Consensus of multi-agent linear dynamic systems via impulsive control protocols

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In this article, we introduce impulsive control protocols for multi-agent linear dynamic systems. First, an impulsive control protocol is designed for network with fixed topology based on the local information of agents. Then sufficient conditions are given to guarantee the consensus of the multi-agent linear dynamic systems by the theory of impulsive systems. Furthermore, how to select the discrete instants and impulsive matrices is discussed. The case that the topologies of networks are switching is also considered. Numerical simulations show the effectiveness of our theoretical results.

Keywords: multi-agent systems; consensus; impulsive control protocols; algebraic graph theory

1. Introduction

Recently, multi-agent systems have been intensively studied in various disciplines, such as mathematical, physical, biological, engineering and social sciences (Vicsek, Czirok, Jacob, Cohen, and Schochet 1995; Jabdabaie, Lin, Morse 2003; Fax and Murray 2004; Savkin 2004; Gao and Cheng 2005; Gu and Hu 2009). Consensus is a kind of typical collective behaviours and basic motions in nature. One of the interesting phenomena in multi-agent systems is the consensus of all agents. The consensus problem is to find the protocols such that all agents can reach an agreement regarding a certain quantity of interest that depends on the states of all agents (Jabdbaie et al. 2003; Fax and Murray 2004; Savkin 2004; Gao and Cheng 2005). Very recently, researchers have discussed some consensus problems, such as directed networks with fixed topology, directed networks with switching topology and undirected networks with time-delay and fixed topology (Hong, Gao, Cheng, and Hu 2007; Xie and Wang 2007; Hong, Chen, and Bushnell 2008; Lin, Jia, Li 2008; Lin and Jia 2008; Porfiri, Stilwell, and Bolli 2008; Ren 2008; Sun, Wang, and Xie 2008; Wang, Chang, and Hu 2008; Liu and Tian 2009; Zhang and Tian 2009). In Olfati-Saber and Murray (2004) and Olfati-Saber, Fax, and Murray (2007), a systematical framework of consensus problem in networks of agents was investigated. The problem of information consensus among multiple agents in the presence of limited and unreliable information exchange with dynamically changing interaction topologies was considered in Ren and Beard (2005). In Hong, Hu, and Gao (2006), the authors considered a multi-agent consensus problem with an active leader and variable interconnection topology. In Xie and Wang (2007), the authors presented the convergence analysis of a consensus protocol for a class of networks of dynamic agents with fixed or switching topology. Consensus algorithms for double-integrator dynamics were proposed in Ren (2008). In Tian and Liu (2008), based on the frequency-domain analysis, the authors studied the consensus problem for multi-agent systems with input and communication delays. In Wang et al. (2008), the authors considered the consensus problem for multi-agent systems, in which all agents have an identical linear dynamic mode that can be of any order.

Meanwhile, the stability analysis and synthesis of systems with impulse have sparked the interest of many researchers (Bainov and Simeonov 1989; Lakshmikantham, Bainov, and Simeonov 1989; Yang 2001; Zhou, Xiang, and Liu 2007; Liu and Liu 2008; Jiang, Yu, and Zhou 2008; Yu, Zhang, Fei, and Jiang 2009; Wu, Xiang, and Zhou 2009). In Zhou et al. (2007), the authors were concerned with the issues of synchronisation dynamics of complex delayed dynamical networks with impulsive effects. In Wu et al. (2009), the authors investigated the problem of average consensus in delayed networks of dynamic agents with impulsive effects. On the other hand, impulsive control provides a new viewpoint when the plant has at least one changeable state variable or when the plant has
impulsive effects. In many cases impulsive control can give an efficient way to deal with plants, which is difficult to provide continuous control or cannot endure continuous control inputs (Yang 2001). It has been shown that impulsive control or synchronisation approach is effective and robust in the control or synchronisation of chaotic systems (Yang 2001; Sun, Zhang, and Wu 2002; Zhang, Jiang, and Bi 2009; Zhang and Sun 2009; Zheng and Chen 2009) and complex dynamical networks (Liu, Liu, and Chen 2005; Cai, Zhou, Xiang, and Liu 2008; Guan and Zhang 2008; Li and Lai 2008; Jiang 2009).

Motivated by the aforementioned discussions, in this article, we introduce impulsive control protocols for multi-agent linear dynamic systems. First, an impulsive protocol is designed for network with fixed topology. The convergence analysis of the consensus problem for multi-agent linear dynamic systems by the theory of impulsive systems. Furthermore, how to select the discrete instants and impulsive matrices is discussed. The case that the topologies of networks are switching is also considered. Numerical simulations show the effectiveness of our theoretical results.

This article is organised as follows. In Section 2, we provide some results in matrix theory and algebraic graph theory. In Section 3, we formulate the consensus problem for multi-agent linear dynamic systems and introduce an impulsive control protocol. The convergence analysis of the consensus problem for network with fixed topology is discussed in Section 4. Section 5 considers the case that the topologies of networks are switching. In Section 6, numerical simulations are included to show the effectiveness of our theoretical results. Some conclusions are drawn in Section 7.

