Structural Controllability of Switched Linear Systems*

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Abstract

In this paper, the structural controllability of switched linear systems is investigated. In particular, switched linear systems with independent subsystem models are studied. The structural controllability is a generalization of the traditional controllability concept for dynamical systems, and purely based on the graphic topologies among state and input vertices. First, two kinds of graphic representations of switched linear systems are proposed. Second, graph theory based necessary and sufficient characterizations of the structural controllability for switched linear systems are presented. Finally, the paper concludes with one illustrative example and discussions on the results and future work.

Index Terms

Structural controllability, switched linear system, graphic interpretation.

I. INTRODUCTION

As a special class of hybrid control systems, a switched linear system consists of several linear subsystems and a rule that orchestrates the switching among them. Switching between different subsystems or different controllers can greatly enrich the control strategies and may accomplish certain control objectives which can not be achieved by conventional dynamical systems. For example, it provided an effective mechanism to cope with highly complex systems and/or systems with large uncertainties [1]. References [2][3] presented good examples that switched

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controllers could provide a performance improvement over a fixed controller. Besides, switched linear systems also have promising applications in control of mechanical systems, aircrafts and satellites and kinds of multi-agents systems [4]. Driven by its importance in both theoretical research and practical applications, switched linear system has attracted considerable attention during the last decade [5]-[15].

Much work has been done on the controllability of switched linear systems. For example, the controllability and reachability for low-order switched linear systems have been presented in [6][7]. Under the assumption that the switching sequence is fixed, references [8][9] introduced some sufficient conditions and necessary conditions for controllability of switched linear systems. Complete geometric criteria for controllability and reachability were established in [10][12]. The authors in [13] studied the controllability of switched bilinear systems using Lie algebraic techniques. In [14], the controllability, reachability and switching sequence design problem of switched systems were deeply investigated.

Up to now, all the previous work mentioned above has been based on the traditional controllability concept of switched linear systems. In this paper, we propose a new notion for the controllability of switched linear system: structural controllability, which may present more practical significance. Actually, when people try to obtain the models of physical processes, a more realistic situation is that most of system parameter values are known only with the approximation of some errors of measurement. Only the zero elements that are fixed either by coordination or by the absence of physical connections among certain parts of the system can be known with 100 percent precision. Thus we will assume here that all the elements of matrices of switched linear systems are fixed zeros or free parameters. Such kind of switched linear systems would represent a large class of parameter dependent switched linear systems. Furthermore, the switched linear system is said to be structurally controllable if one can find a set of values for the free parameters such that the corresponding switched linear system is controllable in the classical sense. For linear structured systems, generic properties including structural controllability have been studied deeply and it turns out that generic properties including structural controllability are true for almost all values of the parameters [16]-[24]. That is also one of the reasons why this kind of structural controllability is so valuable and attracts our great interest.

No matter the traditional controllability or the structural controllability of switched linear systems, all the results achieved were algebraic conditions. However, it remains elusive on what
exactly is the graphic meaning of these algebraic conditions. Graphic conditions can help to understand how the graphic topologies of dynamical systems influence the corresponding generic properties, here especially for the structural controllability. This would be of great significance in many practical applications. For example, in multi-agent systems, graphic interpretations for structural controllability help us to understand the necessary information exchange among agents to make the whole team well-behaved, e.g., controllable. Therefore, this motivates our pursuit on illuminating the structural controllability of switched linear systems from a graph theoretical point of view. In this paper, we propose two graphic representations of switched linear systems and finally, it turns out that the structural controllability of switched linear systems only depends on the graphic topologies of the corresponding systems.

The organization of this paper is as follows: In Section II, we introduce some basic preliminaries, followed by structural controllability study of general switched linear systems in Section III, where several graphic necessary and sufficient conditions for the structural controllability are given. One illustrative example is also presented to illustrate the proposed results. Finally, some concluding remarks are drawn in Section IV.

II. Preliminaries and Problem Formulation

A. Graph Theory Preliminaries

Consider a linear control system:

\[ \dot{x} = Ax(t) + Bu(t), \]  

where \( x(t) \in \mathbb{R}^n \) and \( u(t) \in \mathbb{R}^r \). The matrices \( A \) and \( B \) are structured matrices, which means that their elements are either fixed zeros or free parameters. The structured system \((A, B)\) can be described by a directed graph [16]:

**Definition 1:** The representation graph of structured system \((A, B)\) is a directed graph \( \mathcal{G} \), with vertex set \( \mathcal{V} = X \cup \mathcal{U} \), where \( X = \{x_1, x_2, \ldots, x_n\} \), which is called state vertices set and \( \mathcal{U} = \{u_1, u_2, \ldots, u_r\} \), which is called input vertices set, and edge set \( \mathcal{I} = \mathcal{I}_{UX} \cup \mathcal{I}_{XX} \), where \( \mathcal{I}_{UX} = \{(u_i, x_j) | B_{ji} \neq 0, 1 \leq i \leq r, 1 \leq j \leq n\} \) and \( \mathcal{I}_{XX} = \{(x_i, x_j) | A_{ji} \neq 0, 1 \leq i \leq n, 1 \leq j \leq n\} \) are the oriented edges between inputs and states and between states defined by the interconnection matrices \( A \) and \( B \) above. This directed graph (for notational simplicity, we will use digraph to refer to directed graph) \( \mathcal{G} \) is also called the graph of matrix pair \((A, B)\) and denoted by \( \mathcal{G}(A, B) \).
Note that the total number of vertices in $G(A, B)$ equals to the summation of states number and inputs number. Several important graphic definitions are needed before we proceed forward:

**Definition 2:** *(Stem [16])* An alternating sequence of distinct vertices and oriented edges is called a directed path. A stem is an acyclic, directed path in the state vertex set $X$, that begins in the input vertex set $U$.

