A multiplicative multisplitting method for solving the linear complementarity problem

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1. Introduction

We consider the following finite-dimensional linear complementarity problem (LCP):

\[
\begin{align*}
\text{find} \quad & x \in \mathbb{R}^n, \\
\text{such that} \quad & x \geq \phi, \quad Ax - F \geq 0, \quad (x - \phi)^T(Ax - F) = 0,
\end{align*}
\]

where \( A \in \mathbb{R}^{n \times n} \) is a given matrix, and \( \phi, F \in \mathbb{R}^n \) are given vectors. If all components of vector \( \phi \) are \(-\infty\), then (1.1) reduces to the system of linear equations

\[
Ax = F.
\]

The complementarity problem has important applications in operations research, economic equilibrium models and in the engineering sciences; see, e.g., [1,2]. For this reason, there is growing interest in finding efficient and robust algorithms for solving (1.1). This is reflected in an increasing number of proposals of solution schemes for (1.1) in recent years. In these recent developments an important role has been played by the splitting methods. This class of splitting methods originates from matrix splitting methods such as the Jacobi, Gauss–Seidel and SOR iterations for solving problem (1.2). The first introduction into addressing the linear complementarity problem was by Cottle and Sacher [3], and this was further developed by many authors, e.g., [4–7]. In order to accommodate the requirements of high-speed parallel multiprocessor systems, Machida, Fukushima and Ibaraki [8] presented a class of parallel multisplitting iterative methods for solving the symmetric linear complementarity problems in synchronous parallel computing environments. Bai [9] discussed further the convergence of a variant of these multisplitting methods for some nonsymmetric matrix classes.

In this paper, we assume that \( A \) is an \( H \)-matrix with positive diagonals. A nonsingular matrix \( A \) having all nonpositive off-diagonal entries is called an \( M \)-matrix if the inverse is (entrywise) nonnegative, i.e., \( A^{-1} \geq 0 \); see, e.g., [10]. For any
matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$, its comparison matrix $\langle A \rangle = (\alpha_{ij})$ is defined by

$$\alpha_{ij} = |a_{ij}|, \quad \alpha_{ji} = -|a_{ji}|, \quad j \neq i.$$

A matrix $A$ is said to be an $H$-matrix if $\langle A \rangle$ is an $M$-matrix. $H$-matrices were introduced as a generalization of $M$-matrices. They appear in many applications, e.g., when discretizing certain nonlinear parabolic operators using high order finite elements and sufficiently small time steps [11].

Recently, various Schwarz iterative algorithms for solving finite-dimensional variational inequalities as well as complementarity problems have been presented [12–15]. Such methods are amenable to implementation. Moreover, the convergence rate will not deteriorate with the refinement of the mesh when applied to discretized differential equations. Theory and numerical experiments have shown that the latter advantage is still maintained when the methods are used to solve discretized variational inequalities with an elliptic differential operator [14, 15]. Generally, there are two ways to study the convergence of the Schwarz method for solving LCPs. One is to prove that the method generates a minimizing sequence for some objective function. In this case, the matrix $A$ is often supposed to be symmetric and positive definite. The other way is to prove that the method produces a monotone sequence starting from a super-solution or a sub-solution of the problem. Convergence theorems established in the latter way are often based on the assumption that matrix $A$ is an $M$-matrix. Up to now, there has been no general convergence theory of Schwarz methods for the case where the coefficient matrix $A$ of (1.1) belongs to the $H$-matrix class, since the above two ways are not effective for (1.1) with an $H$-matrix.

The purpose of this paper is to apply a multiplicative multisplitting method to solve (1.1). On one hand, this method is an extension of the multiplicative Schwarz iteration scheme for solving the linear equation (1.2), which was proposed by Benzi, Frommer, Nabben and Szyld [16]. On the other hand, the multiplicative multisplitting method applied to the multiplicative Schwarz method can be thought of as a new way to prove the convergence theorem for (1.1) with an $H$-matrix. We show that the sequence generated by the multiplicative multisplitting method converges to the unique solution of the problem without any restriction on the initial point. The proposed method can be applied to the multiplicative Schwarz method for $H$-compatible splitting. Moreover, we show that the proposed method generates a monotone sequence of iterates if the coefficient matrix $A$ is an $M$-matrix and the initial point is a super-solution of the problem.

