \( \gamma = 1 - \lambda \), it follows that
\[ 0 \leq T\tilde{U}^1 - TU \leq (1 - \lambda)(1 - \lambda)T\tilde{U}^1, \]
i.e.,
\[ 0 \leq \tilde{U}^2 - U \leq (1 - \lambda)^2\tilde{U}^2, \]
and inductively,
\[ 0 \leq \tilde{U}^n - U \leq (1 - \lambda)^n\tilde{U}^0. \]

The proof of estimation (2.39) is identical to that of estimation (2.38) and is omitted.

### 3. THE DISCRETE PROBLEM

Let \( \Omega \) be decomposed into triangles and let \( \tau_h \) denote the set of all those elements; \( h > 0 \) is the mesh size. We assume that the family \( \tau_h \) is regular and quasi-uniform.

Let \( V_h \) denote the standard piecewise linear finite element space,
\[ V_h = \left\{ v \in C(\Omega) \cap H^1_0(\Omega) \text{ such that } \frac{v}{K} \in P_1, \ \forall K \in \tau_h \right\}. \]

Let \( A^i \) be the matrices with generic coefficients
\[ (A^i)_{ls} = a^i(\varphi_l, \varphi_s), \quad 1 \leq i \leq M, \quad 1 \leq l, s \leq m(h). \]
where \( \{ \varphi_l \}, l = 1, 2, \ldots m(h) \) is the basis of \( V_h \), and \( F_l^i \) the approximation of \( f^i \):

\[
F_l^i = (f^i, \varphi_l), \quad l = 1, 2, \ldots m(h). \tag{3.3}
\]

Let \( r_h \) be the usual restriction operator defined by

\[
\forall v \in C(\Omega) \cap H^1_0(\Omega), \quad r_h v = \sum_{l=1}^{m(h)} v_l \varphi_l. \tag{3.4}
\]

In the sequel of the paper, we shall make use of the discrete maximum assumption (d.m.p.). In other words, we shall assume that the matrices \( A^i, 1 \leq i \leq M \) defined in (3.2) are \( M \)-matrices (see [15]).

Under the d.m.p., we shall achieve a similar study to that devoted to the continuous problem. More precisely, we shall show that the qualitative properties and results stated in the previous section are conserved in the discrete case. Their respective proofs will be omitted as they are identical to their continuous analogous ones. As for the continuous problem, we begin by recalling some results related to discrete variational inequalities.

Let \( \zeta_h \in V_h \) be the finite element approximation of \( \zeta \) defined in (2.11):

\[
a(\zeta_h, v - \zeta_h) \geq (f, v - \zeta_h), \quad \forall v \in V_h,
\]

\[
\zeta_h \leq r_h \psi, \quad \psi \leq r_h \psi. \tag{3.5}
\]

Similarly to (2.13), denoting by \( \mathcal{X}_h \) the set of discrete subsolutions, i.e., the set of \( z_h \in V_h \) such that

\[
a(z_h, \varphi_s) \leq (f, \varphi_s), \quad \forall \varphi_s, \quad s = 1, 2, \ldots m(h),
\]

we have Proposition 8.

**Proposition 8.** \( \zeta_h \) is the maximum element of \( \mathcal{X}_h \).

Now let \( \sigma_h \) be a mapping from \( L^\infty(\Omega) \) into \( V_h \), defined by

\[
\zeta_h = \sigma_h(\psi). \tag{3.7}
\]

The mapping \( \sigma_h \) possesses analogous properties to those of the mapping \( \sigma \) defined in (2.14) provided the d.m.p. is satisfied.

**Proposition 9.** \( \sigma_h \) is increasing, concave, and Lipschitz continuous with respect to \( \psi \).

### 3.1. The Discrete Hamilton-Jacobi-Bellman Equation

The discrete Hamilton-Jacobi-Bellman equation consists of solving the following problem. Find \( u_h \in V_h \) solution to

\[
\max_{1 \leq i \leq M} (A^i u_h - F^i) = 0. \tag{3.8}
\]

### 3.2. Approximation of the Discrete HJB Equation by a System of Discrete QVIs

It is shown in [8] that problem (3.8) admits a unique solution and can be approximated by the following system of discrete quasi-variational inequalities (QVIs): find \( (u_{h}^1, \ldots, u_{h}^M) \in (V_h)^M \) solution to

\[
a^i \left( u_h^i, v - u_h^i \right) \geq (f^i, v - u_h^i), \quad \forall v \in V_h, \quad v \leq k + u_h^{i+1},
\]

\[
u_h^i \leq k + u_h^{i+1}, \quad i = 1, 2, \ldots, M,
\]

\[
with \ u_h^{M+1} = u_h^1. \tag{3.9}
\]
Hamilton-Jacobi-Bellman Equations

**THEOREM 3.** (See [8].) Under the d.m.p., there exists a unique solution to the system of QVIIs (3.9). Moreover,

\[
(u_h^1, \ldots, u_h^M) \to (u_h, \ldots, u_h) \text{ uniformly in } \bar{U} \text{ as } k \to 0.
\]  