Notation: Throughout this article, the superscripts ‘−1’ and ‘T’ stand for the inverse and transpose of a matrix, respectively; \( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space; Let \( \mathbb{R}_+ = [0, \infty) \), \( \mathbb{N}_+ = \{0, 1, 2, \ldots\} \), \( \mathbb{N}_+ = \{1, 2, \ldots\} \); \( \mathbb{R}^{n \times m} \) is the set of all \( n \times m \) real matrices. For real symmetric matrices \( X \) and \( Y \), the notation \( X \succeq Y \) (respectively, \( X > Y \)) means that the matrix \( X - Y \) is positive semi-definite (respectively, positive definite); \( I_n \in \mathbb{R}^{n \times n} \) is an identity matrix; \( \lambda_{\min}(P) \) (\( \lambda_{\max}(P) \)) denotes the smallest (largest) eigenvalue of \( P \). For a vector \( x \in \mathbb{R}^n \), let \( \|x\| \) denote the Euclidean vector norm, i.e. \( \|x\| = \sqrt{x^T x} \), and for \( A \in \mathbb{R}^{n \times n} \), let \( \|A\| \) indicate the norm of \( A \) induced by the Euclidean vector norm, i.e. \( \|A\| = \sqrt{\lambda_{\max}(A^T A)} \). Let \( 1_N = (1, 1, \ldots, 1)^T \in \mathbb{R}^N \) and \( e_i \in \mathbb{R}^n \), \( e_i(j) = 1 \), \( e_i(j) = 0 \), \( j \neq i \). For a real-valued vector function \( f: \mathbb{R}^n \to \mathbb{R}^n \), we define \( \Delta f(a) = f(a^+)-f(a^-) \), where \( f(a^+) = \lim_{x \to a^+} f(x) \), \( f(a^-) = \lim_{x \to a^-} f(x) \).

2. Preliminaries

In this section, we provide some results in matrix theory (Horn and Johnson 1985; 1991) and algebraic graph theory (Godsil and Royle 2001, Wang et al. 2008).

The Kronecker product of two matrices \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) and \( B = [b_{ij}] \in \mathbb{R}^{p \times q} \) is denoted by \( A \otimes B \) and is defined as

\[
A \otimes B = \begin{bmatrix}
a_{11}B & a_{12}B & \cdots & a_{1n}B \\
a_{21}B & a_{22}B & \cdots & a_{2n}B \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1}B & a_{m2}B & \cdots & a_{mn}B
\end{bmatrix} \in \mathbb{R}^{np \times nq}.
\]

Let \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \), \( B = [b_{ij}] \in \mathbb{R}^{p \times q} \), \( C = [c_{ij}] \in \mathbb{R}^{n \times k} \), \( D = [d_{ij}] \in \mathbb{R}^{q \times r} \). Then

\[
(A \otimes B)(C \otimes D) = (AC) \otimes (BD).
\]

(1)

If \( A \) and \( B \) are invertible, then so is \( A \otimes B \), and

\[
(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.
\]

(2)

An undirected graph \( \mathcal{G} \) of order \( N \) consists of a vertex set \( \mathcal{V} = \{1, 2, \ldots, N\} \) and an edge set \( \mathcal{E} = \{(i, j): i, j \in \mathcal{V}\} \). The set of neighbours of vertex \( i \) is denoted by \( \mathcal{N}_i = \{(j, i) \in \mathcal{E}: j \neq i\} \). A path between each distinct vertices \( i \) and \( j \) is meant as a sequence of distinct edges of \( \mathcal{G} \) of the form \( (i, k_1), (k_1, k_2), \ldots, (k_{l-1}, k_l), (k_l, j) \). If there is a path between any two vertices of a graph \( \mathcal{G} \), then \( \mathcal{G} \) is connected, otherwise disconnected. A weighted adjacency matrix \( \mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N} \), where \( a_{ii} = 0 \) and \( a_{ij} = a_{ji} \geq 0 \), \( i \neq j \), \( a_{ij} > 0 \) if and only if there is an edge between vertex \( i \) and vertex \( j \). For an unweighted graph \( \mathcal{G} \), \( \mathcal{A} \) is a 0–1 matrix. The out-degree of vertex \( i \) is defined as follows \( \deg_{\text{out}}(i) = \sum_{j=1}^n a_{ij} \). Let \( \mathcal{D} \) be the diagonal matrix with the out-degree of each vertex along the diagonal and call it the degree matrix of \( \mathcal{G} \). The Laplacian matrix of the weighted graph is defined as \( L_{\mathcal{G}} = \mathcal{D} - \mathcal{A} \). For an unweighted graph \( \mathcal{G} \),

\[
L_{\mathcal{G}} = [l_{ij}]_{N \times N},
\]

(3)

where

\[
l_{ij} = \begin{cases} |\mathcal{N}_i|, & i = j, \\ -1, & j \in \mathcal{N}_i, \\ 0, & \text{otherwise}. \end{cases}
\]

(4)

By the definition, every row sum of \( L \) is zero.

