Global bifurcation and nodal solutions for fourth-order problems with sign-changing weight

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ABSTRACT

In this paper, we shall establish unilateral global bifurcation result for a class of fourth-order eigenvalue problems with sign-changing weight. Under some natural hypotheses on perturbation function, we show that \((\mu_k, 0)\) is a bifurcation point of the above problems and there are two distinct unbounded continua, \((C^+_v)\) and \((C^-_v)\), consisting of the bifurcation branch \(C_v^k\) from \((\mu_k, 0)\), where \(\mu_k\) is the \(k\)th positive or negative eigenvalue of the linear problem corresponding to the above problems, \(v \in \{+,-\}\). As the applications of the above result, we study the existence of nodal solutions for a class of fourth-order eigenvalue problems with sign-changing weight. Moreover, we also establish the Sturm type comparison theorem for fourth-order problems with sign-changing weight.

1. Introduction

It is well known that fourth-order problems arise in many applications, see [9,14] and the references therein. Thus, there are many papers concerning the existence and multiplicity of positive solutions or sign-changing solutions addressed by using different methods [3,10,12,13,16]. Problems with sign-changing weight arise from population modeling. In this model, weight function \(m\) changes sign corresponding to the fact that the intrinsic population growth rate is positive at some points and is negative at others, for details, see [4].

Recently, Ma et al. [11] established the existence of the principal eigenvalues of the following linear indefinite weight problem

\[
\begin{aligned}
\begin{cases}
\begin{align*}
u'''(t) &= \lambda g(t)\nu, & t \in (0, 1), \\
u(0) &= u(1) = u'(0) = u'(1) = 0,
\end{align*}
\end{cases}
\end{aligned}
\]

where \(g : [0, 1] \to \mathbb{R}\) is a continuous sign-changing function. They also proved the existence of positive solutions for the corresponding nonlinear indefinite weight problem by Rabinowitz's global bifurcation results [15]. However, there is no any information on the high eigenvalues and the existence of sign-changing solutions for the corresponding nonlinear indefinite weight problem.

It is well known that Dancer’s bifurcation theorem for the compact perturbations of linear operators was a very important step in dealing with boundary value problems, see [7]. In [5], Dai and Ma established a Dancer-type unilateral global bifurcation result for the one-dimensional \(p\)-Laplacian problem. Later, Dai and Ma [6] established the spectrum of the following eigenvalue problem
\[
\begin{aligned}
&\begin{cases}
  u'''' = \mu m(t)u, & t \in (0, 1), \\
  u(0) = u(1) = u''(0) = u''(1) = 0,
\end{cases} \\
\end{aligned}
\]  
\(\text{(1.1)}\)

where \(\mu\) is a real parameter and \(m\) is sign-changing weight. They proved there exists a unique sequence of eigenvalues for the above problem. Each eigenvalue is simple, the eigenfunction corresponding to the \(k\)th positive or negative eigenvalue has exactly \(k - 1\) generalized simple zeros in \((0, 1)\).

In this paper, based the spectral theory of [6], we shall establish unilateral global bifurcation result about the continuum of solutions for the following fourth-order eigenvalue problem

\[
\begin{aligned}
&\begin{cases}
  u'''' = \mu m(t)u + g(t, u, \mu), & t \in (0, 1), \\
  u(0) = u(1) = u''(0) = u''(1) = 0,
\end{cases} \\
\end{aligned}
\]  
\(\text{(1.2)}\)

where \(m\) is a sign-changing function, \(g : (0, 1) \times \mathbb{R}^2 \rightarrow \mathbb{R}\) satisfies the Carathéodory condition and \(g(t, s, 0) \equiv 0\). Let \(I := (0, 1)\) and

\[
M(I) := \{m \in C[\bar{I}] | \text{meas}\{t \in I, m(t) > 0\} \neq 0\},
\]

We also assume that the perturbation function \(g : I \times \mathbb{R}^2 \rightarrow \mathbb{R}\) is continuous with \(g(t, s, 0) \equiv 0\) and satisfies the following hypotheses

\[
\lim_{s \to 0} \frac{g(t, s, \mu)}{|s|} = 0, 
\]

uniformly for \(t \in I\) and \(\mu\) on bounded sets.

Under the conditions of \(m \in M(I)\) and \((1.3)\), we shall show that \((\mu_1', 0)\) is a bifurcation point of \((1.2)\) and there are two distinct unbounded continua, \((C^+_{\mu_1'})\) and \((C^+_{\mu_1'})\), consisting of the bifurcation branch \(C^+_{\mu_1'}\) from \((\mu_1', 0)\), where \(\mu_1'\) is the \(k\)th positive or negative eigenvalue of the linear problem corresponding to \((1.2)\), where \(v \in \{+,-\}\).