**Definition 3:** *(Bud [16])* A bud is an elementary cycle in $X$ with an additional edge that ends in a vertex of the cycle, where an elementary cycle is a closed and simple path in $X$, i.e., a path in $X$ of the form $(v_0, v_1), (v_1, v_2), \ldots, (v_p, v_0)$, consisting of distinct vertices $v_0, v_1, v_2, \ldots, v_p$ and having the same begin and end vertex $v_0$.

Examples of stem and bud can be seen in Fig. 1, 2 respectively. Two graphic properties ‘accessibility’ and ‘dilation’ were proposed by [16], which will serve as the basis of following discussion. We state them as follows:

**Definition 4:** *(Accessibility [16])* A vertex (other than the input vertices) is called nonaccessible if and only if there is no possibility of reaching this vertex through any stem of the graph $G$.

**Definition 5:** *(Dilation [16])* Consider one vertex set $S$ formed by the vertices from the state vertices set $X$ and determine another vertex set $T(S)$, which contains all the vertices $v$ with the property that there exists an oriented edge from $v$ to one vertex in $S$. Then the graph $G$ contains a ‘dilation’ if and only if there exist at least a set $S$ of $k$ vertices in the vertex set of the graph such that there are no more than $k - 1$ vertices in $T(S)$.

![Fig. 1. One stem in a digraph $G$](image1)

![Fig. 2. One bud in a digraph $G$](image2)
B. Switched Linear System, Controllability and Structural Controllability

In general, a switched linear system is composed of a family of subsystems and a rule that governs the switching among them, and is mathematically described by

\[
\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t),
\]

(2)

where \( x(t) \in \mathbb{R}^n \) are the states, \( u(t) \in \mathbb{R}^r \), are piecewise continuous input, \( \sigma : [t_0, \infty) \to \{1, \ldots, m\} \) is the switching signal. System (2) contains \( m \) subsystems \( (A_i, B_i) \), \( i \in \{1, \ldots, m\} \) and \( \sigma(t) = i \) implies that the \( i \)th subsystem \( (A_i, B_i) \) is activated at time instance \( t \).

In the sequel, the following definition of controllability of system (2) will be adopted (definition 7 in [10]):

**Definition 6:** Switched linear system (2) is said to be (completely) controllable if for any initial state \( x_i \) and final state \( x_f \), there exist a time instance \( t_f > t_0 \), a switching signal \( \sigma : [t_0, t_f) \to \{1, \ldots, m\} \) and an input \( u : [t_0, t_f) \to \mathbb{R}^r \) such that \( x(t_0) = x_i \) and \( x(t_f) = x_f \).

For the controllability problem of switched linear systems, a well-known matrix rank condition was given in [11]:

**Lemma 1:** (11) If matrix:

\[
\begin{bmatrix}
B_1, \ldots, B_m, A_1B_1, \ldots, A_mB_1, \ldots, A_mA_1B_1, \ldots, A_m^2B_m, \ldots, A_mA_1B_m, \ldots,
A_1^{n-1}B_1, \ldots, A_mA_1^{n-2}B_1, \ldots, A_1A_m^{n-2}B_m, \ldots, A_m^{n-1}B_m
\end{bmatrix}
\]

(3)

has full row rank \( n \), then switched linear system (2) is controllable, and vice versa.

**Remark 1:** This matrix is called controllability matrix of switched linear system (2) and for simplicity, we will use \( C(A_1, \ldots, A_m, B_1, \ldots, B_m) \) to represent it. If we use \( \text{Im}P \) to represent the range space of arbitrary matrix \( P \), actually, \( \text{Im}C(A_1, \ldots, A_m, B_1, \ldots, B_m) \) is the controllable subspace of switched linear system (2) [10][11]. The above lemma implies that system (2) is controllable if and only if \( \text{Im}C(A_1, \ldots, A_m, B_1, \ldots, B_m) = \mathbb{R}^n \). Besides, controllable subspace can be expressed as \( \langle A_1, \ldots, A_mB_1, \ldots, B_m \rangle \), which is the smallest subspace containing \( \text{Im}B_i \), \( i = 1, \ldots, m \) and invariant under the transformations \( A_1, \ldots, A_m \) [15].

For studying structural controllability, structured switched linear system is defined as:

**Definition 7:** For structured system (2), the elements of all the matrices \( (A_1, B_1, \ldots, A_m, B_m) \) are either fixed zero or free parameters and free parameters in different subsystems \( (A_i, B_i), i \in M \) are independent. A numerically given matrices \( (\tilde{A}_1, \tilde{B}_1, \ldots, \tilde{A}_m, \tilde{B}_m) \) is called an admissible numerical
realization (with respect to \((A_1, B_1, \ldots, A_m, B_m)\)) if it can be obtained by fixing all free parameter entries of \((A_1, B_1, \ldots, A_m, B_m)\) at some particular values.

Similar with the definition of structural controllability of linear system in [23], we have the following definition for structural controllability of switched linear system (2):

**Definition 8:** Switched linear system (2) given by its structured matrices \((A_1, B_1, \ldots, A_m, B_m)\) is said to be structurally controllable if and only if there exists at least one admissible realization \((\tilde{A}_1, \tilde{B}_1, \ldots, \tilde{A}_m, \tilde{B}_m)\) such that the corresponding switched linear system is controllable in the usual numerical sense.