Lemma 2.1: \( \text{ (Godsil and Royle 2001; Wang et al. 2008) } \)

Let \( L \) be the Laplacian of an undirected graph \( \mathcal{G} \) with \( N \) vertices, \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N \) be the eigenvalues
of L. Then

(1) 0 is an eigenvalue of L and \( \mathbf{1}_N \) is the associated eigenvector, that is, \( L \mathbf{1}_N = 0 \);

(2) If \( \mathcal{G} \) is connected, then \( \lambda_1 = 0 \) is the algebraically simple eigenvalue of \( L \) and

(3) If 0 is the simple eigenvalue of \( L \), then it is an \( n \)-multiplicity eigenvalue of \( L \otimes \mathbf{1}_n \) and the corresponding eigenvectors are \( \mathbf{1}_N \otimes e_i \), \( i = 1, 2, \ldots, n \).

3. Problem formulation

Here we consider a system consisting of \( N \) agents indexed by \( i = 1, 2, \ldots, N \). The dynamics of each agent is

\[
\dot{x}^i(t) = Ax^i(t) + u^i(t), \quad x^i(t) \in \mathbb{R}^n, \quad t \geq t_0 \geq 0, \quad i = 1, 2, \ldots, N, \tag{5}
\]

where \( \dot{x}^i(t) = (x^i_1(t), x^i_2(t), \ldots, x^i_n(t))^T \in \mathbb{R}^n \) and \( u^i(t) \in \mathbb{R}^n \) are the state and the control input of agent \( i \) at time \( t \), respectively. \( A \in \mathbb{R}^{n \times n} \) is a constant matrix. We assume that \( \|A\| \neq 0 \).

The control input of agent \( i \) is designed as

\[
u^i(t) = \sum_{k=1}^{\infty} \delta(t - t_k)B_k \sum_{j \in \mathcal{N}_i} (x^j(t) - x^i(t)), \quad k \in \mathbb{N}_+, \quad i = 1, 2, \ldots, N, \tag{6}\]

where the discrete instants \( t_k \) satisfy \( 0 \leq t_0 < t_1 < t_2 < \cdots < t_{k-1} < t_k < \cdots \) and \( \lim_{k \rightarrow \infty} t_k = +\infty \). \( \delta(t) \) is the Dirac delta function, i.e., \( \delta(t) = 0 \) for \( t \neq 0 \), and \( \int_{-\infty}^{\infty} \delta(t)dt = 1 \). The Dirac delta function has the fundamental property that \( \int_{a}^{b} \delta(t)dt = \delta(a) \) for \( \varepsilon \neq 0 \). In many applications, the Dirac delta function is usually used to model a tall narrow spike function (an impulse). \( B_k \in \mathbb{R}^{n \times n} \) are constant matrices to be designed later, \( \mathcal{N}_i \) is the set of neighbours of agent \( i \). Without loss of generality, we assume that \( \lim_{t \rightarrow t_k^+} x^i(t) = x^i(t_k) \), which means that the solution \( x^i(t) \) is right continuous at time \( t_k \).

Adopting a similar approach to that used in Zheng and Chen (2009) and Guan and Zhang (2008), from (5) and (6) we have

\[
\Delta x^i(t_k + \varepsilon) - \Delta x^i(t_k - \varepsilon) = \int_{t_k - \varepsilon}^{t_k + \varepsilon} (Ax^i(s) + u^i(s))ds, \quad \varepsilon > 0
\]

where \( \varepsilon > 0 \) is sufficiently small. As \( \varepsilon \to 0^+ \), this becomes to \( \Delta x^i(t_k) = x^i(t_k^+) - x^i(t_k^-) = B_k \sum_{j \in \mathcal{N}_i} (x^j(t_k) - x^i(t_k)) \), where \( x^i(t_k^-) = \lim_{t \rightarrow t_k^-} x^i(t) \) and \( x^i(t_k^-) = \lim_{t \rightarrow t_k^+} x^i(t) \). This implies that the agent \( i \) will suddenly update its state variable according to its state variables of itself and its neighbours at the instants \( t_k \). Thus the control input \( u^i(t) \) is called an impulsive control protocol.

Under the impulsive control protocol (6), the dynamics of agent \( i \) satisfy the following equations:

\[
\begin{align*}
\dot{x}^i(t) &= Ax^i(t), \quad t \neq t_k, \\
\Delta x^i(t_k) &= x^i(t_k^+) - x^i(t_k^-) = B_k \sum_{j \in \mathcal{N}_i} (x^j(t_k) - x^i(t_k)), \quad i = 1, 2, \ldots, N, \quad k \in \mathbb{N}_+.
\end{align*}
\]

Definition 3.1: For system (5), the consensus is said to be achieved under the impulsive control protocol (6) if

\[
\lim_{t \to +\infty} ||x^i(t)|| = \lim_{t \to +\infty} ||x^i(t) - x^j(t)|| = 0, \quad i, j = 1, 2, \ldots, N. \tag{7}
\]

Remark 1: The impulsive control protocol (6) is only applied to the multi-agent system (5) at certain discrete instants, which is different from continuous control protocol (Olfati-Saber and Murray 2004; Olfati-Saber et al. 2007; Wang et al. 2008). Impulsive control protocol is an effective method in some cases when the multi-agent systems cannot endure continuous control protocol or it is impossible to give continuous control protocol. For example, it is hard to apply the continuous control protocol every time in the distributed attitude consensus problem among multiple networked spacecraft in deep space (Ren 2007). It is worth pointing out that impulsive control protocol for multi-agent systems has received relatively little attention.