(3.10)

3.3. **A Fixed-Point Mapping Associated with System of Discrete QVIIs (3.9)**

We consider the following mapping:

\[
T_h : \mathbb{H}^+ \to (V_h)^M \\
W \to T_h W = (\xi_h^1, \ldots, \xi_h^M),
\]

(3.11)

such that \( \forall i = 1, \ldots, M, \xi_h^i \) is the unique solution of the following discrete VI:

\[
a^i(\xi_h^i, v - \xi_h^i) \geq (f^i, v - \xi_h^i), \quad v \in V_h, \\
\xi_h^i \leq r_h(k + w^{i+1}), \quad v \leq r_h(k + w^{i+1}), \\
\text{with } \xi_h^{M+1} = \xi_h^1.
\]

(3.12)

Denoting by \( \xi_h^i = \sigma_h(k + w^{i+1}), \) we clearly have

\[
T_h W = [\sigma_h(k + w^2), \sigma_h(k + w^3), \ldots, \sigma_h(k + w^i), \ldots, \sigma_h(k + w^1)] .
\]

(3.13)

Let \( \hat{U}_h^0 - (\hat{u}_h^{1,0}, \ldots, \hat{u}_h^{M,0}) \) be the finite element approximation of \( \hat{U}_h^0 \) defined in (2.20):

\[
a^i(\hat{u}_h^{i,0}, v) = (f^i, v), \quad \forall v \in V_h, \quad 1 \leq i \leq M.
\]

(3.14)

By (2.6), it is easy to see that (3.11) admits a unique solution. Moreover, by the d.m.p. and the positivity of \( f^i, \) we have \( \hat{U}_h^0 \geq 0. \)

3.4. **Some Properties of the Mapping \( T_h \)**

**PROPOSITION 10.** The mapping \( T_h \) possesses analogous properties to those of mapping \( T \) defined by (2.16)–(2.19), i.e.,

\[
T_h V < T_h W, \quad \text{whenever } V \leq W.
\]

(3.15)

\[
T_h W \geq 0, \quad \forall W \in \mathbb{H}^+.
\]

(3.16)

\[
T_h W \leq \hat{U}_h^0, \quad \forall W \in \mathbb{H}^+.
\]

(3.17)

Also, by analogy with Propositions 3 and 4, we, respectively, have concavity and Lipschitz dependence properties for \( T_h. \)

**PROPOSITION 11.** \( T_h \) is concave on \( \mathbb{H}^+. \)

**PROPOSITION 12.** \( T_h \) is Lipschitz continuous on \( \mathbb{H}^+, \) i.e.,

\[
\|T_h V - T_h W\|_{\infty} \leq \|V - W\|_{\infty}, \quad \forall V, W \in \mathbb{H}^+.
\]

(3.18)

For the solution of the discrete system of QVIIs (3.9), we have the following iterative scheme.
3.5. A Discrete Iterative Scheme

Starting from $\bar{U}_h^0$ solution of (3.14) (respectively, $\bar{U}_h^0 = (0, 0, \ldots, 0)$), we define

$$\bar{U}_h^{n+1} = T_h \bar{U}_h^n, \quad n = 0, 1, \ldots, \tag{3.19}$$

respectively,

$$\bar{U}_h^{n+1} = T_h' \bar{U}_h^n, \quad n = 0, 1, \ldots. \tag{3.20}$$

Let

$$K_h = \left\{ W \in H^1 \text{ such that } 0 \leq W \leq \bar{U}_h^0 \right\}. \tag{3.21}$$

**Lemma 2.** For $0 < \lambda < \inf(k/\|\bar{U}_h^0\|_\infty, 1)$ we have

$$T_h(0) \geq \lambda \bar{U}_h^0. \tag{3.22}$$

**Proposition 13.** Let $\gamma \in [0; 1]$ such that

$$W - \bar{W} \leq \gamma W, \quad \forall W, \bar{W} \text{ in } K_h. \tag{3.23}$$

Then we have

$$T_h W - T_h \bar{W} \leq \gamma(1 - \lambda) T_h W. \tag{3.24}$$

Then similarly to Proposition 6, we have the following convergence result.

**Proposition 14.** The sequences $(\bar{U}_h^n)$ and $(\bar{U}_h^n)$ are well defined in $K_h$ and converge, respectively, from above and below to the unique solution of system (3.9).

