Fast Lagrange inversion, with an application to factorial numbers

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Received 10 May 1990

Abstract


Suppose $\beta(t)$ and $\gamma(t)$ are a pair of compositional inverse formal power series. Lagrange inversion expresses the coefficient of $t^n$ in $\gamma(t)^k$ in terms of the coefficient of $t^k$ in $\beta(t)^n$. ‘Fast Lagrange inversion’ calculate the latter for invertible power series with nonzero quadratic term, using only positive powers of $\beta$. The result is given for multivariate series, and illustrated by a bivariate generalization of Stirling numbers.

1. Introduction

A delta series is a formal power series of the form $\beta(t) = \beta_1 t + \beta_2 t^2 + \cdots$, $\beta_1 \neq 0$, and any delta series has a compositional inverse $\gamma(t)$, say, where $\beta(\gamma(t)) = t$. It is well known that the coefficient $\langle \gamma^k \rangle_n$ of $t^n$ in $\gamma(t)^k$ equals $(k/n)\langle \beta^{-n} \rangle_{-k}$ for all integers $k \leq n \neq 0$ (see [3]). This Lagrange inversion procedure is usually not attractive, because the coefficients of negative powers of $\beta(t)$ are difficult to obtain in many applications. For example, let $a$ and $b$ be two different real or complex numbers, and define $\beta(t) := e^{at} - e^{bt}$. For nonnegative integers $m$, $\beta(t)^m$ has the coefficients

$$\langle \beta^m \rangle_k = \frac{m!}{k!} F(k, m),$$

where $F(k, m)$ belongs to the family of factorial number of the second kind ($0 \leq m \leq k$). Best known members are the Stirling numbers, where $a = 1$, $b = 0$, and the central factorial numbers of the second kind ($a = 1/2 = -b$). There are combinatorial interpretations of factorial numbers for a large class of integer values of $a$ and $b$ [5]. Applying the binomial theorem to $(e^{at} - e^{bt})^m$ for positive integers $m$ is straightforward, giving

$$m! F(k, m) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j}(ja + (m-j)b)^k.$$
But how can we obtain the coefficients of negative powers, where the binomial theorem does no longer apply? We begin by defining for $1 \leq k \leq n$ the factorial numbers of the first kind as

$$f_{a,b}^n(k) := \frac{(n-1)!}{(k-1)!} \langle \beta^{-n} \rangle_{-k}.$$ 

That they are not so easily expressed in terms of powers and factorials, we know already from the Stirling numbers of the first kind. Yet, such expressions exist, and Lagrange inversion is still practical because of the following observation. There is a large class of invertible power series, where the coefficients $\langle \beta' \rangle_k$ are actually values of some polynomials $b_j(x)$, $\deg b_j = j$. More precisely, $\langle \beta' \rangle_k = \beta^i b_{k-i}(i)$ for all integers $k \geq i$, positive or negative! In that case it is easy to find $\langle \beta^{-n} \rangle_{-k}$ just by extrapolation,

$$\langle \beta^{-n} \rangle_{-k} = \binom{2n-k}{n-k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j \frac{n}{n+j} \beta_1^{-i-n} \langle \beta' \rangle_{j+n-k}.$$ 

For the factorial numbers of the first kind, this ‘fast’ Lagrange inversion gives

$$(a \cdot b)^{n-k} f_{a,b}^n(n, n-k)$$

$$= (n-k) \binom{n+k}{n-k} \sum_{j=0}^{k} \binom{2k}{k-j} \frac{1}{j! (n+j)} \sum_{v=0}^{j} \binom{j}{v} (-1)^v (v + \frac{b}{a-b} i)^{j+k}.$$ 

This is well known for Stirling numbers, and can be found in [2] for central factorial numbers.

Exactly which delta series have coefficients such that extrapolation can be used for fast Lagrange inversion? To answer that question we have to introduce polynomials of binomial type. A sequence $\{p_n(x) \mid n = 0, 1, \ldots\}$ of polynomials is of binomial type, iff $\deg p_n = n$, $p_0(x) = 1$, and

$$p_n(x+y) = \sum_{i=0}^{n} p_i(x)p_{n-i}(y)$$

for all $n = 0, 1, \ldots$. The answer to the above question can be directly derived from results on polynomials of binomial type with polynomial coefficients [6].

**Theorem 1.** Let $\beta(t)$ be a delta series. There exists a sequence $\{b_n(x) \mid n = 0, 1, \ldots\}$ of binomial type such that $\langle \beta' \rangle_k = \beta^i b_{k-i}(i)$ for all integers $i \leq k$ iff $\beta_2$, the coefficient of the quadratic term in $\beta(t)$, is different from zero.