Based on the above result, we investigate the existence of nodal solutions for the following fourth-order problem

\[
\begin{aligned}
&\begin{cases}
  u'''' - \gamma m(t)f(u) = 0, & t \in I, \\
  u(0) = u(1) = u''(0) = u''(1) = 0,
\end{cases} \\
\end{aligned}
\]  
\(\text{(1.4)}\)

where \(f \in C(\mathbb{R}, \mathbb{R}), \gamma\) is a parameter.

The rest of this paper is arranged as follows. In Section 2, we establish the unilateral global bifurcation theory for \((1.2)\). In Section 3, we establish the Sturm type comparison theorem for fourth-order problems with sign-changing weight. In Section 4, we prove the existence of nodal solutions for \((1.4)\) under the linear growth condition on \(f\).

## 2. Unilateral global bifurcation results

We start by considering the following auxiliary problem

\[
\begin{aligned}
&\begin{cases}
  -u'' = e(t), & t \in I, \\
  u(0) = u(1) = 0,
\end{cases} \\
\end{aligned}
\]  
\(\text{(2.1)}\)

for a given \(e \in C(\bar{I})\). It is well known that for every given \(e \in C(\bar{I})\) there is a unique solution \(u \in C^2(\bar{I})\) to the problem \((2.1)\) (see [2]). Let \(\Lambda(e)\) denote the unique solution to \((2.1)\) for a given \(e \in C(\bar{I})\). By the results of [2], we can easily show that \(\Lambda : C^2(\bar{I}) \rightarrow C^{2+k}(\bar{I})\) is continuous for any \(k \geq 0, k \in \mathbb{N} \cup \{0\}\). Hence, \(\Lambda : C(\bar{I}) \rightarrow C^{2+k}(\bar{I})\) is compact.

Let \(E = \{u \in C^4(\bar{I}) | u(0) = u(1) = u''(0) = u''(1) = 0\}\) with the norm\(\|u\| = \max_{t \in I} |u(t)| + \max_{t \in I} |u'(t)| + \max_{t \in I} |u''(t)| + \max_{t \in I} |u'''(t)|\).

Define the operator \(R : \mathbb{R} \times E \rightarrow E\) by

\[
R(\mu, u)(t) := \mu \lambda^2(mu) + \lambda^2 g(t, u, \mu),
\]

Then it is clear that problem \((1.2)\) can be equivalently written as

\[
u = R(\mu, u),
\]

Clearly, \(R\) is completely continuous from \(\mathbb{R} \times E \rightarrow E\) and \(R(\mu, 0) = 0, \forall \mu \in \mathbb{R}\). Let \(\mathcal{S}\) be the closure of the set of nontrivial solution pairs of problem \((1.2)\).

Let

\[
\tilde{g}(t, u, \mu) = \max_{0 \leq |s| \leq u} |g(t, s, \mu)| \quad \text{for} \quad t \in I \quad \text{and} \quad \mu \text{ on bounded sets},
\]

then \(\tilde{g}\) is nondecreasing with respect to \(u\) and
uniformly for $t \in I$ and $\mu$ on bounded sets. Further it follows from (2.2) that
\begin{equation}
\frac{g(t, u, \mu)}{\|u\|} \leq \frac{\tilde{g}(t, |u|, \mu)}{\|u\|} \leq \frac{\tilde{g}(t, \|u\|, \mu)}{\|u\|} \leq 0 \quad \text{as} \quad \|u\| \to 0,
\end{equation}
(2.3)
uniformly for $t \in I$ and $\mu$ on bounded sets. (2.3) implies that $\|A^2 g(t, u, \mu)\|/\|u\| \to 0$ as $\|u\| \to 0$ uniformly for $t \in I$ and $\mu$ on bounded sets. Applying Theorem 2 of [7], we may obtain the following result.

**Theorem 2.1.** Assume (1.3) holds and $m \in M(I)$. Then $(\mu^0_k, 0)$ is a bifurcation point of $S$ and there are two distinct continua $(c^+_k)$ and $(c^-_k)$ emanating from $(\mu^0_k, 0)$ such that either they are both unbounded or $(c^+_k)^+ \cap (c^-_k)^- \neq \{ (\mu^0_k, 0) \}$.

Next, we prove that the first choice of the alternative of Theorem 2.1 is the only possibility. To do it, we give the definitions of nodal solution, generalized simple zero and generalized double zero.