The paper is organized as follows. In Section 2, we present some basic concepts, definitions and some well-known results which will be used later, then propose a multiplicative multisplitting method for the linear complementarity problem. In Section 3, we discuss the convergence of the proposed method for the case where the coefficient matrix belongs to the $H$-matrix with positive diagonals, and establish a general convergence result for the proposed method. In Section 4, the proposed method is applied to $H$-compatible splitting and the multiplicative Schwarz method, separately. These corresponding convergence properties are discussed in detail. Finally, in Section 5, we establish the monotone convergence of the multiplicative multisplitting method under appropriate conditions.

### 2. Preliminaries

In this section, we propose a multiplicative multisplitting method for solving (1.1). First, we start with some notation, definitions and basic results that are useful for the proposed method.

A matrix $A \in \mathbb{R}^{n \times n}$ is called an $H_+\text{-matrix}$ if it is an $H$-matrix having positive diagonal elements; and a $Q\text{-matrix}$ if (1.1) has a solution for any $F \in \mathbb{R}^n$. A sufficient condition for $A \in \mathbb{R}^{n \times n}$ to be a $Q$-matrix is that either $A$ is $H_+\text{-matrix}$ [17] or $A$ is a strictly copositive matrix [18]. For a given matrix $A \in \mathbb{R}^{n \times n}$, let $F, G \in \mathbb{R}^{n \times n}$ be such that $A = F - G$; then $(F, G)$ is called a splitting of the matrix $A$. The splitting $(F, G)$ is called a convergent splitting if the spectral radius of the matrix $(F^{-1}G)$ is less than 1, i.e., $\rho(F^{-1}G) < 1$. It is called a weak regular splitting if $F^{-1} \geq 0$ and $F^{-1}G \geq 0$; a regular splitting if $F^{-1} \geq 0$ and $G \geq 0$; an $M$-splitting if $F$ is an $M$-matrix and $G \geq 0$; an $H$-splitting if $F$ is an $H$-matrix and an $H_\pm$-compatible splitting if $(A) = (F) - |G|$; a $Q$-splitting if $F$ is a $Q$-matrix. In particular, the splitting $(F, G)$ is called an $H_\pm$-splitting and an $H_\pm$-compatible splitting, respectively, with $F$ an $H_\pm$-matrix.

Let $(B, C)$ be a splitting of the matrix $A$ such that $(B)$ is nonsingular, and let $(F, G)$ be a matrix pair. Then we call the matrix pair $(F, G)$ a majorizing pair of the splitting $(B, C)$ if $F$ is an $H$-matrix and it holds that $(B)^{-1} \leq (F)^{-1}$ and $|C| \leq |G|$. In such a case, we say that $(B, C)$ is majorized by $(F, G)$; see [9]. Note that here $(F, G)$ is not necessarily a splitting of the matrix $A$. Evidently, if $A = B - C$ is an $H$-splitting, then $A$ is $H$-matrices and $\rho((B)^{-1}C) \leq \rho((B)^{-1}|C|) < 1$; if it is an $H$-compatible splitting and $A$ is an $H$-matrix, then it is an $H$-splitting and thus convergent; and if $(F, G)$ is a majorizing pair of the splitting $(B, C)$ such that $(F) - |G|$ is an $M$-matrix, then $(B, C)$ is a convergent splitting.

The following concept will play an important role in the subsequent analysis.

**Definition 2.1** ([19]). Let $\omega \in \mathbb{R}^n$ be a positive vector. For a vector $y \in \mathbb{R}^n$, the weighted max-norm is defined by

$$\|y\|_\omega = \max_{1 \leq j \leq n} \frac{|y_j|}{\omega_j}.$$

For a matrix $A \in \mathbb{R}^{n \times n}$, the weighted max-norm is defined by

$$\|A\|_\omega = \sup \{\|Ay\|_\omega : y \in \mathbb{R}^n\}.$$
Obviously, if \( \omega = (1, \ldots, 1)^T \), then the weighted max-norm reduces to the usual maximum norm. The following result deals with the existence and uniqueness of the solution of (1.1) shown in [9]

**Lemma 2.2** ([9]). Let \( A \in \mathbb{R}^{n \times n} \) be an \( H_+ \)-matrix. Then (1.1) has a unique solution for any \( F \in \mathbb{R}^n \).

A multisplitting of a matrix \( A \in \mathbb{R}^{n \times n} \) is a collection of triples \((B_i, C_i, E_i)\), \( i = 1, \ldots, m \), which satisfies:

1. \( A = B_i - C_i \) (\( i = 1, \ldots, m \)) are \( Q \)-splittings.
2. \( E_i \) (\( i = 1, \ldots, m \)) are nonnegative diagonal matrices with \( E_i \leq I \) (the \( n \times n \) identity matrix).