Whereas the theory of multivariate Lagrange inversion is very similar to the univariate case, the actual calculation of an inverse may be tedious. Therefore, ‘fast’ Lagrange inversion can be especially helpful in the presence of several variables. For that reason we prove Theorem 1’ in Section 3, the multivariate version of Theorem 1, and present the whole theory in the multivariate setting. As a reference frame for multivariate Lagrange inversion and Umbral Calculus,
we have chosen the paper ‘Dual Operators and Lagrange Inversion in Several Variables’ by Verde-Star [7]. Other approaches we want to mention are [1, 4, 8–9].

Not every invertible multivariate series $\beta$ is a delta series as defined in the next section. But there always exists an invertible transformation which carries $\beta$ into a delta series. As an example, we invert $(e^{ax} - e^{by}, e^{cx} - e^{dy})$, $ad \neq bs$, in Section 4. This formal power series can be seen as a bivariate version of the factorial generating function $e^{ax} - e^{by}$. Its compositional inverse has relatively simple coefficients, which are just products of two factorial powers.

2. Multivariate Lagrange inversion

In most aspects, our notation follows closely the ‘Dual operators an Lagrange inversion in several variables’ Verde-Star [7]. Let $r$ be a positive integer. On $\mathbb{Z}$ we consider the natural (componentwise) partial order. (In [7], other orders are also considered.) Let $K$ be a field of characteristic 0, and $\mathcal{F}$ be the ring of all Laurent series $\sum_{n} k_n t^n$, where the support of the coefficients $k_n$ is bounded from below, i.e., there exists an $m \in \mathbb{Z}$ such that $k_n \neq 0$ implies $n \geq m$. If $k_m \neq 0$, we say that the series is of order $m$. The Laurent series whose coefficients have a support bounded by 0 constitute a subring $\mathcal{F}_0$ of $\mathcal{F}$. Define $\mathcal{X} := \{ \varphi \in \mathcal{F}_0 \mid \varphi(0) = 1 \}$, so all elements of $\mathcal{X}$ are of order 0. We are mainly interested in vectors $\beta = (\beta_1(t), \ldots, \beta_r(t))$ where $\beta_p(t)/t_p$ is in $\mathcal{X}$ for all $p = 1, \ldots, r$. The set of all such vectors is denoted by $\mathcal{G}$. If $\beta \in \mathcal{G}$, we call $\beta$ a normed delta series. $\beta$ is a delta series, if $\beta(t/a_1, \ldots, t/a_r) \in \mathcal{G}$ for some constant vector $a$ with non-zero components. Every delta series has a compositional inverse, which is another delta series $\gamma$ such that $\beta(t, a_1, \ldots, a_r) = t$, for all $r - 1, \ldots, r$.

Boldfacing of vectors indicates products, i.e., if $y = (y_1, \ldots, y_r)$ we use the notation $y := y_1 \times \cdots \times y_r$, so $t^\beta = t_1^{\beta_1} t_2^{\beta_2} \times \cdots \times t_r^{\beta_r}$, but $t^\beta = (t_1^{\beta_1}, t_2^{\beta_2}, \ldots, t_r^{\beta_r})$. Also, $n! = n_1! \times \cdots \times n_r!$, and

\[
\binom{n}{i} = \frac{n!}{i! (n - i)!}.
\]

To indicate the coefficient of $t^\rho$ in a single Laurent series $\varphi(t)$ we write $\langle \varphi \rangle_\rho$, or simply $\varphi$, if no confusion can occur. In this notation, an inverse pair can be expressed as an orthogonality relation on the coefficients. $\beta$ and $\gamma$ are inverse delta series iff for all $\rho = 1, \ldots, r$, $j \in \mathbb{Z}$

\[
\sum_i \langle \beta_i \rangle_\rho \langle \gamma^j \rangle_\rho = \delta_{\rho,j} \sum_i \langle \gamma_i \rangle_\rho \langle \beta^j \rangle_\rho,
\]

where $e_\rho \in \mathbb{Z}^r$ has the $\rho$th component equal to 1, and all others equal to 0.
Denote the Jacobian determinant of \( \beta \) by \( J\beta \), and the vector of all ones in \( \mathbb{Z}' \) by \( e := (1, \ldots, 1) \). If \( \gamma \) is the compositional inverse of some delta series \( \beta \), then the coefficient \( (\gamma^k)_n \) of \( t^n \) in \( \gamma^k \) equals the coefficient \( \langle \beta^{-n}J\beta \rangle_{-k-e} \) of \( t^{-k-e} \) in \( \beta^{-n}J\beta \), where \( -n - e = (-n_1 - 1) \times \cdots \times (-n_r - 1) \) for all \( k, n \in \mathbb{Z}' \) [7].