**Definition 2.1.** Let $u \in E$ and $t_0 \in I$ such that $u(t_0) = u'(t_0) = u''(t_0) = u'''(t_0) = 0$. We call that $t_0$ is a generalized simple zero if $u'(t_0) \neq 0$ or $u'''(t_0) \neq 0$. Otherwise, we call that $t_0$ is a generalized double zero. If there is no generalized double zero of $u$, we call that $u$ is a nodal solution.

Let $S^k_t$ denote the set of functions in $E$ which have exactly $k-1$ generalized simple zeros in $I$ and are positive near $t = 0$, and set $S_t = S^k_t$, and $S_k = S^k_t \cup S_t$. Clearly, they are disjoint and open in $E$. Finally, let $\Phi^+_k = \mathbb{R} \times S^k_t$ and $\Phi_k = \mathbb{R} \times S_k$ under the product topology.

**Lemma 2.1.** If $(\mu, u)$ is a solution of (1.2) and $u$ has a generalized double zero, then $u \equiv 0$.

**Proof.** Let $u$ be a solution of (1.2) and $t^* \in I$ be a generalized double zero, i.e., $u(t^*) = u'(t^*) = u''(t^*) = u'''(t^*) = 0$. We note that
\begin{equation}
|u(t)| = \left| \int_{t_0}^t \int_{t_0}^\tau \int_{t_0}^\tau \int_{t_0}^\tau (\mu m(\xi)u(\xi) + g(\xi, u(\xi), \mu)) d\xi d\tau d\xi d\tau \right| \leq \int_{t_0}^t \int_{t_0}^\tau \int_{t_0}^\tau (\mu m(\xi)u(\xi) + g(\xi, u(\xi), \mu)) d\xi d\tau d\xi,
\end{equation}
furthermore,
\begin{equation}
|u(t)| \leq \int_{t_0}^t \int_{t_0}^\tau \int_{t_0}^\tau (\mu m(\xi)u(\xi) + g(\xi, u(\xi), \mu)) d\tau d\xi d\tau \leq \int_{t_0}^t \int_{t_0}^\tau \int_{t_0}^\tau (\mu m(\xi)u(\xi) + g(\xi, u(\xi), \mu)) d\tau \leq \int_{t_0}^t \int_{t_0}^\tau \int_{t_0}^\tau \left( \mu m(\xi) + \frac{|g(\xi, u(\xi), \mu)|}{\|u(\xi)\|} \right) d\tau \leq \int_{t_0}^t \int_{t_0}^\tau \left( \mu m(\xi) + \frac{|g(\xi, u(\xi), \mu)|}{\|u(\xi)\|} \right) d\tau,
\end{equation}
In view of (1.3), for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that
\begin{equation}
|g(t, s, \mu)| \leq \varepsilon \cdot |s|,
\end{equation}
uniformly with respect to $t \in I$ and fixed $\mu$ when $|s| \in [0, \delta]$. Hence,
\begin{equation}
|u(t)| \leq \int_{t_0}^t \left( \mu m(\xi) + \varepsilon \max_{s \in [0, \delta]} \left| \frac{g(\xi, s, \mu)}{s} \right| \right) |u(\xi)| \ d\xi,
\end{equation}
By the Gronwall–Bellman inequality [1], we get $u \equiv 0$ on $[0, t^*]$. Similarly, we also can get $u \equiv 0$ on $[t^*, 1]$ and the proof is complete. $\square$

**Lemma 2.2.** Let $c^+_k := (c^+_k)^+ \cup (c^+_k)^-$. If $c^+_k \subset (\Phi_k \cup \{ (\mu^0_k, 0) \})$, then $c^+_k$ cannot contain a pair $(\Pi, 0)$ with $\Pi \neq \mu^0_k$.

**Proof.** Suppose on the contrary, if there exists $(\mu_m, u_m) \to (\mu^0_k, 0)$ when $m \to +\infty$ with $(\mu_m, u_m) \in c^+_k, u_m \neq 0$ and $j \neq k$. Let $w_m := u_m / \|u_m\|$, then $w_m$ should be a solution of problem
\begin{equation}
w(t) = A^2 \left( \mu m + \frac{g(t, u_m, \mu)}{\|u_m\|} \right),
\end{equation}
(2.4)
By (2.3), (2.4) and the compactness of $\Lambda^2$ we obtain that for some convenient subsequence $w_m \to w_0$ as $m \to +\infty$. Now $w_0$ verifies the equation

$$w'''_0 = \mu_1 m(t)w_0,$$

and $\|w_0\| = 1$. Hence $w_0 \in S_j$ which is an open set in $E$, and as a consequence for some $m$ large enough, $u_m \in S$, and this is a contradiction. □