Let \( N_0 \) denote the natural numbers set, and \((B_{k,i}, C_{k,i}, E_{k,i}) \) (\( i = 1, \ldots, m \), \( k \in N_0 \)) be a sequence of multisplittings of the matrix \( A \). Then we consider the following multiplicative multisplitting method for solving (1.1).

**Algorithm 1** (Multiplicative Multisplitting Method). Step 1: Let \( x^0 \in \mathbb{R}^n \) be an arbitrary vector, and set \( k := 0 \).

Step 2: Given \( z^{k,0} = x^k \), for each \( i = 1, 2, \ldots, m \), let \( x^{k,i} \in \mathbb{R}^n \) be an arbitrary solution of the following subproblem:

\[
\begin{align*}
& x \geq \phi^{k,i}, \\
& B_{k,i}x \geq F^{k,i}, \\
& (x - \phi^{k,i})^T(B_{k,i}x - F^{k,i}) = 0,
\end{align*}
\]

(2.1) where \( \phi^{k,i} = \phi - z^{k,i-1} \) and \( F^{k,i} = F - Az^{k,i-1} \), and let

\[
z^{k,i} = z^{k,i-1} + E_{k,i}x^{k,i}.
\]

(2.2)

Step 3: Let \( x^{k+1} = z^{k,m} \). If \( x^{k+1} = x^k \), then stop. Otherwise, set \( k := k + 1 \) and return to Step 2.

Here, the multiple splittings \( A = B_{k,i} - C_{k,i} \) (\( i = 1, \ldots, m \), \( k \in N_0 \)) of the coefficient matrix \( A \in \mathbb{R}^{n \times n} \) are permitted to vary with \( k \), the iteration index, and the weighting matrices \( E_{k,i} \) (\( i = 1, \ldots, m \), \( k \in N_0 \)) are allowed to be arbitrary nonnegative diagonal matrices with \( E_i \leq I \). Moreover, \( \sum_{i=1}^{m} E_{k,i} \) is not necessarily equal to \( I \).

3. Multiplicative multisplitting method

In this section, we will discuss the convergence of Algorithm 1 for the case where the coefficient matrix \( A \in \mathbb{R}^{n \times n} \) belongs to the \( H_+ \)-matrix class and establish a general convergence.

First, we introduce the following splitting (termed the major-multisplitting): let \( A \in \mathbb{R}^{n \times n} \) be an \( H_+ \)-matrix, and for every \( k \in N_0 \), \((B_{k,i}, C_{k,i}, E_{k,i}) \) (\( i = 1, \ldots, m \)) be a multisplitting of the matrix \( A \). Then the multisplitting is called a major-multisplitting of the matrix \( A \) if:

1. \( B_{k,i} \) (\( i = 1, \ldots, m \)) are \( H_+ \)-matrices;
2. for each \( k \), there exists a majorizing pair \((\hat{B}_{k,i}, \hat{C}_{k,i})\) of the splitting \((B_{k,i}, C_{k,i})\) satisfying

\[
\|\hat{H}_k\|_\infty \leq \gamma,
\]

(3.1)

where \( \hat{H}_k = \prod_{i=m}^{1}(I - E_{k,i}(I - (\hat{B}_{k,i})^{-1}(\hat{C}_{k,i}))) \) for some nonnegative constant \( \gamma \in [0, 1) \).

In the following, we show the convergence of Algorithm 1 for the above splittings.

**Lemma 3.1.** Let \( y^{k,i} = z^{k,i-1} + x^{k,i} \); then \( y^{k,i} \) is the solution of the following LCP on \( \mathbb{R}^n \):

\[
\begin{align*}
& y \geq \phi, \\
& B_{k,i}y \geq \bar{F}^{k,i}, \\
& (y - \phi)^T(B_{k,i}y - \bar{F}^{k,i}) = 0,
\end{align*}
\]

(3.2)

where \( \bar{F}^{k,i} = F + C_{k,i}z^{k,i-1} \).