**Remark 2.** In view of Proposition 14, it is easy to see that the solution of system (3.9) is a fixed point of $T_h$, i.e.,

$$U_h = T_h U_h. \tag{3.25}$$

3.6. Discrete Geometric Convergence

Now using the above results, we are able to establish the geometric convergence of sequences $(\bar{U}_h^n)$ and $(\bar{U}_h^n)$.

**Proposition 15.** There exists a positive constant $0 < \mu < 1$ such that

$$\|\bar{U}_h^n - U\|_\infty \leq \mu^n \|\bar{U}_h^0\|_\infty, \tag{3.26}$$

$$\|\bar{U}_h^n - U\|_\infty \leq \mu^n \|\bar{U}_h^0\|_\infty. \tag{3.27}$$

4. THE FINITE ELEMENT ERROR ANALYSIS

This section is devoted to demonstrating that the proposed method is quasi-optimally accurate in $L^\infty(\Omega)$ according to the approximation theory. To this end, we first recall some known $L^\infty(\Omega)$-error estimates results, introduce an auxiliary problem, and prove a fundamental lemma.

**Theorem 4.** (See [16, 17].) Let $\hat{u}^{i,0}$ (respectively, $\hat{u}_h^{i,0}$) be the solution of problem (2.20) (respectively, (3.14)). Then

$$\|\hat{u}^{i,0} - \hat{u}_h^{i,0}\|_{L^\infty(\Omega)} \leq C h^2 |\log h|^{3/2}, \quad \forall i = 1, 2, \ldots, M. \tag{4.1}$$
THEOREM 5. (See [18].) Let the d.m.p. and regularity result (2.12) hold. Then
\[ \| \zeta - \zeta_h \|_{L^\infty(\Omega)} \leq Ch^2 |\log h|^2. \] (4.2)

4.1. An Auxiliary Sequence of Variational Inequalities

We introduce a sequence which consists of finding \( \tilde{U}^n_h = (\tilde{u}^{1,n}_h, \ldots, \tilde{u}^{M,n}_h) \) such that \( \forall n \geq 1, \tilde{u}^{i,n}_h \) is the unique solution of the following VI:
\[
\alpha^i (\tilde{u}^{i,n}_h, v - \tilde{u}^{i,n}_h) \geq (f^i, v - \tilde{u}^{i,n}_h), \quad \forall v \in H^1_0(\Omega), \quad v \leq k + \tilde{u}^{i+1,n-1}_h, \\
\tilde{u}^{i,n}_h \leq k + \tilde{u}^{i+1,n-1}_h, \quad i = 1, 2, \ldots, M,
\]
where \( \tilde{U}^n = (\tilde{u}^{1,n}_h, \ldots, \tilde{u}^{M,n}_h) \) is defined in (2.25). Thus, we clearly have
\[
\tilde{U}^n_h = T_h \tilde{U}^n, \quad \tilde{U}^0_h = \tilde{U}^0. \] (4.4)

PROPOSITION 16. Let assumptions (2.1)-(2.8) and estimation (2.12) hold. Then there exists a constant \( C \) independent of \( n \) such that
\[
\max \left( \| \tilde{u}^{i,n}_h \|_{W^{2,p}(\Omega)}, \| \tilde{u}^{i,n}_h \|_{W^{2,p}(\Omega)} \right) \leq C, \quad \forall i = 1, 2, \ldots, M, \; 1 \leq p < \infty.
\]

PROOF. Since \( \tilde{u}^{i,0} \) is \( W^{2,\infty}(\Omega) \) regular, a simple induction combined with (2.12) leads immediately to the desired result.

We notice that \( \forall i = 1, 2, \ldots, M, \; \tilde{u}^{i,n}_h \) is the finite element approximation of \( \tilde{u}^{i,n}_h \). Therefore, making use of Proposition 16 and Theorem 5, the following error estimate holds.

PROPOSITION 17.
\[
\| \tilde{U}^n - \tilde{U}^n_h \|_\infty \leq Ch^2 |\log h|^2. \] (4.5)

The following lemma will play a crucial role in proving the main result.

LEMMA 3.
\[
\| \tilde{U}^n - \tilde{U}^n_h \|_\infty \leq \sum_{p=0}^{n} \| \tilde{U}^p - \tilde{U}^p_h \|_\infty. \] (4.6)

PROOF. The proof will be carried out by induction.