The differentiation rules for multivariate power series allows us to write the Jacobian of \( \beta^n \) as

\[
J\beta^n = \det(D_\rho \beta_{\rho}^{\rho^o})_{\rho, o=1,\ldots,r} = \det(n_o \beta_{\rho}^{\rho^o-1} D_\rho \beta_{\rho})_{\rho, o=1,\ldots,r} = n \beta^{n-e} \det(D_\rho \beta_{\rho}) = n \beta^{n-e} J\beta.
\]

Hence, we can rephrase the Lagrange inversion result as follows.

If \( \gamma \) is the compositional inverse of \( \beta \), then the coefficient \( (\gamma^k)_n \) of \( t^n \) in \( \gamma^k \) equals the coefficient of \( t^{-k-e} \) in \( (1/(-n))J\beta^{-n} \), if all components of \( n \) are different from zero.

We now explore that last coefficient.

\[
\frac{1}{-n} J\beta^{-n} = \frac{1}{-n} \det \left( \sum_i \langle \beta^{-n_o}_{\sigma} \rangle_{i_o} t^{i-e} \right)_{\rho, o=1,\ldots,r} = \frac{1}{-n} \sum_{\rho \in \mathcal{S}_n} (-1)^{\text{sign}(\pi)} \prod_{o=1}^r \left( \sum_i \langle \beta^{-n_o}_{\sigma} \rangle_{i_o} t^{i-e_o} \right)
\]

So far, we assumed that \( n \) has no zero component. Suppose, \( 0 \leq k \leq n \), and \( n_v = 0 \) is the only vanishing component of \( n \). Then \( k_v = 0 \), and

\[
(\gamma^k)_n = \langle \beta^{-n-e}J\beta \rangle_{-k-e} = \left( \frac{1}{-n + e_v} \det(\hat{\beta}_{\rho, o})_{\rho, o=1,\ldots,r} \right)_{-k-e},
\]

where \( \hat{\beta}_{\rho, o} = D_\rho \beta_{\rho}^{-n_o} \) if \( \sigma \neq v \), and \( \hat{\beta}_{\rho, v} = \beta_v^{-1} D_\rho \beta_v \). For \( \rho \neq v \) one obtains \( \hat{\beta}_{\rho, v} = D_\rho \log(\beta_v/t_v) \) and \( \hat{\beta}_{v, v} = t_v^{-1} + D_v \log(\beta_v/t_v) \) as in [1, p. 320]. Note that \( \hat{\beta}_{v, v} \) is the only element of \( (\hat{\beta}_{\rho, o})_{\rho, o=1,\ldots,r} \) that contains a negative power of \( t_v \). But the \( v \)th component of \(-k - e\) equals \(-1\), so

\[
(\gamma^k)_n = \left( \frac{1}{-n + e_v} t_v^{-1} \det(D_\rho \beta_{\rho}^{-n_o})_{\rho, o=1,\ldots,\rho, v=v, o=1,\ldots, \rho, v=v, o=1,\ldots, r} \right)_{-k-e}.
\]

In other words, \( (\gamma^k)_n \) still equals

\[
\left( \frac{1}{-n} J\beta^{-n} \right)_{-k-e}
\]

if we eliminate all components in the latter expression for which \( n \) has a zero component.
Proposition 1. If \( \gamma \) is the compositional inverse of a delta series \( \beta \), then the coefficient of \( t^k \) in \( \gamma^k \) equals

\[
\langle \gamma^k \rangle_n = \frac{1}{n!} \sum_{i_1+\cdots+i_r=-k} \left( \prod_{\sigma=1}^r \langle \beta_{\sigma}^{-n_\sigma} \rangle_{i_\sigma} \right) \det(i_{\sigma})
\]

for all \( 0 \leq k \leq n \), if \( n \) has no zero component. If \( n \) has vanishing components, compress all vectors on the right-hand side by eliminating those components, where \( n \) is zero.

Of course, for \( r = 1 \) this specializes to the well-known result

\[
\langle \gamma^k \rangle_n = \frac{k}{n} \langle \beta^{-n} \rangle_{-k}.
\]

If one is interested in \( \gamma_\rho \) only \( (k = \epsilon_\rho) \), the formula slightly simplifies. For \( r = 2 \) we obtain the following.