Before proceeding further, we need to introduce the definition of \(g\)-rank of one matrix:

**Definition 9:** The generic rank \((g\text{-rank})\) of a structured matrix \(M\) is defined to be the maximal rank that \(M\) achieves as a function of its free parameters.

Then, we have the following algebraic condition for structural controllability of system (2):

**Lemma 2:** Switched linear system (2) is structurally controllable if and only if

\[
g\text{-rank} C(A_1, A_m, B_1, B_m) = n.
\]

### III. Structural Controllability of Switched Linear Systems

A. Criteria Based on Union Graph

For switched linear system (2), let’s use \(G_i\) with vertex set \(V_i\) and edge set \(I_i\) to represent the underlying graph of subsystem \((A_i, B_i)\), \(i \in \{1, \ldots, m\}\).

As to the whole switched system, one kind of representation graph, which is called union graph, is described in the following definition:

**Definition 10:** Switched linear system (2) can be represented by a union digraph (sometimes named union graph without leading to confusion), defined as a flow structure \(G\). Mathematically, \(G\) is defined as

\[
G_1 \cup G_2 \cup \ldots \cup G_m = \{V_1 \cup V_2 \cup \ldots \cup V_m; I_1 \cup I_2 \cup \ldots \cup I_m\}
\]

For the union graph \(G\), the vertex set is the same as the vertex set of every subgraph \(G_i\). The edge set of \(G\) equals to the union of the edge sets of the subgraphs. Note that there are no multiple edges between any two vertices in \(G\).

**Remark 2:** It turns out that union graph \(G\) is the representation of linear structured system: \((A_1 + A_2 + \ldots + A_m, B_1 + B_2 + \ldots + B_m)\). The reason is that: If the element at position \(a_{ji}(b_{ji})\) in
\[ A_1 + A_2 + \ldots + A_m, B_1 + B_2 + \ldots + B_m \] is a free parameter, this implies that there exist some matrices \([A_p, B_p], p = 1, \ldots, m\) such that the element at position \(a_{ji}(b_{ji})\) is also a free parameter and in the corresponding subgraph \(G_p\), there is an edge from vertex \(i\) to vertex \(j\). According to the definition of union graph, it follows that there is also an edge from vertex \(i\) to vertex \(j\) in union graph \(G\). If the element at position \(a_{ji}(b_{ji})\) in \([A_1 + A_2 + \ldots + A_m, B_1 + B_2 + \ldots + B_m]\) is zero, this implies that for every matrices \([A_p, B_p], p = 1, \ldots, m\), the element at position \(a_{ji}(b_{ji})\) is zero and in the corresponding subgraph \(G_p\), there is no edge from vertex \(i\) to vertex \(j\). It follows that there is also no edge in union graph \(G\) from vertex \(i\) to vertex \(j\).

The following lemma, which will underpin the following analysis on switched linear systems, details the criteria for evaluating structural controllability of linear system \((A, B)\) [16]-[23]:

**Lemma 3:** ([16]-[23]) For a linear system \((A, B)\), the following statements are equivalent:

a) the pair \((A, B)\) is structurally controllable;

b) i) \([A, B]\) is irreducible or not of form I,

   ii) \([A, B]\) has \(g\)-rank \([A, B] = n\) or is not of form II;

c) i) there is no nonaccessible vertex in \(G(A, B)\),

   ii) there is no ‘dilation’ in \(G(A, B)\).

This lemma proposed interesting graphic conditions for structural controllability of linear systems and revealed that the structural controllability is totally determined by the underlying graph topology. However, how about in switched linear systems? Can we also find some kinds of graph which can determine the structural controllability properties of switched linear systems?

With the previous lemmas and definitions, we are in the position to present the first main result of the paper, which is actually one graphic sufficient condition for structural controllability of switched linear systems:

**Theorem 1:** Switched linear system (2) with graphic topologies \(G_i, i \in \{1, \ldots, m\}\), is structurally controllable if the union graph \(G\) satisfies:

i) there is no nonaccessible vertex in \(G\),

ii) there is no ‘dilation’ in \(G\).

**Proof:** Assume the two conditions in this theorem are satisfied. According to remark [2] and lemma [3], the corresponding linear system \((A_1 + A_2 + \ldots + A_m, B_1 + B_2 + \ldots + B_m)\) is structurally controllable. It follows that there exist some scalars for the free parameters in matrices \((A_i, B_i), i =
1, 2, \ldots, m$ such that controllability matrix

$$[B_1 + B_2 + \ldots + B_m, (A_1 + A_2 + \ldots + A_m)(B_1 + B_2 + \ldots + B_m),$$

$$(A_1 + A_2 + \ldots + A_m)^2(B_1 + B_2 + \ldots + B_m), \ldots, (A_1 + A_2 + \ldots + A_m)^{n-1}(B_1 + B_2 + \ldots + B_m)]$$

has full row rank $n$. Expanding the matrix, it follows that matrix

$$[B_1 + B_2 + \ldots + B_m, A_1B_1 + A_2B_1 + \ldots + A_mB_1 + A_1B_2 + A_2B_2$$

$$+ \ldots + A_mB_2 + \ldots + A_1B_m + A_2B_m + \ldots + A_mB_m, A_2B_1, \ldots, A_mB_m, \ldots,$$

$$A_1^{n-1}B_1 + A_2A_1^{n-2}B_1 + \ldots + A_1A_m^{n-2}B_1 + \ldots + A_m^{n-1}B_m, A_2A_1^{n-2}B_1 + \ldots + A_1A_m^{n-2}B_1 + \ldots, A_m^{n-1}B_m].$$