**Proof.** By the definition of \( y^{k,i} \), we have

\[
y^{k,i} - \phi = x^{k,i} + z^{k,i-1} - \phi = x^{k,i} - \phi^{k,i}.
\]

Furthermore,

\[
B_{k,i}y^{k,i} - \bar{F}^{k,i} = B_{k,i}x^{k,i} + B_{k,i}z^{k,i-1} - \bar{F}^{k,i} = B_{k,i}x^{k,i} + B_{k,i}z^{k,i-1} - (F + C_{k,i}z^{k,i-1}) = B_{k,i}x^{k,i} + (B_{k,i} - C_{k,i})z^{k,i-1} - F = B_{k,i}x^{k,i} + Az^{k,i-1} - F = B_{k,i}x^{k,i} - F^{k,i}.
\]

Consequently, (3.2) follows from (2.1).
Lemma 3.2. Let $x^*$ be the unique solution of (1.1), and $y^{k,i} = z^{k,i-1} + x^{k,i}$. Then

$$(B_{k,i})y^{k,i} - x^* \leq |C_{k,i}|z^{k,i-1} - x^*.$$  \hfill (3.3)

Proof. Following the proof of Theorem 3.1 in [9], we can verify (3.3) componentwise. Consider an arbitrary index $j$. We first assume

$$|y^{k,i} - x^*| = (y^{k,i} - x^*)_j,$$

which means that

$$(y^{k,i} - x^*)_j \geq 0.$$  

Thus, if $y^{k,i}_j = \phi_j$, then $x^*_j = \phi_j$. Hence, (3.3) holds for the $j$th component, since the left-hand side is nonpositive while the right-hand side is nonnegative.

If $y^{k,i}_j > \phi_j$, then by Lemma 3.1 we have

$$(B_{k,i}y^{k,i} - C_{k,i}z^{k,i-1} - F)_j = 0.$$  \hfill (3.4)

Furthermore, since $x^*$ be the unique solution of (1.1), we have

$$(B_{k,i}x^* - C_{k,i}x^* - F)_j \geq 0.$$  \hfill (3.5)

Thus, by subtracting (3.5) from (3.4), we get

$$(B_{k,i}(y^{k,i} - x^*))_j \leq (C_{k,i}(z^{k,i-1} - x^*))_j \leq (|C_{k,i}|z^{k,i-1} - x^*)_j.$$  

Note that

$$(B_{k,i}(y^{k,i} - x^*))_j \geq (|B_{k,i}|y^{k,i} - x^*)_j,$$

as $B_{k,i}$ is an $H_+$-matrix. So we have

$$(|B_{k,i}|y^{k,i} - x^*)_j \leq (|C_{k,i}|z^{k,i-1} - x^*)_j.$$  \hfill (3.6)

We next assume

$$|y^{k,i} - x^*| = (x^* - y^{k,i})_j.$$  

In this case, we have

$$(y^{k,i} - x^*)_j \leq 0.$$  

In a similar fashion, we can establish the same inequality, (3.6). Thus inequality (3.3) holds. $\square$

Lemma 3.3. Let $x^*$ be the unique solution of (1.1), and $\varepsilon^{k,i} = z^{k,i} - x^* \ (i = 0, 1, \ldots, m)$. Then $|\varepsilon^{k,i}| \leq (I - E_{k,i}(I - (B_{k,i})^{-1}|C_{k,i}|))|\varepsilon^{k,i-1}|$ for $i = 1, \ldots, m$.

Proof. We deduce from Algorithm 1 that

$$0 \leq |\varepsilon^{k,i}| = |z^{k,i} - x^*| = |z^{k,i-1} + E_{k,i}x^{k,i} - x^*| \leq |z^{k,i-1} - E_{k,i}\varepsilon^{k,i-1}| + |E_{k,i}(\varepsilon^{k,i-1} + x^{k,i})|.$$  

$$(I - E_{k,i})|\varepsilon^{k,i-1}| + E_{k,i}|\varepsilon^{k,i} - x^*|,$$

$$(I - E_{k,i})|\varepsilon^{k,i-1}| + E_{k,i}(B_{k,i})^{-1}|C_{k,i}||\varepsilon^{k,i-1}|$$

$$(I - E_{k,i}(I - (B_{k,i})^{-1}|C_{k,i}|))|\varepsilon^{k,i-1}|,$$

where the second equality follows from $y^{k,i} = z^{k,i-1} + x^{k,i}$, the last inequality follows from Lemma 3.2. $\square$

In the following, we establish a general convergence result by Lemma 3.3.