**Theorem 2.2.** Assume that (1.3) holds and $m \in M(l)$, then from each $(\mu^*_k, 0)$ it bifurcates two distinct unbounded continua $(C^+_k)$ and $(C^-_k)$ of $S$. Moreover, for $\sigma \in \{+, -\}$, we have that

$$(C^+_k)^\sigma \subset \{(\mu^*_k, 0) \cup (\mathbb{R} \times \mathbb{S}_k^\sigma)\}.$$

**Proof.** Noting Theorem 2.1, we only need to prove that $(C^+_k)^\sigma \subset \{(\mu^*_k, 0) \cup (\mathbb{R} \times \mathbb{S}_k^\sigma)\}$. We claim that there exists a neighborhood $\mathcal{O}$ of $(\mu^*_k, 0)$ such that $(\mathcal{O} \cap (C^+_k)^\sigma) \subset \{(\mu^*_k, 0) \cup (\mathbb{R} \times \mathbb{S}_k^\sigma)\}$. Suppose on the contrary, if there exists $(\mu_m, u_m) \to (\mu'_k, 0)$ when $m \to +\infty$ with $(\mu_m, u_m) \in (C^+_k)^\sigma \setminus (\mathbb{R} \times \mathbb{S}_k^\sigma)$ and $u_m \neq 0$. Let $w_m := u_m/\|u_m\|$, then $w_m$ should be a solution of problem

$$w(t) = \Lambda^2 \left( \mu m + \frac{g(t, u, z)}{\|u_m\|} \right), \quad (2.5)$$

By (2.3), (2.4) and the compactness of $\Lambda^2$ we obtain that for some convenient subsequence $w_m \to w_0$ as $m \to +\infty$. Now $w_0$ verifies the equation

$$w'''_0 = \mu_1 m(t)w_0,$$

and $\|w_0\| = 1$. Hence $w_0 \in S_k^r$ which is an open set in $E$, and as a consequence for some $m$ large enough, $u_m \in S_k^r$, and this is a contradiction.

Suppose $(C^+_k)^\sigma \subset \{(\mu^*_k, 0) \cup (\mathbb{R} \times \mathbb{S}_k^\sigma)\}$. Then there exists $(\mu, u) \in (C^+_k)^\sigma \cap (\mathbb{R} \times \mathbb{S}_k^\sigma)$ such that $(\mu, u) \neq (\mu^*_k, 0)$ and $(\mu_m, u_m) \to (\mu, u)$ with $(\mu_m, u_m) \in (C^+_k)^\sigma \cap (\mathbb{R} \times \mathbb{S}_k^\sigma)$. Since $u \in \partial S_k^r$, by Lemma 2.1, $u \equiv 0$. Let $v_n := u_n/\|u_n\|$, then $v_n$ should be a solution of problem

$$v(t) = \Lambda^2 \left( \mu m + \frac{g(t, u, z)}{\|u_n\|} \right). \quad (2.6)$$

By (2.3), (2.5) and the compactness of $\Lambda^2$ we obtain that for some convenient subsequence $v_n \to v_0$ as $n \to +\infty$. Now $v_0$ verifies the equation

$$v'''_0 = \mu m(t)v_0,$$

and $\|v_0\| = 1$. Hence $\mu = \mu^*_j$, for some $j \neq k$. Therefore, $(\mu_n, u_n) \to (\mu^*_j, 0)$ with $(\mu_n, u_n) \in (C^+_k)^\sigma \cap (\mathbb{R} \times \mathbb{S}_k^\sigma)$. This contradicts Lemma 2.2. □

### 3. Sturm type comparison theorem

In this section, we shall establish the Sturm type comparison theorem for fourth-order differential equations with sign-changing weight, which will be used later.

**Lemma 3.1.** Let $b_2(t) > b_1(t) > 0$ for $t \in I$ and $b_i(t) \in C(I), i = 1, 2$. Also let $u_1, u_2 \in E$ be nontrivial solutions of the following differential equations

$$u''' = b_1(t)u, \quad t \in I, \quad i = 1, 2,$$

respectively. If $u_1$ has $k$ generalized simple zeros in $I$, then $u_2$ has at least $k + 1$ generalized simple zeros in $I$.