Since this matrix still has $n$ linear independent column vectors, it follows that it has full row rank $n$. Next, subtracting $B_2, \ldots, B_m$ from $B_1 + B_2 + \ldots + B_m$; subtracting $A_2B_1, \ldots, A_mB_m$ from $A_1B_1 + A_2B_1 + \ldots + A_mB_1 + \ldots + A_mB_m$ and subtracting $A_2A_1^{n-2}B_1, \ldots, A_1A_m^{n-2}B_1, \ldots, A_m^{n-1}B_m$ from $A_1^{n-1}B_1 + A_2A_1^{n-2}B_1 + \ldots + A_1A_m^{n-2}B_1 + \ldots + A_m^{n-1}B_m$, we can get the following matrix:

$$[B_1, B_2, \ldots, B_m, A_1B_1, A_2B_1, \ldots, A_mB_m, \ldots,$$

$$A_1^{n-1}B_1, A_2A_1^{n-2}B_1, \ldots, A_1A_m^{n-2}B_1, \ldots, A_m^{n-1}B_m],$$

which is the controllability matrix for switched linear systems (2). Since column fundamental transformation does not change the matrix rank, this matrix still has full row rank $n$. Hence, the switched linear system (2) is structurally controllable.

It turns out this criterion is not necessary for system (2) to be structurally controllable (see the example in subsection C). This implies that the union graph does not contain enough information for determining structural controllability. This is because edges from different subsystems are not differentiated in union graph. In the following subsection, another graphic representation of switched linear systems is proposed, from which necessary and sufficient conditions for structural controllability arise.
B. Criteria Based on Colored Union Graph

Another graphic representation: ‘colored union graph’ is defined as follows:

Definition 11: Switched linear system (2) can be represented by a colored union digraph (sometimes named colored union graph without leading to confusion), defined as a flow structure \( \tilde{G}(\tilde{V}, \tilde{I}) \), where vertex set \( \tilde{V} = \{V_1 \cup V_2 \cup \ldots \cup V_m \} \) and edge set \( \tilde{I} = \{e \in I_i, i = 1, 2, \ldots, m \} \), i.e., for \( i \in \{1, \ldots, m \} \), to each edge \( e \) we associate index \( i \) in \( \tilde{G} \), if this edge is associated to the subsystem \( i \) (subgraph \( G_i \)). Note that we associate several indexes (several different colors) to an edge \( e \) if it belongs to several subsystems.

Before proceeding further, we need to introduce two definitions which were proposed in [16] for linear system (1) first:

Definition 12: ([16]) The matrix pair \((A, B)\) is said to be reducible or of form I if there exist permutation matrix \( P \) such that they can be written in the following form:

\[
PAP^{-1} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad PB = \begin{bmatrix} 0 \\ B_{22} \end{bmatrix},
\]

where \( A_{11} \in \mathbb{R}^{p \times p}, A_{21} \in \mathbb{R}^{(n-p) \times p}, A_{22} \in \mathbb{R}^{(n-p) \times (n-p)} \) and \( B_{22} \in \mathbb{R}^{(n-p) \times r} \).

Remark 3: Whenever the matrix pair \((A, B)\) is of form I, the system is structurally uncontrollable [16] and meanwhile, the controllability matrix \( Q = [B, AB, \ldots, A^{n-1}B] \) will have at least one row which is identically zero for all parameter values [19]. If there is no such permutation matrix \( P \), we say sthat the matrix pair \((A, B)\) is irreducible.

Definition 13: ([16]) The matrix pair \((A, B)\) is said to be of form II if there exist permutation matrix \( P \) such that they can be written in the following form:

\[
[PAP^{-1}, PB] = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix},
\]

where \( P_2 \in \mathbb{R}^{(n-k) \times (n+r)} \), \( P_1 \in \mathbb{R}^{k \times (n+r)} \) with no more than \( k - 1 \) nonzero columns (all the other columns of \( P_1 \) have only fixed zero entries).

Several graphic properties of colored union graph are introduced in the following lemmas.

Lemma 4: There is no nonaccessible vertex in the colored union graph \( \tilde{G} \) of switched linear system (2) if and only if the following matrix

\[
[A_1 + A_2 + \cdots + A_m, B_1 + B_2 + \cdots + B_m]
\]
is irreducible or not of form I.

**Proof:** One vertex is accessible if and only if it can be reached by a stem. From definitions\[10\] and \[11\] it follows that there is no nonaccessible vertex in the colored union graph if and only if there is no nonaccessible vertex in the union graph. Besides, from remark \[2\] it is clear that the matrix representation of the union graph is $[A_1 + A_2 + \cdots + A_m, B_1 + B_2 + \cdots + B_m]$. According to lemma \[3\] there is no nonaccessible vertex in the union graph if and only if matrix \[6\] is irreducible or not of form I. Consequently the equivalence between accessibility of colored union graph and irreducibility of matrix \[6\] gets proved.

In the colored union graph, a new graphic property ‘$S$-dilation’ needs to be introduced here:

**Definition 14:** In colored union graph $\tilde{G}$, which is composed of subgraphs $G_i, i = 1, 2, \ldots, m$. Consider one vertex set $S$ formed by the vertices from the state vertex set $X$ and determine another vertex set $T(S) = \{v | v \in T_i(S), i = 1, 2, \ldots, m\}$, where $T_i(S)$ is a vertex set in $G_i$ which contains all the vertices $w$ with the property that there exists an oriented edge from $w$ to one vertex in $S$. Then $|T(S)| = \sum_{i=1}^{m} |T_i(S)|$. If $|T(S)| < |S|$, we say that there is a $S$-dilation in the colored union graph $\tilde{G}$.