Theorem 3.4. Let $A \in \mathbb{R}^{n \times n}$ be an $H_+$-matrix, and $(B_{k,i}, C_{k,i}, E_{k,i}) \ (i = 1, \ldots, m)$ be a major-multisplitting of the matrix $A$ for every $k \in \mathbb{N}_0$. Then the sequence $\{x^k\}_{k \in \mathbb{N}_0}$ generated by Algorithm 1 converges to the unique solution $x^*$ of (1.1).
Let \( A \) be an \( M \)-matrix, and a collection of \( m \) triples \((M_i, N_i, E_i)\) be given such that \( 0 \leq E_i \leq I \), \( \sum_{i=1}^{m} E_i \geq I \), and \( A = M_i - N_i \) be a weak regular splitting for \( 1 \leq i \leq m \). Let \( T = (I - E_{m}M_{m}^{-1}A)(I - E_{m-1}M_{m-1}^{-1}A) \ldots (I - E_{1}M_{1}^{-1}A) \). Then for any vector \( \omega = A^{-1}e > 0 \) with \( e > 0 \), \( \rho(T) \leq \|T\|_{\omega} < 1 \).
Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be an $H_+\!$-matrix. For each $i = 1, \ldots, m$, and $k \in N_0$, let $A = B_{k,i} - C_{k,i}$ be an $H_+\!$-compatible splitting, and $E_{k,i}$ be nonnegative diagonal matrices satisfying $E_{k,i} \leq I$ and $\sum_{i=1}^m E_{k,i} \geq I$. We note that $(B_{k,i}) - | C_{k,i} |$ is an $M$-matrix and $B_{k,i}$ is an $H_+\!$-matrix for each $i = 1, \ldots, m$ and $k \in N_0$. Hence, $(A) = (B_{k,i}) - | C_{k,i} |$ is a weak regular splitting for each $i = 1, \ldots, m$ and $k \in N_0$. Now, take

$$
B_{k,i} = B_{k,i}, \quad \hat{C}_{k,i} = C_{k,i}, \quad i = 1, \ldots, m, \quad k \in N_0.
$$

By making use of Lemma 4.1 and Theorem 3.4, we immediately reach the conclusion of the following theorem.

**Theorem 4.2.** Let $A \in \mathbb{R}^{n \times n}$ be an $H_+\!$-matrix and a collection of $m$ triples $(B_{k,i}, C_{k,i}, E_{k,i})$ be given such that $0 \leq E_{k,i} \leq I$, $\sum_{i=1}^m E_{k,i} \geq I$. Assume that for each $k \in N_0$ and $i \in \{1, \ldots, m\}$, $A = B_{k,i} - C_{k,i}$ is an $H_+\!$-compatible splitting. Then the sequence $\{x^k\}$ generated by Algorithm 1 converges to the unique solution of (1.1).

In the sequence, we discuss the convergence of the multiplicative Schwarz method for solving (1.1).

Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be an $H_+\!$-matrix. For each $i = 1, \ldots, m$ and $k \in N_0$, let $A = B_{k,i} - C_{k,i}$ be a $Q$-compatible splitting.

Now, take

$$
B_{k,i} = B_{k,i}, \quad \hat{C}_{k,i} = C_{k,i}, \quad i = 1, \ldots, m, \quad k \in N_0.
$$

For the sake of simplicity, say $B_{k,i} = B_i$ and $C_{k,i} = C_i$ for each $i = 1, \ldots, m$, $k \in N_0$. Let $D = \text{diag}(A)$ and $I_i (i = 1, \ldots, m)$ be the subsets of $S = \{1, \ldots, n\}$ satisfying

$$
\bigcup_{i=1}^m I_i = S, \quad (4.1)
$$

and $I_i = S \setminus I_i$ for each $i = 1, \ldots, m$. Define $B_i (i = 1, \ldots, m)$ as follows:

$$
B_i = \begin{pmatrix}
(B_i)_{ij} &=& A_{ij}, \\
(B_i)_{ij} &=& D_{ij}, \\
(B_i)_{ij} &=& 0, \\
(B_i)_{ij} &=& 0,
\end{pmatrix} \quad (4.2)
$$

where $A_{ij} = (a_{ij})_{\{j\} \in I, \{i\} \in \{k\}}$ is a submatrix of $A$, $A_{ij} = (a_{ij})_{\{j\} \in I, \{i\} \in \{k\}}$ is a principal submatrix of $A$ which is also an $H_+\!$-matrix from [16]. Then we cite a lemma from [20].

**Lemma 4.3** ([20]). Let $A$ be an $H$-matrix and the matrices $B_i$ be of the form (4.2). Then, $A = B_i - C_i \ i = 1, \ldots, m$, are $H$-compatible splittings.