Corollary 1. If \( \gamma = (\gamma_1(s, t), \gamma_2(s, t)) \) is the compositional inverse of a delta series \( \beta = (\varphi(s, t), \psi(s, t)) \), then:

\[
\langle \gamma_1 \rangle_{m,n} = \frac{1}{mn} \sum_{i,j} j \langle \varphi^{-m} \rangle_{i,j} \langle \psi^{-n} \rangle_{-1-i,-j},
\]

\[
\langle \gamma_2 \rangle_{m,n} = -\frac{1}{mn} \sum_{i,j} i \langle \varphi^{-m} \rangle_{i,j} \langle \psi^{-n} \rangle_{-i,-1-j},
\]

\[
\langle \gamma_1 \rangle_{m,0} = \frac{1}{m} \langle \varphi(s, 0)^{-m} \rangle_{-1},
\]

\[
\langle \gamma_2 \rangle_{0,n} = \frac{1}{n} \langle \psi(0, t)^{-n} \rangle_{-1},
\]

for all positive integers \( m \) and \( n \).

Example. Let \( \beta(s, t) = ste^t/(1 + s + t) = \varphi(s, t)\psi(s, t) \), where \( \varphi(s, t) = se^t \), and \( \psi(s, t) = t/(1 + s + t) \). So we find

\[
\langle \varphi^{-m} \rangle_{i,j} = (-m)!/j! \quad \text{if } j \geq 0 \text{ and } i = j - m,
\]

and

\[
\langle \psi^{-n} \rangle_{i,j} = \frac{n!}{(-j-t)!t!(j+n)!} \quad \text{if } i \geq 0 \text{ and } -i \geq j \geq -n.
\]
Corollary 1 gives for positive \( m \) and \( n \)

\[
\langle \gamma_1 \rangle_{m,n} = - \sum \frac{(-m)^j(n-1)!}{j! (2j+3-m)! (m-j-2)! (n-1-j)!},
\]

\[
\langle \gamma_2 \rangle_{m,n} = - \sum \frac{(-m)^{j+1}(n-1)!}{j! (1+2j-m)! (m-1-j)! (n-1-j)!},
\]

\[
\langle \gamma_1 \rangle_{m,0} = \delta_{1,n},
\]

\[
\langle \gamma_2 \rangle_{0,n} = \frac{1}{n} \left( \frac{t}{1+t} \right)^n - 1,
\]

where the summation index \( j \) takes all integer values such that the factorials are defined in the usual sense. Hence,

\[
\gamma_1(s, t) = s - \sum_{j=1} \left( \frac{t}{1-t} \right)^j \sum_{m=0} \frac{(-j-1-m)^{j-1}}{m! (j-m)!} s^{m+j+1}
\]

and

\[
\gamma_2(s, t) = \frac{t}{1-t} - \sum_{j=0} \left( \frac{t}{1-t} \right)^{j+1} \sum_{m=0} \frac{(-j-1-m)^{j-1}}{m! (j-m)!} s^{m+j+1}.
\]

3. Sequences of binomial type

Denote by \( K[x_1, \ldots, x_r] \) the algebra of polynomials over \( K \) in the variables \( x_1, \ldots, x_r \). A polynomial sequence \( \{b_n(x) \mid n \geq 0\} \) has coefficients \( b_{n,i} \), where

\[
b_n(x) = \sum_{i=0}^n \frac{x^i}{i!} b_{n,i}
\]

with \( b_{n,n} \neq 0 \). The range of the summation \( i = 0, \ldots, n \) stands for all integer vectors \( \mathbf{i} \) such that \( 0 \leq i \leq n \). The generating function of a polynomial sequence is a formal power series in \( K[x][[t]] \).

A polynomial sequence \( \{b_n(x) \mid n \geq 0\} \) is of binomial type iff

\[
b_n(x + y) = \sum_{i=0}^n b_i(x)b_{n-i}(y)
\]

for all \( n \in \mathbb{N}^r \). Note that \( b_n(0) = \delta_{0,n} \).

Polynomials of binomial type can also be characterized by their coefficients:

\[
b_{n,n} \neq 0 \quad \text{and} \quad b_{n+i+j+i+j} = \sum_{k=0}^n b_{k+i,j} b_{n-k+i,j} \quad (1)
\]

for all \( i, j, n \in \mathbb{N}^r \).

**Definition.** The polynomial sequence \( \{b_n(x) \mid n \geq 0\} \) has polynomial coefficients iff there exists a polynomial sequence \( \{b_n(x) \mid n \geq 0\} \) such that \( b_{n,i} = \delta_{n-\lambda(i)} \) for all \( 0 \leq i \leq n \).
In order to see which sequences of binomial type have polynomial coefficients, we need a few results from the Umbral Calculus.