Based on this new graphic property, the following lemma can be given:

**Lemma 5:** There is $S$-dilation in the colored union graph $\tilde{G}$ of switched linear system \[2\] if and only if matrix $[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m]$ is of form II. It means that this matrix can be written into: $[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m] = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$, where $P_1 \in \mathbb{R}^{p \times k}$ with no more than $p - 1$ nonzero columns (all the other columns of $P_1$ have only fixed zero entries).

**Proof:** From \[16\]-\[17\] or lemma \[3\] it is known that in linear systems, there is no ‘dilation’ in the corresponding graph if and only if the matrix pair $[A, B]$ can not be of form II or have $g$-rank $n$. From the explanation of this result in \[16\] and definition \[13\] $P_1$ in $[A, B]$ has $p$ rows, which actually represents the $p$ vertices of vertex set $S$ (defined for dilation), and each nonzero element of each row of $P_1$ represents that there is one vertex pointing to the vertex presented by this row. Therefore, the number of nonzero columns in $P_1$ is the number of vertices pointing to some vertex in $S$, and actually equals to $|T(S)|$. Furthermore, by the definition of $S$-dilation, $|T(S)|$ is now the summation of $|T_i(S)|$, $i \in \{1, \ldots, m\}$, in every subgraph. Consequently, it follows that there is $S$-dilation in the colored union graph $\tilde{G}$ if and only if matrix $[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m]$ in of form II.
Before going further to give another algebraic explanation of \textit{S-dilation}, one definition and lemma proposed in [18] must be introduced first:

\textbf{Definition 15:} ([18]) A structured $n \times m'$ ($n \leq m'$) matrix $A$ is of form $(t)$ for some $t$, $1 \leq t \leq n$, if for some $k$ in the range $m' - t < k \leq m'$, $A$ contains a zero submatrix of order $(n + m' - t - k + 1) \times k$.

\textbf{Lemma 6:} ([18]) $g$-rank of $A = t$

i) for $t = n$ if and only if $A$ is not of form $(n)$;
ii) for $1 \leq t < n$ if and only if $A$ is of form $(t + 1)$ but not of form $(t)$.

From the above definition and lemma, another lemma is proposed here:

\textbf{Lemma 7:} There is no \textit{S-dilation} in the colored union graph $\tilde{G}$ of switched linear system (2) if and only if the following matrix

$$[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m] \quad (7)$$

has $g$-rank $n$.

\textbf{Proof:} \textit{Necessity:} If matrix (7) has $g$-rank $< n$, from lemma 6 it follows that matrix (7) is of form $(n)$. Then referring to definition 15 matrix (7) must have a zero submatrix of order $(n + m' - t - k + 1) \times k$. Here, $t$ can be chosen as $n$, then (7) has a zero submatrix of order $(m' - k + 1) \times k$. For this $(m' - k + 1)$ rows, there are only $(m' - k)$ nonzero columns. Consequently, matrix (7) is of form II and by lemma 5 there is \textit{S-dilation} in the colored union graph $\tilde{G}$ of switched linear system (2).

\textit{Sufficiency:} If there is \textit{S-dilation} in the colored union graph $\tilde{G}$, by lemma 5 matrix (7) is of form II, then obviously $P_1$ in (7) can not have row rank equal to $k$ and furthermore, matrix (7) can not have $g$-rank $= n$.

With the above lemmas, a graphic necessary and sufficient condition for switched linear system to be structurally controllable can be proposed here:

\textbf{Theorem 2:} Switched linear system (2) with graphic representations $\mathcal{G}_i$, $i \in \{1, \ldots, m\}$, is structurally controllable if and only if the colored union graph $\tilde{G}$ should always satisfy the following two conditions:

i) there is no nonaccessible vertex in the colored union graph $\tilde{G}$,
ii) there is no \textit{S-dilation} in the colored union graph $\tilde{G}$.

\textbf{Proof:} \textit{Necessity:}(i) If there exist nonaccessible vertices in $\tilde{G}$, by lemma 4 the matrix $[A_1 + A_2 + \cdots +$
$A_m, B_1 + B_2 + \cdots + B_m$] is reducible or of form I. It follows that the controllability matrix

\[
[B_1 + B_2 + \ldots + B_m, (A_1 + A_2 + \ldots + A_m)(B_1 + B_2 + \ldots + B_m),
\]

\[
(A_1 + A_2 + \ldots + A_m)^2(B_1 + B_2 + \ldots + B_m), \ldots, (A_1 + A_2 + \ldots + A_m)^{n-1}(B_1 + B_2 + \ldots + B_m)]
\]

always has at least one row that is identically zero (remark 3). It is clear that every component of the matrix, such as $B, A_i B_j$ and $A^p_i A^q_j B_r$ has the same row always to be zero. As a result, the controllability matrix

\[
[B_1, \ldots, B_m, A_1 B_1, \ldots, A_mB_m, A_1^2 B_1, \ldots, A_mB_m A_1, \ldots, A_m^2 B_m, \ldots, A_mB_m A_1 B_m, \ldots,
\]

\[
A_1^{n-1} B_1, \ldots, A_mA_1^{n-2} B_1, \ldots, A_1 A_m^{n-2} B_m, \ldots, A_m^{n-1} B_m
\]

always has one zero row and can not be of full rank $n$. Therefore, switched linear system (2) is not structurally controllable.

