Randomized Consensus Processing over Random Graphs: Independence and Threshold

Guodong Shi and Karl Henrik Johansson

Abstract

In this paper, we study randomized consensus processing over general random graphs. At time step $k$, each node will follow the standard consensus algorithm, or stick to current state by a simple Bernoulli trial with success probability $p_k$. Connectivity-independent and arc-independent graphs are defined, respectively, to capture the fundamental independence of random graph processes with respect to a consensus convergence. Sufficient and/or necessary conditions are presented on the success probability sequence for the network to reach a global a.s. consensus under various conditions of the communication graphs. Particularly, for arc-independent graphs with simple self-confidence condition, we show that $\sum_k p_k = \infty$ is a sharp threshold corresponding to a consensus $0 \rightarrow 1$ law, i.e., the consensus probability is 0 for almost all initial conditions if $\sum_k p_k$ converges, and jumps to 1 for all initial conditions if $\sum_k p_k$ diverges. Convergence rates are established by lower and upper bounds of $\epsilon$-computation time. Finally, a belief evolution model in social networks is investigated and convergence condition is given for an opinion agreement, as a simple application of previous result.

Keywords: Random Graphs, Dynamics Randomization, Consensus, Threshold

I. INTRODUCTION

In recent years, there has been considerable research interest on distributed algorithms for exchanging information, for estimating and for computing over a network of nodes, due to a variety of potential applications in sensor, peer-to-peer and wireless networks. Targeting the design of simple decentralized algorithms for computation or estimation, where each node exchanges information only in a neighboring view, distributed averaging or simple consensus seeking serves as a primitive toward more sophisticated information processing algorithms [16], [17], [18], [34], [31].

The investigation of consensus problem actually had a long history, which can be traced back to 1950s on the study of ergodicity of non-homogeneous Markov chains described as product of stochastic matrices [10], [9]. Stochastic matrices’ product had a natural distributed structure, and was then used to model standard consensus, or distributed averaging algorithms in discrete-time fashion among different research areas such as computer science [12], [13], engineering [15], [14], [23] and social science [11], [40], [41]. Deterministic consensus algorithms have been extensively studied for both time-invariant and time-varying communication graphs in the literature, in which efforts were typically devoted to finding proper connectivity conditions which can ensure a desired collective convergence for the considered network [24], [15], [14], [23], [20], [21]. More than that, researchers were also interested in the design of averaging algorithms to reach a faster consensus, or reach a consensus asynchronously [31], [30], [22], [34], [35].

The underlying communication graph based on which consensus algorithms are carried out may be random. Among various random graph processes, Erdős–Rényi model, usually denoted as $G(n, p)$, is a classical model, in which each edge exists randomly and independent of other edges with probability $p \in (0, 1]$ over a network with $n$ nodes. In [25], the authors studied linear consensus dynamics with communication graphs as a sequence of independent, identically distributed (i.i.d.) Erdős–Rényi random graphs, and almost sure convergence was shown. Then in [26], the analysis was generalized to directed Erdős-Rényi graphs.

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Moreover, mean-square performance for consensus algorithms over i.i.d. random graphs was studied in [28], and the influence of random packet drop was investigated in [29]. Some other results on distributed consensus over random graphs with graph independence showed up in [27], [37]. The independence of the switching communication graphs may be missing. In [33], the communication graph was described as a finite-state Markov chain where each topology corresponds to one state, and almost sure consensus was studied by investigating the connectivity of the closed positive recurrent sets of the Markovian random graph. In [36], convergence to consensus was studied under more general setting for linear consensus algorithms, where the random update and control matrices were determined by possibly non-stationary stochastic matrix processes coupled with disturbances.

However, classical random graph theory suggested that many important properties of graphs appear suddenly from a probabilistic point of view [5]. To be precise, there is usually a function \( f_0(n) \) called threshold for the edge existence probability \( p(n) \), such that if \( p(n) \) grows faster than \( f_0(n) \), almost every graph has certain property like whether the graph is connected, \( k \)-connected, Hamiltonian, etc. While if \( p(n) \) grows slower than \( f_0(n) \), almost every graph fails to have such a property. This phenomena was called a 0–1 law. For instance, it has been shown that \( \ln n/n \) is such a threshold function for the connectivity of \( G(n,p) \) [8]. Moreover, threshold function was also been shown for random geometric graph as a general model of wireless networks in [7].