Remark 2: In this article, the objective is to design an impulsive control protocol such that the system (5) can achieve the consensus. However, the objective in a general complex network consisting of \( N \) linear dynamics nodes with impulsive coupling is to analyse the synchronisation problem (Zhou et al. 2007; Wu et al. 2009).

Remark 3: In the study of the standard impulsive control systems, we usually design an impulsive controller and obtain some conditions which guarantee the stability of the overall impulsive control system. However, in the study of the consensus problem of multi-agent systems under the impulsive control protocol, we should select the discrete instants and impulsive matrices to make the multi-agent systems achieve the consensus, which is different from the standard impulsive control systems. Besides, in the real word, the communication topologies of the multi-agent systems are dynamically changing over time. Therefore, the consensus problem of multi-agent systems via the impulsive control protocol in the
presence of the real-world communication constraints is a major challenge.

Before proceeding, we recall some preliminaries on the impulsive systems which will be used throughout the proofs of our main results.

Consider the following impulsive system:

\[
\begin{align*}
\dot{x}(t) &= Ax(t), \quad t \neq t_k, \\
\Delta x(t_k) &= B_k x(t_k^-), \quad k \in \mathbb{N}_+,
\end{align*}
\]

(9)

where \(x(t) \in \mathbb{R}^n\), \(A \in \mathbb{R}^{n \times n}\), \(B_k \in \mathbb{R}^{n \times n}\). The discrete instants \(t_k\) satisfy \(0 \leq t_0 < t_1 < t_2 \ldots < t_{k-1} < t_k < \ldots\) and \(\lim_{k \to +\infty} t_k = +\infty\). \(\Delta x(t_k) = x(t_k^+) - x(t_k^-)\), \(x(t_k^+) = \lim_{t \to t_k^+} x(t)\) and \(x(t_k^-) = \lim_{t \to t_k^-} x(t)\). Without loss of generality, we assume that \(\lim_{t \to t_k^-} x(t) = x(t_k^-)\), which means that the solution \(x(t; t_0, x_0)\) is right continuous at time \(t_k\). Then

\[
x(t; t_0, x_0) = e^{A(t-t_0)} \prod_{i=1}^{k} (I_n + B_i)e^{A(t_{i-1}-t_{i-1})}x_0,
\]

(10)

where \(t_k \leq t < t_{k+1}, k \in \mathbb{N}_+\).

The following result was established in Bainov and Simeonov (1989) and Yang (2001).

**Lemma 3.2:** All solutions of system (9) are asymptotically stable if the conditions (H1) and (H2) hold,

(H1) \(0 < \theta_1 \leq t_k - t_{k-1} \leq \theta_2 < +\infty, k \in \mathbb{N}_+\),

(H2) \(\|U_k\| \leq q < 1, k \in \mathbb{N}_+\),

where \(U_k = (I_n + B_k)e^{A(t_{k-1}-t_{k-1})}\).

4. Network with fixed topology

In this section, we provide the convergence analysis of the consensus problem for network with fixed topology, i.e. \(G(t) = G\) for time \(t\).

Let \(x(t) = (x^1(t), x^2(t), \ldots, x^N(t))^T\), then the system (7) can be described as

\[
\begin{align*}
\dot{x}(t) &= (I_N \otimes A)x(t), \quad t \neq t_k, \\
\Delta x(t_k) &= (I_N \otimes B_k)(-L \otimes I_n)x(t_k^-), \quad k \in \mathbb{N}_+.
\end{align*}
\]

(11)

Since \(L\) is symmetric, there is an orthogonal matrix \(Y \in \mathbb{R}^{N \times N}\) such that

\[
YLY^{-1} = YLY^T = D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N),
\]

where \(\{\lambda_1, \lambda_2, \ldots, \lambda_N\} = \sigma(L)\) is the spectrum of \(L\).

Inspired by Wang et al. (2008), let

\[
\tilde{x}(t) = (Y \otimes I_n)\tilde{x}(t).
\]