(ii) Suppose that switched linear system (2) is structurally controllable, i.e., the controllability matrix satisfies $g$-rank $C(A_1, \ldots, A_m, B_1, \ldots, B_m) = n$. Specifically,

\[
\text{Im}[B_1, \ldots, B_m, A_1 B_1, \ldots, A_mB_m, A_1^2 B_1, \ldots, A_m^{n-1} B_m] = \mathbb{R}^n.
\]

Since $\forall M \in \mathbb{R}^{n \times r}$, $\text{Im}(A_i M) \subseteq \text{Im}(A_i)$, we have that

\[
\text{Im}[B_1, \ldots, B_m, A_1 B_1, \ldots, A_mB_m, A_1^2 B_1, \ldots, A_m^{n-1} B_m] \subseteq \text{Im}[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m] \subseteq \mathbb{R}^n.
\]

Thus $g$-rank $C(A_1, \ldots, A_m, B_1, \ldots, B_m) = n$ requires that $\text{Im}[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m] = \mathbb{R}^n$ and therefore $g$-rank $[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m] = n$. However, if there is S-dilation in the colored union graph $\mathcal{G}$, by lemma 5 $g$-rank $[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m] < n$. Consequently, the switched linear system (2) is not structurally controllable.

Sufficiency: Firstly, consider a structured linear system:

\[
\dot{x}(t) = Ax(t) + Bu(t)
\]

(8)

It is well known that the system is structurally controllable if and only if $g$-rank $(sI - A B) = n, \forall s \in \mathbb{C}$. Otherwise, the PBH test [25] states that system (8) is uncontrollable if and only if there exists generically a vector $q \neq 0$ such that $Aq = s_0 q, s_0 \in \mathbb{C}$ and $Bq = 0$, where $g$-rank $(s_0 I - A B) < n$.

On one hand, if $g$-rank $(sI - A B) = n, \forall s \in \mathbb{C} \setminus \{0\}$, then the uncontrollability of system (8) implies necessarily that there exists generically a vector $q \neq 0$ such that $Aq = 0$ and $Bq = 0$. 

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On the other hand, lemma 14.1 of [23] states that, if in the digraph associated to [8], every state vertex is an end vertex of a stem (accessible), then $g$-rank $(sI - A B) = n, \forall s \in \mathbb{C} \setminus \{0\}$. Assume that the two conditions in theorem 2 are satisfied. Using lemma 14.1 of [23], as all the parameters of matrices $A_1, \ldots, A_m, B_1, \ldots, B_m$ are assumed to be free, we can state that condition (i) of theorem 2 implies that, for almost all vector values $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_m)$, we have $g$-rank $(sI - (\bar{u}_1 A_1 + \ldots + \bar{u}_m A_m), (\bar{u}_1 B_1 + \ldots + \bar{u}_m B_m)) = n, \forall s \neq 0$. On the other hand, if switched linear system (2) is structurally uncontrollable, then for all constant values $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_m)$, linear systems defined by matrices $(\bar{A}, \bar{B})$ are also uncontrollable, where $\bar{A} = \sum_{i=1}^{m} \bar{u}_i A_i$ and $\bar{B} = \sum_{i=1}^{m} \bar{u}_i B_i$. This is due to the fact that for all constant values $\bar{u}$, $\text{Im}(C(\bar{A}, \bar{B}) \subseteq \text{Im}(C(A_1, \ldots, A_m, B_1, \ldots, B_m))$. Therefore, if switched linear system is structurally uncontrollable, we have that for each matrix pair $(\bar{A}, \bar{B})$, there exists a nonzero vector $q$ such that $\bar{A}q = 0$ and $\bar{B}q = 0$. Since this statement is true for almost all the values $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_m)$, we have that for almost all $n \cdot m$-tuple values $\bar{u}^j = (\bar{u}_1^j, \ldots, \bar{u}_m^j), j = 1, \ldots, n \cdot m$, we can find nonzero vectors $q_j$ such that the following system is satisfied:

$$
\begin{cases}
\sum_{i=1}^{m} \bar{u}_i^j A_i q_j = 0, \\
\sum_{i=1}^{m} \bar{u}_i^j B_i q_j = 0.
\end{cases}
$$

Obviously, there can not exist more than $n$ linear independent vectors $q_j$. Let us denote $q_1, q_2, \ldots, q_n$ the vectors such that $\text{span} \ (q_1, q_2, \ldots, q_{n \cdot m}) \subseteq \text{span} \ (q_1, q_2, \ldots, q_n)$. All the vectors $q_j, j = n + 1, \ldots, n \cdot m$ are linear combinations of $q_1, q_2, \ldots, q_n$. Therefore, system (9) contains the following equations:

$$
\begin{cases}
\sum_{k=1}^{n} \sum_{i=1}^{m} a_{i,k}^j (\bar{u}) A_i q_k = 0 & j = 1, \ldots, n \cdot m \\
\sum_{k=1}^{n} \sum_{i=1}^{m} a_{i,k}^j (\bar{u}) B_i q_k = 0 & j = 1, \ldots, n \cdot m
\end{cases}
$$

where $a_{i,k}^j (\bar{u})$ are linear functions of $\bar{u}^j, j = 1, \ldots, n \cdot m$. Since system (9) is satisfied for almost all the values, we can generically find $\bar{u}^j, j = 1, \ldots, n \cdot m$ such that