**Proposition 2** [7, Proposition 4.3(i)]. Suppose \( \{ b_n(x) \mid n \geq 0 \} \) is a polynomial sequence, and let

\[
\beta_\rho(t) = \sum_{n=0}^{\infty} b_{n,\rho} t^n
\]

(\( \rho = 1, \ldots, r \)). \( \{ b_n(x) \mid n \geq 0 \} \) is of binomial type iff \( \beta_\rho \) is of order \( e_\rho \), and

\[
\beta(t)^m = \sum_{n=m}^{\infty} b_{n,m} t^n
\]

for all \( m \in \mathbb{N}^\ast \).

**Lemma 1** [7, (4.15)]. The polynomial sequence \( \{ b_n(x) \mid n \geq 0 \} \) is of binomial type iff

\[
\sum_{n=0}^{\infty} b_n(x) t^n = \exp[x \cdot \beta(t)],
\]

where \( \beta(t)^e_\rho = \beta_\rho(t) \) has order \( e_\rho \) (\( \rho = 1, \ldots, r \)).

Proposition 2 shows us how a sequence \( \{ b_{n,\rho} \mid n \geq e_\rho, \rho = 1, \ldots, r \} \) of scalars determines the whole sequence of binomial type. In the following lemma we show that the same is true for the scalar sequence \( \{ b_n(\rho) \mid n \geq e_\rho, \rho = 1, \ldots, r \} \).

**Lemma 2.** Let \( a_{n,\rho} \in K \) for all \( n \in \mathbb{N}^\ast, \rho = 1, \ldots, r \). There exists a sequence \( \{ b_n(x) \mid n \geq 0 \} \) of binomial type such that \( b_n(\rho) = a_{n,\rho} \) for all \( n \in \mathbb{N}^\ast, \rho = 1, \ldots, r \) iff \( a_{0,\rho} = 1 \) and \( a_{e_\rho,\rho} \neq 0 \) for all \( \rho = 1, \ldots, r \).

**Proof.** If \( \{ b_n(x) \} \) is of binomial type, then we know from Lemma 1 that for \( x = e_\rho \)

\[
\sum_{n=0}^{\infty} b_n(e_\rho) t^n = \sum_{i=\mathbb{N}^\ast} \frac{e_\rho^i}{i!} \beta(t)^i - \sum_{i=\mathbb{N}^\ast} \beta(t)^{e_\rho i}/i!
\]

\[
= \sum_{i=\mathbb{N}^\ast} \frac{1}{i!} \sum_{n=i e_\rho} b_{n,i e_\rho} t^n = \sum_{n=0}^{\infty} t^n \sum_{i=0}^{\infty} b_{n,i e_\rho}/i!
\]

by Proposition 2. So we see that \( a_{0,\rho} = b_{0,0} = 1 \), and \( a_{e_\rho,\rho} = b_{e_\rho,0} + b_{e_\rho,e_\rho} - b_{e_\rho,e_\rho} \neq 0 \). Vice versa, if \( b_n(\rho) = a_{n,\rho} \) we have to find a power series \( \beta_\rho(t) \) of order \( e_\rho \), the ‘logarithm’ of \( \sum_{n=0}^{\infty} b_n(e_\rho) t^n \), such that

\[
\sum_{n=0}^{\infty} b_n(e_\rho) t^n = \sum_{i=\mathbb{N}^\ast} \beta_\rho(t)^i/i!.
\]

Let \( \beta_\rho(t) = \sum_{n=0}^{\infty} d_n t^n \). Choosing \( t = 1 \) gives \( d_\rho = b_\rho(e_\rho) = a_{e_\rho,\rho} \neq 0 \), so \( \beta_\rho(t) \) will be of order \( e_\rho \). For \( n > e_\rho \) we calculate \( a_{n,\rho} = d_n + \text{sum of products of coefficients} \)
Lemma 2 is instrumental for characterizing polynomial sequences that have polynomial coefficients. The following theorem generalizes the univariate characterization theorem \[6, \text{Theorem 1}\] to the multivariate case as far as it is needed for fast Lagrange inversion. The proof is the obvious analog of the univariate case, and is only given for completeness.

**Theorem 2.** Let \(\{b_n(x) \mid n \geq 0\}\) be a sequence of polynomial type such that \(b_{n,n} = 1\) for all \(n \in \mathbb{N}'\). The following statements are equivalent:

(i) \(\{b_n(x)\}\) has polynomial coefficients.