Let $n_i = |I_i|$ denote the cardinality of the set $I_i$ and $x_i = (x_i)_{\{j\} \in I_i}$ denote the subvector of $x \in \mathbb{R}^n$. We consider the following multiplicative Schwarz method for solving (1.1), where similar multiplicative Schwarz methods for obtaining the $M$-matrix were proposed in [14, 15].

**Algorithm 2** (Multiplicative Schwarz Method). Step 1: Let $x^0 \in \mathbb{R}^n$ be an arbitrary vector, and set $k = 0$.

Step 2: Given $x^{k,0} = x^k \in \mathbb{R}^n$, for each $i = 1, 2, \ldots, m$, let $z^{k,i}_i \in \mathbb{R}^{|I_i|}$ be an arbitrary solution of the following subproblem:

$$
\begin{cases}
\sum_{j \in I_i} (z_{j} - \phi_{i,j}) = 0, \\
A_{ij}z_{j} + A_{ij}z^{k,i-1}_{j} - F_{ij} \geq 0, \\
(z - \phi_{i,j})^T (A_{ij}z_{j} + A_{ij}z^{k,i-1}_{j} - F_{ij}) = 0, \\
z_{j}^{k,i} = z^{k,i-1}_{j}.
\end{cases} \quad (4.3)
$$

and $z^{k,i}_i = z^{k,i-1}_i$.

Step 3: Let $x^{k+1} = z^{k,m}$. If $x^{k+1} = x^k$, then stop. Otherwise, set $k := k + 1$ and return to Step 2.

The following lemma holds true from Lemma 3.1.

**Lemma 4.4.** For each splitting $A = B_i - C_i$ ($i = 1, \ldots, m$), let $B_i$ be of the form (4.2). Then problem (3.2) is equivalent to the following problem: Find $y^{k,i} \in \mathbb{R}^n$ such that

$$
\begin{cases}
y \geq \phi, \\
A_{ij}y_{j} \geq \tilde{F}^{k,i}_{ij}, \\
D_{ij}y_{j} \geq \tilde{F}^{k,i}_{ij}, \\
(y_{j} - \phi_{i,j})^T (A_{ij}y_{j} - \tilde{F}^{k,i}_{ij}) = 0, \\
(y_{j} - \phi_{i,j})^T (D_{ij}y_{j} - \tilde{F}^{k,i}_{ij}) = 0
\end{cases} \quad (4.4)
$$

where $\tilde{F}^{k,i} = F + C_{i}z^{k,i-1}$.
For each $i = 1, \ldots, m$ and $k \in N_0$, let $E_{k,i} = E_i$ be defined as follows:

$$E_i = (e_{ij}) = \begin{cases} 
1, & j = l \in l_i, \\
0, & \text{others.}
\end{cases} \quad (4.5)$$

Then $E_i$ are nonnegative diagonal matrices satisfying $0 \leq E_i \leq I$ and $\sum_{i=1}^{n} E_i \geq I$, with equality if and only if there is no overlap.

**Lemma 4.5.** Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be an $H_+\text{-matrix}$. For each $i = 1, \ldots, m$, $A = B_i - C_i$ where $B_i$ is of the form (4.2). Let $z^{k,i} = z^{k,i-1} + E_i(y^{k,i} - z^{k,i-1})$ where $E_i$ is defined in (4.5). Then $z^{k,i}_i$ is the unique solution of subproblem (4.3) and $z^{k,i}_i = z^{k,i-1}_i$.

**Proof.** From (2.2) and Lemma 4.4, we have $z^{k,i} = z^{k,i-1} + E_i(y^{k,i} - z^{k,i-1})$. Hence, if $E_i$ is defined in (4.5), then in subproblem (4.4) we need only calculate $y^{k,i}_i$, that is to say,

$$z^{k,i} = (z^{k,i}_j) = \begin{cases} 
y^{k,i}_j, & j \in l_i, \\
z^{k,i-1}_j, & \text{others}.
\end{cases} \quad (4.6)$$

Moreover, the $y^{k,i}_i \in \mathbb{R}^n$ satisfy the following low-dimensional linear complementarity problem:

$$\begin{cases} 
y_i \geq \phi_i, \\
A_{ii} y_i - F^{k,i}_i \geq 0, \\
(y_i - \tilde{y}_i)^T (A_{ii} y_i - F^{k,i}_i) = 0.
\end{cases} \quad (4.7)$$

It is obvious that problem (4.7) is equivalent to problem (4.3). Hence $z^{k,i}_i = y^{k,i}_i$ is a solution of problem (4.3), which is unique since $A_{ii}$ is an $H_+\text{-matrix}$. □

The following theorem is a direct consequence of Lemma 4.5.