Therefore, naturally, one may wonder, would there be any threshold condition which leads to a similar 0–1 law with respect to collective convergence of consensus dynamics over random graphs? Although various sufficient conditions have been established to guarantee a global consensus with probability 1 [25], [26], [28], [29], [36], and many of them are even sufficient and necessary [27], [33], the literature still lacks of a model and analysis for consensus dynamics over random graphs which can accurately describe the fundamental independence of graph processes and whether such a 0–1 law could exist.

To this end, in this paper, we study standard consensus algorithm on random graphs with simple Bernoulli randomization. The communication graph is assumed to be a general random digraph process independent with probability such that node \( i \) is asleep or broken at time \( k \), which usually comes from the unpredictability of the environment and the unreliability of the communication network [28], [37].

### A. Problem Definition

Consider a network with node set \( \mathcal{V} = \{1, 2, \ldots, n\} \). A (simple) directed graph (digraph), \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) consists of a node set \( \mathcal{V} \) and an arc set \( \mathcal{E} \), where each element \( e = (i,j) \in \mathcal{E} \) is an ordered pair of two different nodes in \( \mathcal{V} \) from node \( i \) to node \( j \) [4]. Then there are as many as \( 2^{n(n-1)} \) different digraphs with node set \( \mathcal{V} \). We label these graphs from 1 to \( 2^{n(n-1)} \) by an arbitrary order. In the following, we will identify an integer in \([1, 2^{n(n-1)}]\) with the corresponding graph in this order. Denote \( \Omega = \{1, \ldots, 2^{n(n-1)}\} \) as the graph set.

The communication graph of the network over time, is model as a sequence of random variables, \( \{\mathcal{G}_k(\omega) = (\mathcal{V}, \mathcal{E}_k(\omega))\}_{k=0}^\infty \), which take value in \( \Omega \). Where there is no possible confusion, we write \( \mathcal{G}_k(\omega) \) as \( \mathcal{G}_k \).

We call node \( j \) a neighbor of \( i \) if there is an arc from \( j \) to \( i \) in graph \( \mathcal{G} \), and each node is supposed to be a neighbor of itself. Denote the random set \( \mathcal{N}_i(k) = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}_k\} \cup \{i\} \) as the neighbor set of node \( i \) at time \( k \). The agent dynamics is described as follows:

\[
x_i(k + 1) = \begin{cases} 
\sum_{j \in \mathcal{N}_i(k)} a_{ij}(k)x_j(k), & \text{with probability } p_k \\
x_i(k), & \text{with probability } 1 - p_k
\end{cases}
\] (1)

where \( 0 \leq p_k < 1 \) and \( a_{ij}(k) \) denotes the weight of arc \((j, i)\). For \( a_{ij}(k) \), we assume the following weights rule as our standing assumptions [17], [20], [21], [39].
A1. For all \( i \) and \( k \), we have \[ \sum_{j \in \mathcal{N}(k)} a_{ij}(k) = 1. \]

A2. There exists a constant \( \eta > 0 \) such that \( \eta \leq a_{ij}(k) \) for all \( i, j \) and \( k \).

Denote
\[
H(k) = \max_{i=1,...,n} x_i(k), \quad h(k) = \min_{i=1,...,n} x_i(k)
\]
as the maximum and minimum states among all nodes, respectively, and define \( \mathcal{H}(k) = H(k) - h(k) \) as the consensus metric. Our interest is in the consensus convergence of the randomized consensus algorithm and in the (absolute) time it takes for the network to reach a consensus [31].

**Definition 1:** A global a.s. consensus of (1) is achieved if
\[
\mathbb{P}(\lim_{k \to \infty} \mathcal{H}(k) = 0) = 1
\]
for any initial condition \( x(0) = (x_1(0) \ldots x_n(0))^T \in \mathbb{R}^n \). Moreover, for any \( 0 \leq \epsilon < 1 \), the \( \epsilon \)-computation time is denoted by \( T_{\text{com}}(\epsilon) \), and is defined as
\[
T_{\text{com}}(\epsilon) = \sup_{x(0)} \left\{ k : \mathbb{P}\left( \frac{\mathcal{H}(k)}{\mathcal{H}(0)} \geq \epsilon \right) \leq \epsilon \right\}. \tag{3}
\]

**B. Main Results**

We first present an impossibility conclusion.