(ii) There exists a sequence of binomial type \(\{\tilde{b}_n(x) \mid n \geq 0\}\) such that \(b_{n,i} = \tilde{b}_{n,i}(i)\) for all \(0 \leq i \leq n\).

(iii) \(b_{\rho,\rho} \neq 0\) for all \(\rho = 1, \ldots, r\).

(iv) There exists a sequence of binomial type \(\{\tilde{b}_n(x) \mid n \geq 0\}\) such that \(b_{n,e_\rho} = \tilde{b}_{n-e_\rho}(e_\rho)\) for all \(n > 0\).

(v) There exist power series \(\tilde{\beta}_\rho(t)\) of order 1 such that \(\beta_\rho(t) = t_\rho \exp[\tilde{\beta}_\rho(t)]\) for all \(\rho = 1, \ldots, r\).

**Proof.** (i) \(\Rightarrow\) (ii). For all \(i, j, n \in \mathbb{N}'\)

\[
\tilde{b}_n(i + j) = b_{n+i+j,i+j} = \sum_{k=0}^{n} b_{k+i,j} b_{n-k+j,j} \quad \text{(see (1))}
\]

\[
= \sum_{k=0}^{n} \tilde{b}_k(i) b_{n-k}(j).
\]

Using the fact that \(\{\tilde{b}_n(x)\}\) is a sequence of polynomials shows that the above identity holds for the formal variables \(x\) and \(y\) in place of \(i\) and \(j\). Hence \(\{\tilde{b}_n(x) \mid n \geq 0\}\) is of binomial type.

(ii) \(\Rightarrow\) (iii). From \(\tilde{b}_{e_\rho}(0) = 0\) follows that

\[
\tilde{b}_{e_\rho}(x) = \tilde{b}_{e_\rho,e_\rho} x^{e_\rho} = \tilde{b}_{e_\rho,e_\rho} x^{e_\rho}.
\]

Hence,

\[
b_{2e_\rho} = \tilde{b}_{e_\rho}(e_\rho) = \tilde{b}_{e_\rho,e_\rho} \neq 0.
\]

(iii) \(\Rightarrow\) (iv). Let \(a_{n,0} = b_{n+e_\rho,e_\rho}\) for all \(n \geq 0\) and \(\rho = 1, \ldots, r\). From Lemma 2 we conclude the existence of a sequence \(\{\tilde{b}_n(x)\}\) of binomial type such that \(\tilde{b}_n(e_\rho) = a_{n,\rho}\).

(iv) \(\Rightarrow\) (v).

\[
\beta_\rho(t) = \sum_{n\geq0} b_{n,\rho} t^n = \sum_{n\geq0} \tilde{b}_{n-e_\rho}(e_\rho) t^n = t^\rho \sum_{n\geq0} \tilde{b}_n(e_\rho) t^n
\]

\[
= t^\rho \exp[\tilde{\beta}_\rho(t)].
\]
(v) $\Rightarrow$ (i). By Proposition 2,
$$\sum_{n \geq m} b_{n,m} t^n = \beta(t)^m = t^m \exp[m \cdot \tilde{\beta}(t)] = \sum_{n \geq 0} \tilde{b}_n(m) t^{n+m} \quad \text{(Lemma 1)}.$$ 
Comparing coefficients shows that $b_{n,m} = \tilde{b}_{n-m}(m)$. □

The multivariate generalization of Theorem 1 is a simple consequence of Theorem 2.

**Theorem 1'**. If $\beta(t)$ is a delta series such that $\langle \beta_{\rho} \rangle_{2e_\rho} \neq 0$ for all $\rho = 1, \ldots, r$, then there exists a sequence $\{\tilde{b}_n(x) \mid n \geq 0\}$ of polynomial type such that
$$\langle \tilde{\beta}^{(k)}_{\rho} \rangle_n = \tilde{b}_{n-k_{\rho}}(k_{\rho} \rho),$$
for all integers $k_{\rho}$, and $n \in \mathbb{N}^r + k_{\rho} e_{\rho}$, $\rho = 1, \ldots, r$. Furthermore,
$$\sum_{n \geq 0} \tilde{b}_n(k_{\rho} e_{\rho}) t^n = (\beta_{\rho}(t)/t_{\rho})^{k_{\rho}}.$$