Using the properties (1) and (2) of Kronecker product, we have when \(t \in (t_k, t_{k+1})\), \(k \in \mathbb{N}_+\),

\[
\tilde{x}(t) = (Y \otimes I_n)\tilde{x}(t) = (Y \otimes I_n)(I_N \otimes A)x(t) = (Y \otimes I_n)(I_N \otimes A)(Y \otimes I_n)^{-1}\tilde{x}(t)
\]

\[
= (Y \otimes I_n)(I_N \otimes A)(Y \otimes I_n)^{-1}\tilde{x}(t) = (YI_NY^{-1}) \otimes (I_NA)\tilde{x}(t) = (I_N \otimes A)\tilde{x}(t)
\]

and

\[
\Delta \tilde{x}(t_k) = (Y \otimes I_n)\Delta x(t_k) = (Y \otimes I_n)(I_N \otimes B_k)(-L \otimes I_n)x(t_k^-)
\]

\[
= (Y \otimes I_n)(I_N \otimes B_k)(-L \otimes I_n)(Y \otimes I_n)^{-1}\tilde{x}(t_k^-)
\]

\[
= (Y \otimes I_n)(I_N \otimes B_k)(-L \otimes I_n)(Y \otimes I_n)^{-1}(YI_NY^{-1}) \otimes (I_NA)\tilde{x}(t_k^-)
\]

\[
= (-YI_NLY^{-1}) \otimes (I_NB_kI_NI_N)\tilde{x}(t_k^-) = (-D \otimes B_k)\tilde{x}(t_k^-).
\]

Thus (11) becomes

\[
\begin{cases}
\dot{\tilde{x}}(t) = (I_N \otimes A)\tilde{x}(t), \quad t \neq t_k, \\
\Delta \tilde{x}(t_k) = (-D \otimes B_k)\tilde{x}(t_k^-), \quad k \in \mathbb{N}_+.
\end{cases}
\]

(13)

Therefore

\[
\begin{cases}
\dot{\tilde{x}}(t) = A\tilde{x}(t), \quad t \neq t_k, \\
\Delta \tilde{x}(t_k) = -\lambda_iB_k\tilde{x}(t_k^-), \quad i = 2, \ldots, N, \quad k \in \mathbb{N}_+.
\end{cases}
\]

(14)

**Theorem 4.1:** Consider system (5). Assume that the graph \(G\) of the network is connected. If there exist discrete instants \(t_k\) and impulsive matrices \(B_k\) such that the conditions (H3) and (H4) hold, then the consensus is achieved under the impulsive control protocol (6).

(H3) \(0 < \theta_1 \leq t_k - t_{k-1} \leq \theta_2 < +\infty, \quad k \in \mathbb{N}_+\),

(H4) \(\|V_k\| \leq q < 1, \quad i = 2, \ldots, N, \quad k \in \mathbb{N}_+\),

where \(V_k = (I_n - \lambda_iB_k)e^{A(t_{k-1}-t_{k-1})}\).

**Proof:** Since the graph \(G\) is connected, by Lemma 2.1, \(\lambda_1 = 0\) is the algebraically simple eigenvalue of \(L\), and all the other eigenvalues of \(L\) are positive. Then we have

\[
0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N.
\]

By Lemma 3.2, it follows that if there exist discrete instants \(t_k\) and impulsive matrices \(B_k\) such that the conditions (H3) and (H4) hold, then the system (14) is asymptotically stable, i.e. \(\tilde{x}(t) \to 0, \ t \to +\infty, \ i = 2, \ldots, N\).
It can be verified that
\[
(L \otimes I_n)x(t) = (Y \otimes I_n)^{-1}(Y \otimes I_n)(L \otimes I_n)(Y^{-1} \otimes I_n)\tilde{x}(t)
\]
\[
= (Y \otimes I_n)^{-1}(D \otimes I_n)\tilde{x}(t)
\]
\[
= (Y \otimes I_n)^{-1}\begin{bmatrix}
0 & \lambda_2 \tilde{x}_2^{(t)} \\
\vdots & \vdots \\
\lambda_N \tilde{x}_N^{(t)}
\end{bmatrix}.
\]
Hence \((L \otimes I_n)x(t) \to 0, t \to \infty\). Since the graph \(\mathcal{G}\) is connected, by Lemma 2.1, 0 is the eigenvalue of \(L\) with multiplicity \(n\). The \(n\) linearly independent eigenvectors associated with the eigenvalue 0 of \(L \otimes I_n\) are \(1_N \otimes e_i, i = 1, 2, \ldots, n\). Therefore \(x \to 1_N \otimes s, t \to +\infty\), where \(s = \sum_{i=1}^{n} \xi_i e_i \in \mathbb{R}^n, \xi_i \in \mathbb{R}, i = 1, 2, \ldots, n\). Thus system (5) achieves the consensus under the impulsive control protocol (6). This completes the proof. \(\square\)

**Remark 4:** Theorem 4.1 gives the sufficient conditions to guarantee the consensus for multi-agent system (5) under the impulsive control protocol (6). However, for any multi-agent system in the form of (5), if the graph \(\mathcal{G}\) of the network is connected, we can always choose proper discrete instants \(t_k\) and impulsive matrices \(B_k\) such that the conditions (H3) and (H4) hold. For simplicity, we choose the equidistant impulsive interval \(\Delta_k = t_k - t_{k-1} = \Delta, k \in \mathbb{N}_+\). The impulsive matrices \(B_k, k \in \mathbb{N}_+,\) are chosen as \(p I_n, p \in \mathcal{H}\). If \(0 < p < 1/\lambda_2\) and
\[
0 < \Delta < \frac{\ln\frac{1}{1-\lambda_2 p}}{\|A\|},
\]
then \(\|V_{\tilde{h}}\| \leq (1 - \lambda_2 p)e^{\|A\|\Delta} = q < 1\).