**Proof.** Let $c$ and $d$ be any two consecutive generalized simple zeros of $u_1$ on $I$. Firstly, we claim that $u_1 \neq 0$ in $(c, d)$. Suppose on the contrary that there exists a point $t_0 \in (c, d)$ such that $u_1(t_0) = 0$. By an argument similar to that of Theorem 3.3 of [6], we can show that $u'_1(t_0) = 0$. If $u'_1(t_0) = u''_1(t_0) = 0$, we have that $u \equiv 0$; this is a contradiction. Otherwise, $t_0$ is generalized simple zero of $u_1$ on $I$, which contradicts the fact that $c$ and $d$ are two consecutive generalized simple zeros of $u_1$ on $I$.

If $u_2$ has a generalized simple zero in $(c, d)$, then the conclusion has done. Otherwise, by an argument similar with the above, we can show that $u_2 \neq 0$ in $(c, d)$. Without loss of generality, we can assume that $u_1(t) > 0, u_2(t) > 0$ in $(c, d)$. Then an easy calculation shows that
The left-hand side of (3.1) equals
\[ u''_1(d)u_2(d) - u''_1(c)u_2(c) + u'_1(d)u'_2(d) - u'_1(c)u'_2(c), \]
Next, we shall show that
\[ u''_1(d)u_2(d) - u''_1(c)u_2(c) + u'_1(d)u'_2(d) - u'_1(c)u'_2(c) \geq 0, \]
In fact, if this occurs, we arrive a contradiction. We divide the proof into two steps.

**Step 1**: We show that \( u''_1(d)u_2(d) - u''_1(c)u_2(c) \geq 0 \).

Let \( \nu := u'_1 \). We consider the following system
\[
\begin{cases}
\nu^2 = \nu, & t \in I, \\
\nu' = b_1u_1,
\end{cases}
\]
By simple computation, one has that
\[ u'_1(t)^2 = \frac{t^2}{2} + \frac{bu_1^2}{2} + C, \]
for any constant \( C \). Let \( t_0 \in (c, d) \) be the point satisfying
\[ u_1(t_0) = \max_{t \in [c, d]} u_1(t), \]
Then (3.2) implies that
\[ 0 = \frac{t_0^2}{2} + \frac{bu_1^2}{2} + C, \]
It follows \( C < 0 \). Putting \( c \) into (3.2), we have that
\[ u'_1(c)u''_1(c) = C < 0, \]
Using this and the fact \( u'_1(c) \geq 0 \), we get that \( u''_1(c) < 0 \). Similarly, we can show that \( u''_1(d) > 0 \). Hence, we have that
\[ u''_1(d)u_2(d) - u''_1(c)u_2(c) \geq 0. \]
**Step 2**: We show that \( u''_1(d)u_2(d) - u''_1(c)u_2(c) \geq 0 \).

It suffices to show that \( u''_1(c) \leq 0 \) and \( u''_1(d) \leq 0 \) since the facts \( u'_1(c) \geq 0 \) and \( u'_1(d) \leq 0 \). Suppose on the contrary that \( u''_1(c) > 0 \) or \( u''_1(d) < 0 \), we shall deduce a contradiction.

Let \( u_\epsilon := u_2(t) + 1. \) Then \( u''_\epsilon = b_2u_2 \) and \( u_\epsilon \geq 1 \) in \((c, d)\). For some \( \epsilon > 0 \) small enough, let \( \tilde{u} \in C^4([-\epsilon, 1 + \epsilon]) \) and \( \tilde{b} \geq 0 \) be such that \( \tilde{u}(-\epsilon) = \tilde{u}(1 + \epsilon) = \tilde{u}'(-\epsilon) = \tilde{u}'(1 + \epsilon) = 0, \tilde{u}'_\epsilon = u_\epsilon \) and \( \tilde{u}''_\epsilon = \tilde{b}\tilde{u} \). Then we have
\[
\begin{cases}
\tilde{u}'(t) = \tilde{b}\tilde{u}, & t \in (-\epsilon, 1 + \epsilon), \\
\tilde{u}(-\epsilon) = \tilde{u}(1 + \epsilon) = \tilde{u}'(-\epsilon) = \tilde{u}'(1 + \epsilon) = 0,
\end{cases}
\]
Set \( a := (2c - \epsilon)/2 \) and \( b := (2d + \epsilon)/2 \). Let \( \pi \in C^4([a, b]) \) and \( \bar{b} \geq 0 \) be such that \( \pi'_{\epsilon} = \bar{b}\pi \geq 0 \) in \((a, b)\) and \( \pi(a) = \pi(b) = \pi'(a) = 0 \) and \( \pi''_{\epsilon} = \bar{b}\pi \). Set \( w := \pi' \), then \( w \) should be a solution of the following problem
\[
\begin{cases}
w' = \bar{b}w, & t \in (a, b), \\
w(a) = w(b) = 0,
\end{cases}
\]
The Strong Maximum Principle implies that \( w < 0 \) in \((a, b)\). This follows that \( u''_1 \leq 0 \) in \([c, d] \). \( \square \)