**Proof.** By Proposition 2, $\langle \beta_{\rho} \rangle_{2e_\rho} = b_{2e_\rho, e_{\rho}}$, where $\{b_n(x) \mid n \geq 0\}$ is the corresponding sequence of binomial type. Hence, we know from Theorem 2 that
$$\beta_{\rho}(t)^{k_{\rho}} = t^{k_{\rho}} \exp[k_{\rho} \tilde{\beta}_{\rho}(t)] = t^{k_{\rho}} \exp[k_{\rho} e_{\rho} \cdot \tilde{\beta}(t)]$$
for some delta series $\tilde{\beta}$. Let $\{\tilde{b}_n(x) \mid n \geq 0\}$ be the corresponding sequence of binomial type. Then
$$(\beta_{\rho}(t)/t_{\rho})^{k_{\rho}} = \sum_{n \geq 0} \tilde{b}_n(k_{\rho} e_{\rho}) t^n. \quad \Box$$

4. **Example**

A bivariate generalization of $e^{ax} - e^{bt}$ can be constructed as the pair $\mu_1(s, t) := e^{as} - e^{bt}$, and $\mu_2(s, t) := e^{cs} - e^{dt}$. $\mu$ has a compositional inverse iff $ad - bc \neq 0$, but $\mu$ is not a delta series. Hence, we transform $\mu$ into a delta series by considering $\beta_1 := \mu_1 \circ \lambda$ and $\beta_2 := \mu_2 \circ \lambda$, where $\lambda_1(s, t) := ds + bt$, and $\lambda_2(s, t) := cs + at$. This gives
$$\beta_1(s, t) = e^{abt}(e^{ads} - e^{bct}), \quad \text{and} \quad \beta_2(s, t) = e^{cds}(e^{bct} - e^{adv}).$$

The transformed series factors, which simplifies our calculations! For positive integers $m$, $n$, $i$ and $j$,
$$\langle \beta_1^m \rangle_{i,j} = \langle (e^{ads} - e^{bct})^m \rangle_{i,j} \frac{(abm)!}{i! j!} F_1(i, m),$$
and
$$\langle \beta_2^n \rangle_{i,j} = \langle (e^{bct} - e^{adv})^n \rangle_{i,j} \frac{(cdn)!}{i! j!} F_2(j, n).$$
where \( F_i(i, m) \) and \( F_j(j, n) \) are the factorial numbers of the second kind as defined in the introduction, corresponding to \((e^{ads} - e^{bcx})^m\) and \((e^{bcx} - e^{ad})^n\), respectively. Thus, \( F_i(j, n) = (-1)^n F_i(j, n) \). By Theorem 1', the coefficients \( \langle \beta^m_i \rangle_{i,j} \) are values of a sequence \( \{b_{m,n}(x) \mid m, n \geq 0\} \) of binomial type, so that for all integers \( i_1, j_1 \geq m, j_2 \geq 0 \)

\[
(ad - bc)^{-i_1} \langle \beta^m_i \rangle_{i_1,j_1} = \hat{b}_{i_1-m,j_1}(m, 0) \quad (bc - ad)^{-j_2} \langle \beta^m_j \rangle_{i_2,j_2} = \hat{b}_{i_2-j_2-n}(0, n)
\]

and therefore

\[
\langle \beta^m_i \rangle_{i_1,j_1} = \hat{b}_{i_1-m,j_1}(m) = \delta^{(m)}_{i_1,j_1} = \frac{(abm)^{i_1}}{j_1!}, \quad \langle \beta^m_j \rangle_{i_2,j_2} = \hat{b}_{i_2-j_2-n}(0) = \frac{(-1)^n c_{i_2,j_2}^{i_2}}{i_2!} = \delta^{(n)}_{i_2,j_2},
\]

where the univariate sequences \( \{\hat{b}^1_n(x) \mid n \in \mathbb{N}\} \) and \( \{\hat{b}^2_n(x) \mid n \in \mathbb{N}\} \) have generating functions \(((a^{ads} - e^{-bcx})^{l})^{x} \) and \(((e^{bcx} - e^{ad})^{i^{c}})^{x} \), respectively. In terms of factorial numbers of the first kind (as defined in the introduction),

\[
\frac{(k - 1)!}{(m - 1)!} f^{ad, bc}(m, k) = \hat{b}^{1}_{m-k}(m) = (-1)^n \hat{b}^{2}_{m-k}(m) \quad \text{for } 0 \leq k \leq m.
\]