5. Networks with switching topology

In this section, we provide the convergence analysis of the consensus problem for networks with switching topology.

Here we consider \(m\) graphs indexed by \(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m\). We define a switching signal \(\sigma: [t_0, +\infty) \to \{1, 2, \ldots, m\}\). The switching signal is a piecewise constant right continuous function.

The following impulsive control protocol is applied to agent \(i\):
\[
\dot{u}_i(t) = \sum_{k=1}^{\infty} \delta(t-t_k)B_k \sum_{j \in A_i(t)} (x_j(t) - x_i(t)), \quad k \in \mathbb{N}_+,
\]
(15)
where the discrete instants \(t_k\) satisfy
\[
0 \leq t_0 < t_1 < t_2 < \cdots < t_{k-1} < t_k < \cdots \text{ and } \lim_{k \to \infty} t_k = +\infty, B_k \in \mathbb{R}^{p \times n}\]
are constant matrices to be designed later. \(\mathcal{N}_i(t)\) is the set of neighbours of agent \(i\) at time \(t\). Suppose that \(\sigma(t_k) = \tau_k^i\) and its graph is \(\mathcal{G}_{\tau_k^i}\) with the Laplacian \(L_{\tau_k^i}\), where \(\tau_k^i \in \{1, 2, \ldots, m\}, k \in \mathbb{N}_+\).

Under the impulsive control protocol (15), the dynamics of agent \(i\) satisfies the following equations:
\[
\left\{\begin{array}{l}
\dot{x}_i(t) = A x_i(t), \quad t \neq t_k, \\
\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = B_k \sum_{j \in A_{\tau_k^i}} (x_j(t_k^+) - x_i(t_k^-)),
\end{array}\right.
\]
(16)
\[
\quad i = 1, 2, \ldots, N, \quad k \in \mathbb{N}_+.
\]
Let \(x(t) = (x^1(t), x^2(t), \ldots, x^N(t))^T\), then the system (16) can be described as
\[
\left\{\begin{array}{l}
\dot{x}(t) = (I_N \otimes A)x(t), \quad t \neq t_k, \\
\Delta x(t_k) = (I_N \otimes B_k)(-L_{\tau_k^i} \otimes I_n)x(t_k^-), \quad k \in \mathbb{N}_+.
\end{array}\right.
\]
(17)
Since \(L_{\tau_k^i}\) is symmetric, there is an orthogonal matrix \(Y_{\tau_k^i} \in \mathbb{R}^{N \times N}\) such that
\[
Y_{\tau_k^i} L_{\tau_k^i} Y_{\tau_k^i}^{-1} = Y_{\tau_k^i} L_{\tau_k^i} Y_{\tau_k^i}^T = D_{\tau_k^i} = \text{diag} \{\lambda_1^{\tau_k^i}, \lambda_2^{\tau_k^i}, \ldots, \lambda_N^{\tau_k^i}\},
\]
where \(\{\lambda_1^{\tau_k^i}, \lambda_2^{\tau_k^i}, \ldots, \lambda_N^{\tau_k^i}\} = \sigma(L_{\tau_k^i})\) is the spectrum of \(L_{\tau_k^i}\).

Let
\[
\tilde{x}(t) = (Y_{\tau_k^i} \otimes I_n)x(t), \quad t \in [t_k, t_{k+1}), k \in \mathbb{N}_+,
\]
(18)
then (17) can be transformed into
\[
\left\{\begin{array}{l}
\dot{\tilde{x}}(t) = (I_N \otimes A)\tilde{x}(t), \quad t \neq t_k, \\
\Delta \tilde{x}(t_k) = (-D_{\tau_k^i} \otimes B_k)\tilde{x}(t_k^-), \quad k \in \mathbb{N}_+.
\end{array}\right.
\]
(19)
Therefore
\[
\left\{\begin{array}{l}
\dot{\tilde{x}}(t) = A \tilde{x}(t), \quad t \neq t_k, \\
\Delta \tilde{x}(t_k) = -\lambda_i^{\tau_k^i} B_k \tilde{x}(t_k^-), \quad i = 2, \ldots, N, \quad k \in \mathbb{N}_+.
\end{array}\right.
\]
(20)

**Theorem 5.1:** Consider system (5). Assume that the networks are switching and the graphs \(\mathcal{G}_i, i = 1, 2, \ldots, m,\) are connected. If there exist discrete instants \(t_k\) and impulsive matrices \(B_k\) such that the conditions (H5) and (H6) hold then the consensus is achieved under the impulsive control protocol (15).

(H5) \(0 < \theta_1 \leq t_k - t_{k-1} \leq \theta_2 < \infty, \quad k \in \mathbb{N}_+\),

(H6) \(\|W_{\tilde{h}}\| \leq q < 1, \quad i = 2, \ldots, n, \quad k \in \mathbb{N}_+\),

where \(W_{\tilde{h}} = (I_n - \lambda_i^{\tau_k^i} B_k)e^{\|A(t_k-t_{k-1})\|}\).