**Lemma 3.2.** Assume \( m \in M(I) \). Let \( \bar{I} = [a, b] \) be such that \( \bar{I} \subset I^+ \) and
\[ \operatorname{meas} \bar{I} > 0. \]
Let \( g_n : I \to (0, +\infty) \) be continuous function and such that
\[ \lim_{n \to +\infty} g_n(t) = +\infty \text{ uniformly on } \bar{I}, \]
Let \( y_n \) be a solution of the equation
\[
\begin{align*}
\begin{cases}
y_n''(t) = m(t)g_n(t)y_n, & t \in I, \\
u(0) = u(1) = u'(1) = u''(1) = 0,
\end{cases}
\end{align*}
\]
Then the number of zeros of \( y_n \) in \( I \) goes to infinity as \( n \to +\infty \).

**Proof.** After taking a subsequence if necessary, we may assume that
\[
m(t)g_n(t) \geqslant \lambda_j, \quad t \in \bar{I},
\]
as \( j \to +\infty \), where \( \lambda_j \) is the \( j \)th eigenvalue of the following problem
\[
\begin{align*}
\begin{cases}
u''(t) = \lambda u(t), & t \in \bar{I}, \\
u(a) = u(b) = u''(a) = u''(b) = 0,
\end{cases}
\end{align*}
\]
Let \( \phi_j \) be the corresponding eigenvalue of \( \lambda_j \). It is easy to check that the distance between any two consecutive zeros of \( \phi_j \) is \( (b - a)/j \) (also see [8]). Hence, the number of zeros of \( \phi_j(y) \) goes to infinity as \( j \to +\infty \). By Lemma 3.1, one has that the number of zeros of \( y_n \) goes to infinity as \( n \to +\infty \). It follows the desired results. \( \square \)

Similarly, we also have the following lemma.

**Lemma 3.3.** Assume that \( m \in M(I) \). Let \( \bar{I} = (c, d) \) be such that \( \bar{I} \subset I^- \) and \( \operatorname{meas} \bar{I} > 0 \),
\[
\text{Let } g_n : I \to (-\infty, 0) \text{ be continuous function and such that} \lim_{t \to \pm \infty} g_n(t) = -\infty \text{ uniformly on } \bar{I}.
\]
Let \( y_n \) be a solution of the equation
\[
\begin{align*}
y_n''(t) = m(t)g_n(t)y_n, & \quad t \in I, \\
u(0) = u(1) = u'(1) = u''(1) = 0,
\end{align*}
\]
Then the number of zeros of \( y_n \) goes to infinity as \( n \to +\infty \).

**4. Existence of nodal solutions for (1.4)**

In this section, we shall investigate the existence and multiplicity of nodal solutions for problem (1.4) under the linear growth condition on \( f \).

Firstly, we suppose that:
\[(H_1) \ f \in C(\mathbb{R}, \mathbb{R}) \text{ with } f(s)s > 0 \text{ for } s \neq 0; \]
\[(H_2) \text{ there exist } f_0, f_\infty \in (0, +\infty) \text{ such that} \]
\[
\begin{align*}
f_0 &= \lim_{|s| \to 0} \frac{f(s)}{s}, & f_\infty &= \lim_{|s| \to +\infty} \frac{f(s)}{s}.
\end{align*}
\]
Let \( \mu_k^+ \) be the \( k \)th positive or negative eigenvalue of (1.1). Applying Theorem 2.2, we shall establish the existence of nodal solutions of (1.4) follows.

**Theorem 4.1.** Let \( (H_1), (H_2) \) hold and \( m \in M(I) \). Assume that for some \( k \in \mathbb{N} \), either
\[
\gamma \in \left( \frac{\mu_k^+}{f_\infty}, \frac{\mu_k^-}{f_0} \right) \cup \left( \frac{\mu_k^+}{f_0}, \frac{\mu_k^-}{f_\infty} \right),
\]
or
\[
\gamma \in \left( \frac{\mu_k^+}{f_\infty}, \frac{\mu_k^-}{f_\infty} \right) \cup \left( \frac{\mu_k^+}{f_0}, \frac{\mu_k^-}{f_0} \right).
\]
Then (1.4) has two solutions \( u_\gamma^+ \) and \( u_\gamma^- \) such that \( u_\gamma^+ \) has exactly \( k - 1 \) generalized simple zeros in \( I \) and is positive near 0, and \( u_\gamma^- \) has exactly \( k - 1 \) generalized simple zeros in \( I \) and is negative near 0.