**Theorem 1:** If \( \sum_{k=0}^{\infty} p_k < \infty \), then global a.s. consensus cannot be achieved for Algorithm (1). Moreover, a general lower bound for \( T_{\text{com}}(\epsilon) \) can be given by
\[
T_{\text{com}}(\epsilon) \geq \sup \left\{ k : \sum_{i=0}^{k-1} \log(1 - p_i)^{-1} \leq \frac{\log \epsilon^{-1}}{n} \right\}.
\]

Note that, Theorem 1 holds for all possible graph processes. Plus a simple self-confidence assumption, this impossibility claim can be improved as follows.

**Theorem 2:** Assume that \( a_{ij}(k) \geq \gamma_0 \) for all \( i \) and \( k \), where \( \gamma_0 > 1/2 \) is a constant. If \( \sum_{k=0}^{\infty} p_k < \infty \), then for almost all initial conditions, Algorithm (1) achieves consensus with probability 0.

In order to establish possibility answers to a global consensus, we need independence and connectivity of the graph processes. A digraph \( G \) is said to be quasi-strongly connected if \( G \) has a root [6]. Then we introduce the following definition.

**Definition 2:** Let \( \{G_k\}_{0}^{\infty} \) be a random graph process. Then \( \{G_k\}_{0}^{\infty} \) is called to be

(i) connectivity-independent if events \( C_k \doteq \{G_k \text{ is quasi-strongly connected} \} \), \( k = 0, 1, \ldots \), are independent.

(ii) arc-independent if there exists a (nonempty) deterministic graph \( G^* = (V, E^*) \) such that events \( A_{k, r} \doteq \{(i_r, j_r) \in G_k\} \), \( (i_r, j_r) \in E^*, k = 0, 1, \ldots \), are independent. In this case \( G^* \) is called a basic graph of this random graph process.

The sufficiency results for consensus convergence are stated in the following, respectively, for connectivity-independent and arc-independent graphs.

**Theorem 3:** Suppose \( \{G_k\}_{0}^{\infty} \) is connectivity-independent and there exists a constant \( 0 < q < 1 \) such that \( \mathbb{P}(G_k \text{ is quasi-strongly connected}) \geq q \) for all \( k \). Assume in addition that \( p_{k+1} \leq p_k \). Then Algorithm (1) achieves a global a.s. consensus if \( \sum_{s=0}^{\infty} p_k^{s-1} = \infty \). Moreover, an upper bound for \( T_{\text{com}}(\epsilon) \) can be given by
\[
T_{\text{com}}(\epsilon) \leq \inf \left\{ M : \sum_{i=1}^{M} \log \left( 1 - \frac{(q^n)(n-1)^2}{2} \cdot p_i^{n-1} \right) \geq \log \epsilon^{-2} \right\} \times (n - 1)^2.
\]
Consensus appears suddenly for arc-independent graphs with $a_{ii}(k) \geq \gamma_0$.

**Theorem 4:** Suppose $\{G_k\}_{0}^{\infty}$ is arc-independent with a quasi-strongly connected basic graph, and there exists a constant $0 < \theta_0 < 1$ such that $P((i,j) \in E_k) \geq \theta_0$ for all $k$ and $(i, j) \in E^*$. Then Algorithm (1) achieves a global a.s. consensus if and only if $\sum_{k=0}^{\infty} p_k = \infty$. In this case, we have

$$T_{com}(\epsilon) \leq \inf \left\{ k : \sum_{i=0}^{k-1} (1 - (1 - p_i)^n) \geq \frac{(n - 1)}{\log A} \log \left( \frac{A\epsilon^2}{n} \right) \right\}$$

(4)

where $A = 1 - \left( \frac{\eta\theta_0}{n} \right)^{(n-1)|E^*|}$ with $|E^*|$ as the number of elements in $E^*$.

Connectivity is a global property for a graph, and it indeed does not rely on any specific arc. We believe that the convergence condition given in Theorem 3 is quite tight since the probability that all the links function in Algorithm (1) at time $k$ is $p_k^n$, and connectivity can be lost easily by losing any single link. Moreover, the convergence conditions given in Theorems 3 and 4 are consistent with the widely-used decreasing gain condition in the study of stochastic approximations on various adaptive algorithms [3].

Combing Theorems 2 and 4, we see that $\sum_{k=0}^{\infty} p_k = \infty$ is a sharp threshold for Algorithm (1) to reach consensus with arc-independent graphs and self-confidence assumption (see Fig. 1). In other words, a similar $0 - 1$ law is established for consensus dynamics on random graphs as the classical random graph theory [5].

**C. Paper Organization**

The rest of the paper is organized as follows.

In Section II, we prove the impossibility claims, i.e., Theorems 1 and 2.