**Proof:** Since the graph \(\mathcal{G}_{\tau_k^i}\) is connected, it follows that
\[
0 = \lambda_1^{\tau_k^i} < \lambda_2^{\tau_k^i} \leq \cdots \leq \lambda_N^{\tau_k^i}.
\]
Let $\bar{x}(t; t_0, x_0)$ be the solution of (20), then

$$
\bar{x}(t; t_0, x_0) = e^{(t-t_0)k} \left( \sum_{j=1}^{k} \left( I_n - \lambda_j^{-1} B_j \right) e^{(t_t-j-1)k} \right) x_0,
$$

where $t_k \leq t < t_{k+1}, k \in \mathbb{N}_+$. From (21), we know that if there exist discrete instants $t_k$ and impulsive matrices $B_k$ such that the conditions (H5) and (H6) hold, then the system (20) is asymptotically stable, i.e., $\bar{x}(t) \to 0, t \to +\infty, i = 2, \ldots, N$. Therefore, for any given $\varepsilon > 0$, there exists a constant $T$, such that $\|\bar{x}(t)\| < \varepsilon, t > T$.

Inspired by Wang et al. (2008), define the synchronisation manifold

$$
\mathcal{M} = \{ x \in \mathbb{R}^{mN}: x^1 = x^2 = \cdots = x^N \} = \{ I_N \otimes s| s \in \mathcal{M} \}.
$$

For any $x(t) \in \mathbb{R}^{mN}$, we can decompose it by

$$
x(t) = (I_N \otimes s(t)) + \eta(t),
$$

where $I_N \otimes s(t) \in \mathcal{M}, \eta(t) \in \mathcal{M}^\perp$, $\mathcal{M}^\perp$ is the orthogonal complement of $\mathcal{M}$. Then we have, when $t \in [t_k, t_{k+1})$,

$$
\left( L_{t_k}^{-1} \otimes I_n \right) (I_N \otimes s(t)) = \left( L_{t_k}^{-1} \otimes I_n \right) (I_N \otimes (s(t) - s(t_0))) = 0.
$$

For some $t > T$, one can choose an integer $k'$ such that $t \in [t_{k'}, t_{k'+1})$. It can be verified that

$$
\left( L_{t_k}^{-1} \otimes I_n \right) \eta(t) = \left( Y_{t_k}^{-1} \otimes I_n \right) (L \otimes I_n) \bar{x}(t) \left( Y_{t_k}^{-1} \otimes I_n \right)^{-1} (Y_{t_k}^{-1} \otimes I_n) \bar{x}(t) = \left( Y_{t_k}^{-1} \otimes I_n \right)^{-1} (D_{t_k} \otimes I_n) \bar{x}(t) = \left( Y_{t_k}^{-1} \otimes I_n \right)^{-1} \left[ \begin{array}{c} \lambda_{t_k}^{-1} \bar{x}(t) \\ \vdots \\ \lambda_{N t_k}^{-1} \bar{x}(t) \end{array} \right].
$$

Since the graph $\mathcal{G}_{t_k}$ is connected, by Lemma 2.1, 0 is the eigenvalue of $L_{t_k}^{-1} \otimes I_n$ with multiplicity $n$. The $n$ linearly independent eigenvectors associated with the eigenvalue 0 of $L_{t_k}^{-1} \otimes I_n$ are $I_N \otimes e_i, i = 1, 2, \ldots, n$. Since $\eta(t) \in \mathcal{M}^\perp$, then $\eta(t) \perp (I_N \otimes e_i), i = 1, 2, \ldots, n$. We have

$$
\alpha \|\eta(t)\| \leq \lambda_{t_k}^{-1} \|\eta(t)\| \leq \|(L_{t_k}^{-1} \otimes I_n) \eta(t)\| \leq \beta \|\bar{x}(t)\|,
$$

where $\alpha = \min_{i=1}^{m} \lambda_i^{1/2}, \beta = \max_{i=1}^{m} \lambda_i^{1/2}, \nu = \max_{i=1}^{m} \|Y_i^{-1}\|$. Therefore

$$
\|\eta(t)\| \leq \frac{\beta \|\bar{x}(t)\|}{\alpha} \leq \frac{\beta \|I_N \otimes (s(t) - s(t_0))\|}{\alpha} \leq \frac{\beta \|N \otimes (s(t) - s(t_0))\|}{\alpha} \leq \frac{\beta \|N \otimes (s(t) - s(t_0))\|}{\alpha} \leq \frac{\beta \gamma (N-1)}{\alpha} \varepsilon.
$$

Hence $\eta(t) \to 0, t \to +\infty$. Thus system (5) achieves the consensus under the impulsive control protocol (15). This completes the proof.}