$$
\det \begin{bmatrix}
a_{0,1}^1(\bar{u}) & a_{0,2}^1(\bar{u}) & \cdots & a_{m,1}^1(\bar{u}) \\
a_{0,1}^2(\bar{u}) & a_{0,2}^2(\bar{u}) & \cdots & a_{m,2}^2(\bar{u}) \\
\vdots & \vdots & \ddots & \vdots \\
a_{0,1}^{n}(\bar{u}) & a_{0,2}^{n}(\bar{u}) & \cdots & a_{m,n}^{n}(\bar{u})
\end{bmatrix} \neq 0, \quad \det \begin{bmatrix}
b_{0,1}^1(\bar{u}) & b_{0,2}^1(\bar{u}) & \cdots & b_{m,1}^1(\bar{u}) \\
b_{0,1}^2(\bar{u}) & b_{0,2}^2(\bar{u}) & \cdots & b_{m,2}^2(\bar{u}) \\
\vdots & \vdots & \ddots & \vdots \\
b_{0,1}^{n}(\bar{u}) & b_{0,2}^{n}(\bar{u}) & \cdots & b_{m,n}^{n}(\bar{u})
\end{bmatrix} \neq 0.
$$

In this case, the only solution of (10) is $A_1 q_k = \ldots = A_m q_k = B_1 q_k = \cdots = B_m q_k = 0, k = 1, \ldots, n$. Obviously, if the switched linear system is structurally uncontrollable, then at
least one vector $q_k, k = 1, \ldots, n$ is nonzero. Consequently, the switched linear system \( \text{(2)} \) is structurally uncontrollable only if there exists at least one nonzero vector $q$ such that $A_1q = \ldots = A_m q = B_1 q = \ldots = B_m q = 0$. However, if condition $ii$ of theorem \( \text{(2)} \) is satisfied, then $g$-rank $[A_1, \ldots, A_m, B_1, \ldots, B_m] = n$ and therefore there does not exist a vector $q \neq 0$ such that $A_1 q = \ldots = A_m q = B_1 q = \ldots = B_m q = 0$. Hence, the two conditions are sufficient to ensure the structural controllability of switched linear system \( \text{(2)} \).

Actually, using the terminologies ‘dilation’ and ‘$S$-dilation’ as graphic criteria is not so numerically efficient. For example, to check the second condition of theorem \( \text{(2)} \) we need to test for all the possible vertex subsets to see whether there exist $S$-dilation in the union colored graph or not. Consequently, we will adopt another notion ‘$S$-disjoint edges’ to form more numerically efficient graphic interpretation of structural controllability of system \( \text{(2)} \).

**Definition 16:** In the colored union graph $\hat{G}$, consider $k$ edges $e_1 = (v_1, v'_1), e_2 = (v_2, v'_2), \ldots, e_k = (v_k, v'_k)$. We define for $i = 1, \ldots, k, S_i$ as the set of integers $j$ such that $v_j = v_i$, i.e., $S_i = \{ 1 \leq j \leq k | v_j = v_i \}$. $e_1, e_2, \ldots, e_k$ are $S$-disjoint if the following two conditions are satisfied:

i) edges $e_1, e_2, \ldots, e_k$ have distinct end vertices,

ii) for $i = 1, \ldots, k, S_i = \{ i \}$ or there exist $r$ distinct integers $i_1, i_2, \ldots, i_r$ such that $e_{j_1} \in I_{i_1}, e_{j_2} \in I_{i_2}, \ldots, e_{j_r} \in I_{i_r},$ where $j_1, j_2, \ldots, j_r$ are all the elements of $S_i$.

Roughly speaking, $k$ edges are $S$-disjoint if their end vertices are all distinct and if all the edges which have the same begin vertex can be associated to distinct indexes $i$. For this new graphic property, the following lemma can be given:

**Lemma 8:** Considering switched linear system \( \text{(2)} \), there exist $n$ $S$-disjoint edges in associated colored union graph $\hat{G}$ if and only if $[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m]$ has $g$-rank $= n$.

**Proof:** *Necessity:* If there exist $n$ $S$-disjoint edges in $\hat{G}$, matrix $[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m]$ contains at least $n$ free parameters. Since the $n$ $S$-disjoint edges have distinct end vertices, the corresponding $n$ free parameters lie on $n$ different rows. Besides, the $n$ $S$-disjoint edges have distinct begin vertices or have same begin vertex that can be associated to distinct indexes $i$. This implies that these $n$ free parameters lie on $n$ different columns. Keep these $n$ free parameters and set all the other free parameters to be zero. We can see that matrix $[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m]$
has the following form: 
\[
\begin{bmatrix}
0 & \lambda_1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \lambda_2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda_n & 0 & 0 & 0 & \ldots & 0 
\end{bmatrix},
\]
which has g-rank \(n\).

**Sufficiency:** From the definition 12.3 and the following discussions of [23], for a structured matrix \(Q\), g-rank \(Q = s\)-rank \(Q\). where \(s\)-rank of \(Q\) is defined as the maximal number of free parameters that no two of which lie on the same row or column. If matrix \([A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m]\) has g-rank \(= n\), it follows that there exists \(n\) free parameters from \(n\) different rows, which implies that the corresponding \(n\) edges have different end vertices, from \(n\) different columns, which implies that these \(n\) edges start from different vertices or start from same vertices but can be associated to different indexes. Hence the condition that the matrix has g-rank \(n\) is sufficient to ensure the existence of \(n\) \(S\)-disjoint edges.