Then in Section III, convergence analysis for connectivity-independent graphs is presented. In fact, we will study some generalized cases which only rely on the independence and connectivity of joint graphs on different time intervals. This concept of joint connectivity has been widely studied in the literature for deterministic consensus algorithms [16], [17], [24], [20], [21]. We will investigate directed, bidirectional and acyclic communications, respectively, and different convergence conditions will be shown. In this way, the proof of Theorem 3 will be obtained as a simple special case.

In Section IV, we turn to arc-independent graphs. The proof of Theorem 4 will be carried out using a stochastic matrix argument.
We discuss some application of the result to belief evolution model in social networks in Section V and concluding remarks are given in Section VI.

II. IMPOSSIBILITY ANALYSIS

This section focuses on the proof of Theorems 1 and 2.

The following lemma is well-known.

**Lemma 1:** Suppose $0 \leq b_k < 1$ for all $k$. Then $\sum_{k=0}^{\infty} b_k = \infty$ if and only if $\prod_{k=0}^{\infty} (1 - b_k) = 0$.

**A. Proof of Theorem 1**

From algorithm (1), if $\sum_{k=0}^{\infty} b_k < \infty$, we have
\[
P(x_i(k+1) = x_i(k), \ k = 0, 1, \ldots) \geq \prod_{k=0}^{\infty} (1 - p_k) = r_0,
\]
where $0 < r_0 < 1$ is a well-defined constant according to Lemma 1. Then it is straightforward to see that the impossibility claim of Theorem 1 holds.

Next, we define a scalar random variable $\varpi(k)$, by that $\varpi(k) = H(k+1)/H(k)$ when $H(k) > 0$, and $\varpi(k) = 1$ when $H(k) = 0$. Obviously, $h(k)$ is non-decreasing, and $H(k)$ is non-increasing. Thus, it always holds that $\varpi(k) \leq 1$. We see from the considered algorithm that
\[
P(\varpi(k) = 1) \geq (1 - p_k)^n. \tag{5}
\]
As a result, we obtain
\[
P\left(\frac{H(k)}{H(0)} \geq \epsilon\right) \geq P(\varpi(j) = 1, \ j = 0, \ldots, k-1)
\]
\[
\geq \prod_{j=0}^{k-1} (1 - p_j)^n, \tag{6}
\]
and then the lower bound for the $\epsilon$-computation given in Theorem 1 can be easily obtained. The proof of Theorem 1 is completed.

**B. Proof of Theorem 2**

In order to prove Theorem 2, we need the following lemma.

**Lemma 2:** Assume that $a_{ii}(k) \geq \gamma_0 > 1/2$ for all $i$ and $k$. Then
\[
H(k+1) \geq (2\gamma_0 - 1)H(k)
\]
for all $k$.

**Proof.** Suppose $x_m(k) = h(k)$ for some $m \in V$. Then we have
\[
\sum_{j \in \mathcal{N}_m(k)} a_{mj}(k)x_j(k) = a_{mm}(k)x_m(k) + \sum_{j \in \mathcal{N}_m(k) \setminus \{m\}} a_{mj}(k)x_j(k)
\]
\[
\leq a_{mm}(k)h(k) + (1 - a_{mm}(k))H(k)
\]
\[
\leq \gamma_0 h(k) + (1 - \gamma_0)H(k),
\]
which implies
\[
h(k+1) \leq \gamma_0 h(k) + (1 - \gamma_0)H(k). \tag{7}
\]
A symmetric argument leads to
\[ H(k + 1) \geq (1 - \gamma_0) h(k) + \gamma_0 H(k). \] (8)

Based on (7) and (8), we obtain
\[
\mathcal{H}(k + 1) = H(k + 1) - h(k + 1)
\geq (1 - \gamma_0) h(k) + \gamma_0 H(k) - \left[ \gamma_0 h(k) + (1 - \gamma_0) H(k) \right]
\geq (2\gamma_0 - 1) \mathcal{H}(k).
\] (9)

The desired conclusion follows.

Noting the fact that Lemma 2 holds for all possible communication graphs, we see that
\[
P \left( 2\gamma_0 - 1 \leq \varpi(k) \leq 1 \right) = 1
\] (10)
and
\[
P \left( \varpi(k) < 1 \right) \leq P \left( \text{at least one node takes averaging at time } k \right)
= 1 - (1 - p_k)^n
\] (11)
where \( \varpi(k) \) follows the definition in the proof of Theorem 1.