**Remark 5:** Theorem 5.1 gives the sufficient conditions to guarantee the consensus for multi-agent system (5) with switching topology under the impulsive control protocol (15). However, for any multi-agent system in the form of (5), if the networks are switching and the graphs $\mathcal{G}_i, i = 1, 2, \ldots, m$, are connected, we can always choose proper discrete instants $t_k$ and impulsive matrices $B_k$ such that the conditions (H5) and (H6) hold. For simplicity, we choose the equidistant impulsive interval $\Delta t_k = t_k - t_{k-1} = \Delta, k \in \mathbb{N}_+$. The impulsive matrices $B_k, k \in \mathbb{N}_+$, are chosen as $p I_n, p \in \mathbb{R}$. Let $\alpha = \min_{i=1}^{m} \lambda^{1/2}, \beta = \max_{i=1}^{m} \lambda^{1/2}$. If $0 < p < 1/\beta$ and

$$
0 < \Delta < \frac{1}{\beta ap} \|A\|_\mathbb{R} \leq \frac{1}{\beta ap} \|A\|_\mathbb{R} = q < 1.
$$

**Remark 6:** We are only concerned with the topologies of networks at discrete instants $t_k^-$. Thus the dwell time does not need any lower bound (Wang et al. 2008).

### 6. Simulations

**Example 6.1:** Consider the following multi-agent system (Wang et al. 2008):

$$
\dot{x}(t) = A x(t) + u(t), \quad x(t) \in \mathbb{R}^3, \quad i = 1, 2, 3, 4,
$$

where

$$
A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}.
$$

The initial values are chosen as

$$
x(0) = \begin{bmatrix} 4 \\ -1 \\ -4 \end{bmatrix}, \quad x^2(0) = \begin{bmatrix} -4 \\ 6 \\ 3 \end{bmatrix}, \quad x^3(0) = \begin{bmatrix} -5 \\ 5 \\ 7 \end{bmatrix}, \quad x^4(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.
$$

A fourth-order Runge–Kutta method with step size 0.01 is used in the simulation. Simulation results for the multi-agent system (22) with $u(t) = 0$ are shown in Figures 1–3. From Figures 1–3, it is shown that the multi-agent system (22) with $u(t) = 0$ cannot achieve consensus.
From Figure 4, the Laplacian matrices of graphs $G_i$, $i=1, 2, 3, 4$ are obtained as follows:

$L_1 = \begin{bmatrix}
1 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 1
\end{bmatrix}$

$L_2 = \begin{bmatrix}
2 & -1 & 0 & -1 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
-1 & 0 & -1 & 2
\end{bmatrix}$

$L_3 = \begin{bmatrix}
3 & -1 & -1 & -1 \\
-1 & 2 & -1 & 0 \\
-1 & -1 & 3 & -1 \\
-1 & 0 & -1 & 2
\end{bmatrix}$

$L_4 = \begin{bmatrix}
3 & -1 & -1 & -1 \\
-1 & 3 & -1 & -1 \\
-1 & -1 & 3 & -1 \\
-1 & -1 & -1 & 3
\end{bmatrix}$

Consider the graph $G_1$. For simplicity, the impulsive matrices $B_k$, $k \in \mathbb{N}_+$, are chosen as $p \cdot I_4$, where $0 < p = 0.29 < 1/\lambda_4 = 1/3.4142 = 0.2929$. Choose the equidistant impulsive interval $\Delta t_k = t_k - t_{k-1} = \Delta = 0.1 < (\ln \frac{1}{1-2p})/\|A\| = 0.1075$, where $\lambda_4 = 0.5858$, $\|A\| = 1.7321$, $k \in \mathbb{N}_+$. It is easy to check that $\|V_k\| \leq (1 - \lambda_2 p)e^{k\|A\|} = 0.9871 < 1$. Thus the conditions (H3) and (H4) of Theorem 4.1 are satisfied. Simulation results for $G_1$ are shown in Figures 5–7. The simulation
results show that the impulsive protocol is efficient to solve the consensus problem for network with fixed topology.

In the following, we give simulation results of the consensus problem for networks with switching topology. Here we consider four graphs indexed by \( G_1, G_2, G_3, G_4 \). We define a switching signal \( \sigma: [t_0, +\infty) \rightarrow \{1, 2, 3, 4\}, \sigma(t) = (100t \mod 4) + 1 \). For simplicity, the impulsive matrices \( B_k, k \in \mathbb{N}_4 \), are chosen as \( p_4I_4 \), where \( \beta = \max_{i=1}^4 \lambda_{i}^j = 4, \quad 0 < p = 0.245 < 1, \beta = 0.25 \). Choose the equidistant impulsive interval \( \Delta t_k = t_k - t_{k-1} = \Delta = 0.08 < (\ln(1/\alpha))/\|A\| = 0.0894 \), where \( \alpha = \min_{i=1}^4 \lambda_i^j = 0.5858, \quad k \in \mathbb{N}_4 \). It is easy to check that \( \|W_i\| \leq (1 - \alpha p)\alpha^{1/4}\Delta = 0.9838 < 1 \). Thus the conditions (H5) and (H6) of Theorem 5.1 are satisfied. Simulation results are shown in Figures 8–10. It can be seen from Figures 5–10 that the switching topology affects the performance of the consensus of multi-agent systems heavily.

7. Conclusion
In this article, we have introduced impulsive control protocols for multi-agent linear dynamic systems. Convergence analysis of the impulsive control protocol
for the networks with fixed or switching topology is presented. Numerical simulations are given to show the effectiveness of our theoretical results.

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