We find the inverse \( \gamma \) of \( \beta \) from Proposition 1. For positive integers \( n \) and \( m \), and \( 0 \leq l \leq m, 0 \leq k \leq n \),

\[
\langle \gamma^l \rangle_{m,n} = \frac{1}{mn} \sum_{i_1 + j_1 = l, j_3 + j_3 = -k} \langle \beta^{-m}_{i_1,j_1} \rangle_{i_1,j_1} \langle \beta^{-n}_{i_2,j_2} \rangle_{i_2,j_2} \frac{\det(i_1, i_2)}{i_1! j_1! j_2!} = \begin{cases} (-1)^{n-m-l-k} \hat{b}^{1}_{m-l-k}(m, l) \frac{(-abm)^{j_1}}{j_1!} (-cdn)^{i_1} \frac{(l + j + 1)!}{l!} \\ \times \hat{b}^{1}_{n-k-j}(n, j) \end{cases}
\]

\[
= \begin{cases} (-1)^{n-m-l-k} \hat{b}^{1}_{n-j-k}(n, j) \frac{(l + j)!(l + j + 1)!}{j!} \frac{(-cdn)^{i_1}}{i_1!} \\ \times \frac{(abm)^{j_1}}{j_1!} \end{cases}
\]

\[
\langle \gamma^l \rangle_{m,n} = \frac{1}{m} ((e^{ads} - e^{bcx})^{-m})^l = \frac{(l - 1)!}{m!} f^{ad, bc}(m, l),
\]

and

\[
\langle \gamma^k \rangle_{0,n} = \frac{1}{n} ((e^{bcx} - e^{ad})^{-n})^k = \frac{(-1)^n (k - 1)!}{n!} f^{ad, bc}(n, k).
\]

The above expression for \( \langle \gamma^l \rangle_{m,n} \) can be simplified, using an identity which is easily derived as follows. The factorial numbers of the first kind are the connection coefficients between the powers \( \xi^x \) and the polynomials of binomial type \( \{p_{v}(\xi) \mid v = 0, 1, \ldots\} \) (see [5]), satisfying the system of difference equations

\[
p_{v}(\xi + ad) - p_{v}(\xi + bc) = p_{v-1}(\xi), \quad v = 1, 2, \ldots
\]
Thus,

\[ p_\nu(\xi) = \frac{1}{\nu!} \sum_{i=1}^{\nu} f^{ad,bc}(\nu, i)\xi^i = \xi q^{ad,bc}_\nu(\xi) \]

where

\[ q^{ad,bc}_\nu(\xi) = (ad - bc)^{-\nu} \frac{1}{\nu!} \prod_{i=1}^{\nu-1} (\xi - iad - (\nu - i)bc). \]

From \( q^{ad+u/v, bc+u/v}_\nu(\xi) = q^{ad,bc}_\nu(\xi - u) \) follows that

\[ f^{ad+u/m, bc+u/m}(m, l) = \sum_{i=0}^{m-l} \binom{l + i - 1}{i} f^{ad,bc}(m, l + i)(-u)^{-i}. \]

To shorten the notation, we think of \( ad \) and \( bc \) as fixed, \( u, m, \) and \( l \) as variable, and write \( f(m, l; u) \) for the above factorial number. In this notation,

\[ \langle y_1, y_2 \rangle_{m,n} = (-1)^n \frac{l! k!}{m! n!} \cdot [f(m, l; cdn)f(n, k; abm) \]

\[ - abm f(m, l; cdn)f(n, k + 1; abm) - cdn f(m, l + 1; cdn)f(n, k; abm)]. \]

Especially,

\[ \langle y_1 \rangle_{m,n} = \frac{(-1)^{n-1}}{m! n!} abm f(m, 1; cdn)f(n, 1; abm) \]

\[ = (-1)^{n-1} cdn q^{ad,bc}_n(-abm)q^{ad,bc}_m(-cdn), \]

and \( \langle y_1 \rangle_{m,0} = q^{ad,bc}_m(0), \langle y_2 \rangle_{0,n} = (-1)^n q^{ad,bc}_n(0). \) Hence, \( cdn \langle y_1 \rangle_{m,n} = abm \langle y_2 \rangle_{m,n}. \) The compositional inverse of \( \mu \) is now obtained as \( (\lambda_1 \circ \gamma, \lambda_2 \circ \gamma), \)

where

\[ \langle \lambda_1 \circ \gamma \rangle_{m,n} = \langle ay_1 + by_2 \rangle_{m,n} \]

\[ = (-1)^{n-1} bd(am + cn)q^{ad,bc}_n(-abm)q^{ad,bc}_m(-cdn), \]

and

\[ \langle \lambda_2 \circ \gamma \rangle_{m,n} = \langle cy_1 + a y_2 \rangle_{m,n} \]

\[ = (-1)^{n-1} a c(bm + dn)q^{ad,bc}_n(-abm)q^{ad,bc}_m(-cdn). \]

References