Next, by Lemma 1, it is not hard to find
\[
\sum_{k=0}^{\infty} p_k < \infty \iff \prod_{k=0}^{\infty} (1 - p_k) > 0
\iff \prod_{k=0}^{\infty} (1 - p_k)^n > 0
\iff \sum_{k=0}^{\infty} (1 - (1 - p_k)^n) < \infty,
\] (12)
where the last equivalence is obtained by taking \( b_k = 1 - (1 - p_k)^n \) in Lemma 1.

Therefore, if \( \sum_{k=0}^{\infty} p_k < \infty \), applying the First Borel-Cantelli Lemma on (11), it follows immediately that
\[
P \left( \varpi(k) < 1 \text{ for infinitely many } k \right) = 0.
\] (13)

Furthermore, based on (10), we eventually have
\[
P \left( \lim_{k \to \infty} \mathcal{H}(k) = 0 \text{ for } \mathcal{H}(0) > 0 \right) \leq P \left( \varpi(k) < 1 \text{ for infinitely many } k \right) = 0.
\] (14)

Since \( \{x(0) : \mathcal{H}(0) = 0\} \) has zero measure in \( \mathbb{R}^n \), Theorem 2 follows and this ends the proof.

III. CONNECTIVITY-INDEPENDENT GRAPHS

In this section, we present the convergence analysis for connectivity-independent graphs. We are going to study some more general cases relying on the joint graphs only.

Joint connectivity has been widely studied in the literature on consensus seeking [16], [17]. The joint graph of \( \mathcal{G}_k \) on time interval \([k_1, k_2]\) for \( 0 \leq k_1 \leq k_2 \leq +\infty \), is denoted by
\[
\mathcal{G}_{[k_1, k_2]} = \left( \mathcal{V}, \bigcup_{k \in [k_1, k_2]} \mathcal{E}_k \right).
\]

Then we introduce the following connectivity definition.

**Definition 3:** \( \{\mathcal{G}_k\}_{k=0}^{\infty} \) is said to be
(i) *stochastically uniformly quasi-strongly connected*, if there exist two constants \( B \geq 1 \) and \( 0 < q < 1 \) such that \( \{G_{[mB,(m+1)B-1]}\}_{m=0}^{\infty} \) is connectivity-independent and

\[
P\left(G_{[mB,(m+1)B-1]} \text{ is quasi-strongly connected} \right) \geq q, \quad m = 0, 1, \ldots,
\]

(ii) *stochastically infinitely quasi-strongly connected*, if there exist a sequence \( 0 = c_0 < \cdots < c_m < \cdots \) and a constant \( 0 < q < 1 \) such that \( \{G_{[c_m,c_{m+1}]}\}_{m=0}^{\infty} \) is connectivity-independent and

\[
P\left(G_{[c_m,c_{m+1}]} \text{ is quasi-strongly connected} \right) \geq q, \quad m = 0, 1, \ldots.
\]

Roughly speaking, uniform (or infinite) joint-connections are defined on the union graphs in bounded (or boundless) time intervals.

In Subsection III.A, we will study stochastically uniformly jointly quasi-strongly connected graph processes, and Theorem 3 will be obtained as a special case. Then in Subsections III.B and III.C, two special cases, i.e., bidirectional and acyclic graph processes are investigated, respectively, for which the convergence condition is shown to be different with the general directed graphs.

### A. Bounded Joint Connections

The following result is for consensus seeking on stochastically uniformly quasi-strongly connected graphs.

**Proposition 1:** Suppose \( \{G_k\}_{k=0}^{\infty} \) is stochastically uniformly quasi-strongly connected. Algorithm (1) achieves a global a.s. consensus if \( \sum_{s=0}^{\infty} \bar{p}_s = \infty \), where

\[
\bar{p}_s = \inf \left\{ \prod_{l=1}^{n-1} p_{\alpha_l} : s(n-1)^2 B \leq \alpha_1 < \cdots < \alpha_{n-1} < (s+1)(n-1)^2 B \right\}.
\]

Moreover, in this case, we have

\[
T_{\text{com}}(\epsilon) \leq \inf \left\{ M : \sum_{i=0}^{M-1} \log \left( 1 - \frac{(q \eta)^{(n-1)^2} B}{2} \cdot \bar{p}_i \right) \right\} \geq \log \epsilon^{-2} \times (n-1)^2 B.
\]

In order to prove Theorem 1 we first establish a lemma characterizing a useful property of stochastically uniformly jointly quasi-strongly connected graphs.