**Theorem 4.6.** Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$. Then the sequence $\{x^t\}_{k \in N_0}$ generated by Algorithm 2 converges to the unique solution of (1.1).

**Proof.** $A = B_i - C_i$ $(i = 1, \ldots, m)$ are $H$-compatible splittings from Lemma 4.3. By making use of Theorem 4.2 and Lemma 4.5, we immediately reach the conclusion of this theorem. □

**Remark 4.1.** For the system of linear equation (1.2), let $B_{k,i} = B_i$ and $C_{k,i} = C_i$ for each $i = 1, \ldots, m$, $k \in N_0$, the matrices $B_i$ be of the form (4.3) and the matrices $E_i$ be of the form (4.5). Then Algorithm 1 reduces to the overlapping multiplicative Schwarz method proposed in [20]. Moreover, if the coefficient matrix of (1.2) is an $M$-matrix, then Algorithm 1 reduces to the overlapping multiplicative Schwarz method proposed in [16].

**Remark 4.2.** Convergence theorems of the Schwarz method are generally based on the assumption that matrix $A$ is an $M$-matrix; see, e.g., [14, 15]. Generally speaking, these convergence theorems are to prove that the method produces a monotone sequence starting from a super-solution or a sub-solution of the problem, which is not effective for solving (1.1) with an $H_+\text{-matrix}$. Hence, in this paper the multiplicative multisplitting method applied to the multiplicative Schwarz method can be thought of as a new way to prove the convergence theorem for the $H_+\text{-matrix}$.

5. Monotone convergence for the $M$-matrix

We know that Algorithm 1 can be thought of as a multisplitting iteration method. In [21], Bai studied the monotone convergence property of the multisplitting iteration method for LCPs. So we show in this section the similar monotone convergence property of Algorithm 1 when the coefficient matrix $A$ is an $M$-matrix and $A = B_{k,i} - C_{k,i}$ is an $M$-splitting for each $i = 1, \ldots, m$, $k \in N_0$. We first recall the concept of a super-solution [22]. The super-solution set of (1.1) is the set

$$W = \{y \in \mathbb{R}^n \mid y \geq \phi, Ay - F \geq 0\}. \quad (5.1)$$

This set is also called the feasible set of (1.1) in the LCP literature (see, e.g., [23]). It is well known that the solution of (1.1) is the minimal element of the super-solution $W$ if $A$ is an $M$-matrix, but this is not the case if $A$ is an $H$-matrix [2]. Hence, in the following, the coefficient matrix $A$ is always taken as an $M$-matrix.

Let $A$ be an $M$-matrix. For each $i = 1, \ldots, m$ and $k \in N_0$, let $A = B_{k,i} - C_{k,i}$ be an $M$-splitting and the weighting matrix $E_{k,i}$ ($0 \leq E_{k,i} \leq I$, $\sum_{i=1}^{m} E_{k,i} \geq I$) be a nonnegative diagonal matrix satisfying the following condition:

$$B_{k,i} E_{k,i} = E_{k,i} B_{k,i}. \quad (5.2)$$

In fact, this condition gives us a lot of freedom in choosing $B_{k,i}$. Let $B_{k,i}$ be of the form (4.2) and $E_{k,i}$ be of the form (4.5). Then this choice clearly satisfies our condition (5.2).

The following lemma shows that if, at some step $k$, $z^{k,i-1}$ coincides with the unique solution of (1.1), then $0 \in \mathbb{R}^n$ is the unique solution of (2.1).
**Lemma 5.1.** Let \( x^* \) be the unique solution of (1.1). If \( z^{k,i-1} = x^* \), then we have \( x^k = 0 \) for some \( i \in \{1, 2, \ldots, m\} \).

**Proof.** Since \( x^* - \phi \geq 0, Ax^* - F \geq 0 \) and \((x^* - \phi)^T(Ax^* - F) = 0\), we have

\[
\begin{aligned}
0 - \phi^k &= x^* - \phi \\ B_k,i0 - F^k &= Ax^* - F \\
\end{aligned}
\]

(5.3)

Multiplying these two inequalities, we have

\[
0 \leq (0 - \phi^k)(B_k,i0 - F^k) = (x^* - \phi)^T(Ax^* - F) = 0,
\]

and hence,

\[
(0 - \phi^k)^T(B_k,i0 - F^k) = 0.
\]

(5.4)

It follows from (5.3) and (5.4) that \( x^k = 0 \) is a solution of (2.1), which is unique by \( B_k,i \) being an \( M \)-matrix. \( \square \)

**Lemma 5.2.** Let \( x^k \) be the solution of (2.1). If \( z^{k,i-1} \in W \), then inequality \( x^k \leq 0 \) holds for each \( i = 1, \ldots, m \).