**Proof.** We only prove the case of \( \gamma > 0 \). The case of \( \gamma < 0 \) is similar. Consider the problem
\[
\begin{align*}
u''(t) &= \mu \gamma m(t)f(u), & t \in I, \\
u(0) &= u(1) = u'(1) = u''(1) = 0,
\end{align*}
\]
(4.1)
Let $\zeta \in C(\mathbb{R}, \mathbb{R})$ be such that
\[ f(s) = f_0 s + \zeta(s), \]
with
\[ \lim_{|s| \to 0} \frac{\zeta(s)}{s} = 0. \]
Hence, the condition (1.3) holds. Using Theorem 2.2, we have that there are two distinct unbounded continua, \((C_k^+)\) and \((C_k^-)\), consisting of the bifurcation branch \(c_k^+\) from \((\mu_k^+ / f_0, 0)\), such that
\[ (C_k^+)^c \subset \left\{ (\mu_k^0, 0) \right\} \cup (\mathbb{R} \times L^1) \]
It is clear that any solution of (4.1) of the form \((1, u)\) yields a solutions \(u\) of (4.1). We shall show that \((C_k^+)^c\) crosses the hyperplane \((1) \times E\) in \(\mathbb{R} \times E\). To this end, it will be enough to show that \((C_k^+)^c\) joins \((\mu_k^0 / f_0, 0)\) to \((\mu_k^+ / f_0, +\infty)\). Let \((\mu_n, y_n) \in (C_k^+)^c\) satisfy
\[ \mu_n + \|y_n\| \to +\infty. \]
We note that \(\mu_n > 0\) for all \(n \in \mathbb{N}\) since \((0, 0)\) is the only solution of (4.1) for \(\mu = 0\) and \((C_k^+)^c \cap \{(0) \times E\} = \emptyset\).

Case 1: \(\mu_k^+ / f_\infty < \gamma < \mu_k^+ / f_0\).

In this case, we only need to show that
\[ \frac{\mu_k^+}{f_\infty} - \frac{\mu_k^+}{f_0} \subseteq \left\{ \mu \in \mathbb{R} : (\mu, u) \in (C_k^+)^c \right\}. \]
We divide the proof into two steps.

Step 1: We show that if there exists a constant \(M > 0\) such that
\[ \mu_n < (0, M), \]
for \(n \in \mathbb{N}\) large enough, then \((C_k^+)^c\) joins \((\mu_k^0 / f_0, 0)\) to \((\mu_k^+ / f_\infty, +\infty)\).

In this case it follows that
\[ \|y_n\| \to +\infty. \]
Let \(\xi \in C(\mathbb{R}, \mathbb{R})\) be such that
\[ f(s) = f_n s + \xi(s), \]
Then
\[ \lim_{|s| \to \infty} \frac{\xi(s)}{s} = 0, \]
Let
\[ \bar{\xi}(u) = \max_{0 \leq |s| \leq u} |\xi(s)|, \]
Then \(\bar{\xi}\) is nondecreasing and
\[ \lim_{u \to +\infty} \frac{\bar{\xi}(u)}{u} = 0, \quad (4.2) \]
We divide the equation
\[ y_n^\gamma = \mu_n y_n^\gamma f_\infty y_n + \mu_n^\gamma m(t) \xi(y_n), \]
by \(\|y_n\|\) and set \(y_n = y_n / \|y_n\|\). Since \(y_n\) is bounded in \(E\), after taking a subsequence if necessary, we have that \(y_n \to y\) for some \(y \in E\). Moreover, from (4.2) and the fact that \(\bar{\xi}\) is nondecreasing, we have that
\[ \lim_{n \to +\infty} \frac{\bar{\xi}(y_n(t))}{\|y_n\|} = 0, \quad (4.3) \]
since
\[ \frac{\bar{\xi}(y_n(t))}{\|y_n\|} \leq \frac{\bar{\xi}(\|y_n(t)\|)}{\|y_n\|} \leq \frac{\bar{\xi}(\|y_n(t)\|)}{\|y_n\|} \leq \frac{\bar{\xi}(\|y_n(t)\|)}{\|y_n\|}. \]
By the continuity and compactness of \(\Lambda^2\), it follows that
\[ \bar{y}^\gamma = \bar{y}^\gamma m(t) f_\infty \bar{y}, \]
where \(\bar{y} = \lim_{n \to +\infty} \mu_n^\gamma y\), again choosing a subsequence and relabeling it if necessary. We claim that
\[ \bar{y} \in (C_{k}^{+})^\sigma, \]