STEP 1. Indeed, from (3.19) and (4.4) we have
\[
\tilde{U}^1_h = T_h \tilde{U}^0, \quad \tilde{U}^1_h = T_h \tilde{U}^0.
\]

Then
\[
\| \tilde{U}^1 - \tilde{U}^1_h \|_\infty \leq \| \tilde{U}^1 - \tilde{U}^0 \|_\infty + \| \tilde{U}^0_h - \tilde{U}^1_h \|_\infty \leq \| \tilde{U}^1 - \tilde{U}^0 \|_\infty + \| T_h \tilde{U}^0 - T_h \tilde{U}^0 \|_\infty,
\]
since \( T_h \) is Lipschitz continuous (see Proposition 12), it follows that
\[
\| \tilde{U}^1 - \tilde{U}^1_h \|_\infty \leq \| \tilde{U}^1 - \tilde{U}^0 \|_\infty + \| \tilde{U}^0 - \tilde{U}^0_h \|_\infty \leq \sum_{p=0}^{1} \| \tilde{U}^p - \tilde{U}^p_h \|_\infty.
\]

STEP n. Assume that
\[
\| \tilde{U}^{n-1} - \tilde{U}^{n-1}_h \|_\infty \leq \sum_{p=0}^{n-1} \| \tilde{U}^p - \tilde{U}^p_h \|_\infty.
\]
Then, since
\[ \tilde{U}_h^n = T_h \tilde{U}_n - 1, \quad \tilde{U}_h^n = T_h \tilde{U}_n - 1, \]
applying Proposition 12 again, we obtain
\[
\left\| \tilde{U}_h^n - \tilde{U}_h^n \right\| \leq \left\| \tilde{U}_n - \tilde{U}_h^n \right\| + \left\| \tilde{U}_h^n - \tilde{U}_h^n \right\| \\
\leq \left\| \tilde{U}_n - \tilde{U}_h^n \right\| + \left\| T_h \tilde{U}_n - T_h \tilde{U}_h^n \right\| + \left\| \tilde{U}_n - \tilde{U}_h^n \right\| + \left\| \tilde{U}_h^n - \tilde{U}_h^n \right\| \\
\leq \left\| \tilde{U}_n - \tilde{U}_h^n \right\| + \sum_{p=0}^{n-1} \left\| \tilde{U}_p - \tilde{U}_h^n \right\| \leq \sum_{p=0}^n \left\| \tilde{U}_p - \tilde{U}_h^n \right\|.
\]
This completes the proof.

Now guided by Propositions 7, 15, 17, Lemma 3, and Theorem 4, we are in a position to prove our main result.

### 4.2. $L^\infty$-Error Estimate for the System of QVIs (2.15)

**Theorem 6.**
\[
\| U - U_h \| \leq C h^2 | \log h |^2.
\]

**Proof.**
\[
\| U - U_h \| \leq \| U - \tilde{U}_n \| + \| \tilde{U}_n - \tilde{U}_h^n \| + \| \tilde{U}_h^n - U_h \| \\
\leq \mu^n \| \tilde{U}_n \| + \mu^n \| \tilde{U}_h^n \| + \| \tilde{U}_n - \tilde{U}_h^n \| + \sum_{p=0}^{n-1} \| \tilde{U}_p - \tilde{U}_h^n \| \\
\leq \mu^n \| \tilde{U}_n \| + \mu^n \| \tilde{U}_h^n \| + C h^2 | \log h |^{3/2} + n C h^2 | \log h |^2,
\]
where we have made use of estimations (2.38), (3.26), (4.1), (4.6), and (4.5), respectively. Finally, taking $\mu^n = h^2$, we obtain the desired result.

### 4.3. $L^\infty$-Error Estimate for the HJB Equation (1.1)

**Theorem 7.**
\[
\| u - u_h \| \leq C h^2 | \log h |^2.
\]

**Proof.** Indeed,
\[
\| u - u_h \| \leq \max_{1 \leq i \leq M} \left( \| u^i - u \|_{L^\infty(\Omega)} + \| u^i - u_h \|_{L^\infty(\Omega)} + \| u_h^n - u_h \|_{L^\infty(\Omega)} \right) \\
\leq \max_{1 \leq i \leq M} \left( \| u^i - u \|_{L^\infty(\Omega)} \right) + \max_{1 \leq i \leq M} \left( \| u^i - u_h^n \|_{L^\infty(\Omega)} \right) + C h^2 | \log h |^3,
\]
where we have used Theorem 6. Thus, by Theorems 2 and 3, we get the desired result.

## 5. CONCLUSION

- We have established a convergence order of the standard finite element method for the HJB equation (1.1). As far as we know, it is the first finite element error estimate obtained for this problem.

However, it is worth noting that the approach developed in this paper relies on the coercivity of the variational forms $a^\varepsilon(u, v)$. For general second-order elliptic operator, such a condition is satisfied if the constant $c_0$ in (2.3) is sufficiently large. This means that this approach is restricted to operators $A^\varepsilon$ with zero-order terms $a_0(x)$ strictly positive.
The method relies also on the discrete maximum principle. This assumption imposes a restriction on the differential operator as well as the triangulation. In 2-D, for example, angles of triangles must be acute (cf. [15]).

The error estimate obtained contains a logarithmic factor with an extra power of $|\log h|$ than expected. We think that this may be due to the followed approach.

REFERENCES