**Proof.** Since \( A_{z,k,i-1} - F \geq 0 \) and \( z_{k,i-1} \geq \phi \), we have \( 0 \geq \phi - z_{k,i-1} = \phi^k \) and \( B_k,i0 \geq F - Az_{k,i-1} = F^k \). This implies that \( 0 \in \mathbb{R}^n \) is a super-solution of problem (2.1). Since for each \( i = 1, \ldots, m \), \( B_k,i \) is also an \( M \)-matrix, it follows that \( x^k \) is the minimal element of the super-solution of (2.1) and hence we have \( x^k \leq 0 \).

**Lemma 5.3.** If \( z^{k,i-1} \in W \) for each \( i = 1, \ldots, m, k \in N_0 \), then \( z^{k,i} \in W \).

**Proof.** Since \( z^{k,i-1} \in W \), we have \( z^{k,i-1} \geq \phi \) and \( A_{z,k,i-1} = 0 \). It then follows that

\[
\begin{aligned}
z^{k,i} &= z^{k,i-1} + E_{k,i}x^{k,i} \\
&\geq z^{k,i-1} + E_{k,i}(\phi - z^{k,i-1}) \\
&= \phi + (I - E_{k,i})z^{k,i-1} - (I - E_{k,i})\phi \\
&= \phi + (I - E_{k,i})(z^{k,i-1} - \phi) \\
&\geq \phi,
\end{aligned}
\]

where the first inequality follows from \( 0 \geq z^{k,i-1} \geq \phi - z^{k,i-1} \) and the last inequality follows from \( 0 \leq E_i \leq I \). Furthermore,

\[
\begin{aligned}
A_{z,k,i} - F &= A(z^{k,i-1} + E_{k,i}x^{k,i}) - F \\
&= A_{z,k,i} - F + E_{k,i}E_{k,i}x^{k,i} - C_{k,i}E_{k,i}x^{k,i} \\
&= A_{z,k,i} - F + E_{k,i}E_{k,i}x^{k,i} - C_{k,i}E_{k,i}x^{k,i} \\
&\geq A_{z,k,i} - F - E_{k,i}(A_{z,k,i} - F) - C_{k,i}E_{k,i}x^{k,i} \\
&= (I - E_{k,i})(A_{z,k,i} - F) - C_{k,i}E_{k,i}x^{k,i} \\
&\geq 0
\end{aligned}
\]

where the first inequality follows from the second line of (2.1), the third equality follows from (5.2) and the last equality follows from Lemma 5.2 and \( z^{k,i-1} \in W \). Inequality \( z^{k,i} \geq \phi \) together with inequality \( A_{z,k,i} - F \geq 0 \) implies \( z^{k,i} \in W \). \( \square \)

The following theorem shows the monotone convergence of Algorithm 1 when the coefficient matrix \( A \) is an \( M \)-matrix and \( A = B_{k,i} - C_{k,i} \) is an \( M \)-splitting for each \( i = 1, \ldots, m, k \in N_0 \).

**Theorem 5.4.** Let \( A \) be an \( M \)-matrix and \( A = B_{k,i} - C_{k,i} \) be an \( M \)-splitting for each \( i = 1, \ldots, m, k \in N_0 \). If \( x^0 \in W \), then the sequence \( \{x^k\} \in N_0 \) generated by Algorithm 1 converges to the unique solution \( x^* \) of (1.1). Moreover, for any \( k \in N_0 \),

\[
x^k \in W \quad \text{and} \quad x^* \leq x^{k+1} \leq x^k.
\]

(5.5)

**Proof.** Since \( x^0 \in W \), it follows that \( x^k \in W \) holds from the deduction of Algorithm 1 and Lemma 5.3. Moreover, it follows from (2.2) and Lemma 5.2 that \( x^{k+1} \leq x^k \) for all \( k \in N_0 \). In particular, \( \{x^k\} \) converges to \( x^* \) from Theorem 4.2. Clearly, we have \( x^* \leq x^k \) for all \( k \in N_0 \). \( \square \)

**References**