It is clear that \( \bar{y} \in (C_{k}^{+})^\sigma \subseteq (C_{k}^{+})^\sigma \) since \( (C_{k}^{+})^\sigma \) is closed in \( \mathbb{R} \times E \). Hence, \( \bar{y} \in \Pi_{\infty} = \mu^{+}_n \), so that

\[ \bar{\Pi} = \frac{\mu^{+}_n}{f_{\infty}}. \]

Therefore, \( (C_{k}^{+})^\sigma \) joins \( (\mu^{+}_n/f_0, 0) \) to \( (\mu^{+}_n/f_{\infty}, +\infty) \).

Step 2: We show that there exists a constant \( M \) such that \( \mu_n \in (0, M] \) for \( n \in \mathbb{N} \) large enough.

On the contrary, we suppose that

\[ \lim_{n \to +\infty} \mu_n = +\infty. \]

Since \( (\mu_n, y_n) \in (C_{k}^{+})^\sigma \), it follows that

\[ y_n^m = \gamma \mu_n m(t) \tilde{f}_n(t) \varphi(y_n), \]

where

\[ \tilde{f}_n(t) = \begin{cases} \frac{f(y_n(t))}{y_n(t)}, & \text{if } y_n(t) \neq 0, \\ f_0, & \text{if } y_n(t) = 0. \end{cases} \]

Conditions \( (H_4) \) and \( (H_5) \) imply that there exists a positive constant \( \gamma \) such that \( \tilde{f}_n(t) > \gamma \) for any \( t \in I \) and all \( n \in \mathbb{N} \). Then Lemma 3.2 follows that \( y_n \) has more than \( k \) zeros in \( I \) for \( n \) large enough, and this contradicts the fact that \( y_n \) has exactly \( k \) zeros in \( I \).

Case 2: \( \mu^{+}_n/f_0 < \gamma < \mu^{+}_n/f_{\infty} \).

In this case, we have that

\[ \frac{\mu^{+}_n}{f_0} < 1 < \frac{\mu^{+}_n}{f_{\infty}}. \]

Assume that \( (\mu_n, y_n) \in (C_{k}^{+})^\sigma \) is such that

\[ \lim_{n \to +\infty} (\mu_n + ||y_n||) = +\infty, \]

In view of Step 2 of Case 1, we have known that there exists \( M > 0 \), such that for \( n \in \mathbb{N} \) sufficiently large,

\[ \mu_n \in (0, M]. \]

Applying the same method used in Step 1 of Case 1, after taking a subsequence and relabeling it if necessary, it follows that

\[ (\mu_n, y_n) \to (\frac{\mu^{+}_n}{f_{\infty}}, +\infty) \text{ as } n \to +\infty. \]

Thus, \( (C_{k}^{+})^\sigma \) joins \( (\mu^{+}_n/f_0, 0) \) to \( (\mu^{+}_n/f_{\infty}, +\infty) \). □

Using the similar proof with that of Theorem 4.1, we can obtain the more general results as follows.

**Theorem 4.2.** Let \( (H_4), (H_5) \) hold and \( m \in M(I) \). Assume that for some \( k, n \in \mathbb{N} \) with \( k \leq n \), either

\[ \gamma \in \left( \frac{\mu^{+}_n}{f_{\infty}}, \frac{\mu^{+}_n}{f_0} \right), \quad \left( \frac{\mu^{+}_n}{f_{\infty}}, \frac{\mu^{+}_n}{f_0} \right). \]

or

\[ \gamma \in \left( \frac{\mu^{+}_n}{f_0}, \frac{\mu^{+}_n}{f_{\infty}} \right), \quad \left( \frac{\mu^{+}_n}{f_0}, \frac{\mu^{+}_n}{f_{\infty}} \right). \]

Then (1.4) has \( n - k + 1 \) pairs solutions \( u_j^+ \) and \( u_j^- \) for \( j \in \{k, \ldots, n\} \) such that \( u_j^+ \) has exactly \( j - 1 \) generalized simple zeros in \( I \) and is positive near 0, and \( u_j^- \) has exactly \( j - 1 \) generalized simple zeros in \( I \) and is negative near 0.

**Remark 4.1.** Clearly, Theorem 1.1 of [11] is the corollary of Theorem 4.1.

**References**