Then we can get another necessary and sufficient condition for structural controllability of system (2).

**Theorem 3:** Switched linear system (2) with graphic representations \(G_i\), \(i \in \{1, \ldots, m\}\), is structurally controllable if and only if the colored union graph \(\tilde{G}\) should always satisfy the following two conditions:

i) there is no nonaccessible vertex in the colored union graph \(\tilde{G}\),

ii) there exist \(n\) \(S\)-disjoint edges in the colored union graph \(\tilde{G}\).

**Proof:** Lemma 5 and lemma 8 show that there exist \(n\) \(S\)-disjoint edges in the colored union graph \(\tilde{G}\) if and only if there is no \(S\)-dilation in \(\tilde{G}\). Then this theorem follows immediately. 

**Remark 4:** Compared with condition using ‘\(S\)-dilation’, this condition using ‘\(S\)-disjoint edges’ does not require to check all the vertex subsets, which is a more efficient criterion. The maximal number of ‘\(S\)-disjoint edges’ can be calculated using bipartite graphs. For example, we can use the algorithm in [26], which allows to compute the cardinality of maximum matching into a bipartite graph. A bipartite graph is a graph whose vertices can be divided into two disjoint sets \(U\) and \(W\) such that every edge connects a vertex in \(U\) to one in \(W\). To build a bipartite graph in directed subgraph \(G_i(V_i, I_i)\), what we need to do is adding some vertices and make \(U_i = \{v \in V_i | \exists (v, v') \in I_i\}\), which implies that cardinality \(|U_i|\) equals to the number of nonzero columns in matrix \([A_i, B_i]\). Besides, \(W_i = X_i\), i.e., the state vertex set. Then it follows that the maximum matching in this bipartite graph is the same as the maximal \(S\)-disjoint edge set in \(G_i(V_i, I_i)\).
According to definition of \( S\)-disjoint edges, the beginning vertex from different subgraphs should be differentiated when building the bipartite graph for colored union graph \( \tilde{G} \). Therefore for the bipartite graph of \( \tilde{G} \), \( \mathcal{U} = \{ v \mid \exists (v, v') \in I_i, i = 1, 2, \ldots, m \} \), which implies that cardinality \(|\mathcal{U}|\) equals to the number of nonzero columns in matrix \([A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m]\). And \( W = X \), i.e., the state vertex set. Similarly, the maximum matching in this bipartite graph is the same as the maximal \( S\)-disjoint edge set in colored union graph. Therefore the complexity order of algorithm using method in [26] is \( O(\sqrt{p + n \cdot q}) \), where \( q \) is the number of edges in colored union graph, i.e., the number of free parameters in all system matrices, \( p \) is the number of nonzero columns in matrix \([A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m]\) and \( n \) is number of state variables.

Compared with condition (ii), condition (i) is easier to check. We have to look for paths which connect each state vertex with one of the input vertex. This is a standard task of algorithmic graph theory. For example, depth-first search or breadth-first search algorithm for traversing a graph can be adopted and the complexity order is \( O(|V| + |E|) \), where \(|V|\) and \(|E|\) are cardinalities of vertex set and edge set in union graph.

C. Numerical Examples

Consider a switched linear system with two subsystems as depicted by the graphic topologies in Fig. 3(a)-(b). In colored union graph \( \tilde{G} \) (Fig. 3(d)), thin lines represent edges from subgraph (a) and thick lines represent the edges from subgraph (b). It turns out that the colored union graph \( \tilde{G} \) has no nonaccessilbe vertex and no \( S\)-dilation. Besides, the three edges are \( S\)-disjoint edges since they have different end vertices and one edge begins at vertex 3 and two edges begin at vertex 0 but they come from different subsystems.

According to theorem [2] or [3] this switched linear system is structurally controllable. On the
other hand, the system matrices of each subsystem of corresponding subgraph are:

\[
A_1 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
0 \\
0 \\
\lambda_1 \\
\end{bmatrix}; \quad A_2 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & \lambda_2 \\
0 & 0 & 0 \\
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
\lambda_3 \\
0 \\
0 \\
\end{bmatrix}.
\]

the controllability matrix \( \mathbf{S} \) can be calculated and can be shown to have \( g-rank=3 \). In addition, there exist a \textit{dilation} in union graph Fig. 3(c), where \( S = \{1, 2, 3\} \) and \( T(S) = \{0, 3\} \), which shows that the condition in theorem \([1]\) is not necessary for structural controllability.

IV. CONCLUSIONS AND FUTURE WORK

In this paper, a more ‘practical’ concept of controllability: structural controllability for switched linear systems has been investigated. Combining the knowledge in the literature of switched linear systems and graph theory, several graphic necessary and sufficient conditions for the structurally controllability of switched linear systems have been proposed. These graphic interpretations provide us better understanding on how the graphic topologies of switched linear systems will influence or determine the structural controllability of switched linear systems and therefore, would be of great practical significance for different kinds of physical systems or processes. This shows us a great perspective that we can design the switching algorithm to make the switched linear system structurally controllable conveniently just having to make sure some properties of the corresponding graph (union or colored union graph) are kept during the switching process.

In this paper, the parameters in different subsystem models are assumed to be independent. A more general hypothesis is that some free parameters remain the same among different subsystems switching, i.e., assuming dependence between models of subsystem. It turns out that our necessary and sufficient condition derived in this paper would be a necessary condition under this dependence hypothesis. The structural controllability considering the dependence of subsystem models is an important further investigation topic that we are working on.

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