**Lemma 3:** Assume that \( G_k \) is stochastically uniformly quasi-strongly connected. Then for any \( s = 0, 1, \ldots \), we have

\[
P\left( \exists i_0 \in \mathcal{V} \text{ and } s(n-1)^2 \leq \tau_1 < \cdots < \tau_{n-1} < (s+1)(n-1)^2 \text{ s.t. } i_0 \text{ is a center of } G_{[\tau_1B, (\tau_1+1)B-1]} \right) \geq q^{(n-1)^2}.
\]

**Proof:** Since \( G_k \) is stochastically uniformly quasi-strongly connected, the probability that each graph \( G_{[\tau B, (\tau+1)B-1]} \) for \( \tau = s(n-1)^2, \ldots, (s+1)(n-1)^2 - 1 \), has a center is no less than \( q^{(n-1)^2} \). We count a time whenever there is a center node in \( G_{[\tau B, (\tau+1)B-1]} \). The time count for a center node will lead to at least \( (n-1)^2 \) counts. However, the total number of the nodes is \( n \). Thus, at least one node is counted more than \( (n-2) \) times. The conclusion follows.

We are now ready to prove Proposition 1.

**Proof of Proposition 1** Denote \( k_s = s(n-1)^2 B \) for \( s = 0, 1, \ldots \). Let \( i_0 \) be the center node defined in Lemma 3 such that the probability that \( i_0 \) is a center of \( G_{[\tau_jB, (\tau_j+1)B-1]} \) for \( j = 1, \ldots, n-1 \) with \( k_s \leq \tau_j B \leq k_{s+1} - 1 \) is no less than \( q^{(n-1)^2} \).
Assume that
\[ x_{i_0}(k_s) \leq \frac{1}{2} h(k_s) + \frac{1}{2} H(k_s). \] (15)

With the weights rule, we see that
\[ \sum_{j \in \mathcal{N}_{i_0}(k_s)} a_{i_0j}(k_s) x_j(k_s) = a_{i_0i_0}(k_s) x_{i_0}(k_s) + \sum_{j \in \mathcal{N}_{i_0}(k_s) \setminus \{i_0\}} a_{i_0j}(k_s) x_j(k_s) \leq a_{i_0i_0}(k_s) \left( \frac{1}{2} h(k_s) + \frac{1}{2} H(k_s) \right) + \left( 1 - a_{i_0i_0}(k_s) \right) H(k_s) \leq \frac{\eta}{2} h(k_s) + (1 - \frac{\eta}{2}) H(k_s). \] (16)

Thus, with \( \eta < 1 \), we obtain
\[ x_{i_0}(k_s + 1) \leq \frac{\eta}{2} h(k_s) + (1 - \frac{\eta}{2}) H(k_s). \] (17)

Continuing the estimates, we know that for any \( q = 0, 1, \ldots, \)
\[ x_{i_0}(k_s + q) \leq \frac{\eta^q}{2} h(k_s) + (1 - \frac{\eta^q}{2}) H(k_s). \] (18)

When \( i_0 \) is a center of \( \mathcal{G}_{(\tau_2 B; (\tau_2 + 1) B - 1)} \), there will be a node \( i_1 \in \mathcal{V} \) different with \( i_0 \) and a time instance \( \hat{k}_1 \in [\tau_1 B, (\tau_1 + 1) B - 1] \) such that \( (i_0, i_1) \in \mathcal{E}_{\hat{k}_1} \). Denote \( \hat{k}_1 = k_s + \varsigma \) with \( \tau_1 B - k_s \leq \varsigma \leq \tau_1 B - k_s + B - 1 \). If \( i_1 \) takes the consensus option at time step \( \hat{k}_1 + 1 \), with (13), we obtain
\[ x_{i_1}(k_s + \varsigma + 1) = a_{i_0i_1}(k_s + \varsigma) x_{i_0}(k_s + \varsigma) + \sum_{j \in \mathcal{N}_{i_1}(k_s + \varsigma) \setminus \{i_0\}} a_{i_1j}(k_s + \varsigma) x_j(k_s + \varsigma) \leq \eta^{\varsigma} \frac{\eta^q}{2} h(k_s) + \left( 1 - \frac{\eta^q}{2} \right) H(k_s) \leq \frac{\eta^{\varsigma + 1}}{2} h(k_s) + \left( 1 - \frac{\eta^{\varsigma + 1}}{2} \right) H(k_s), \] (19)

which leads to that for any \( q = (\tau_1 + 1) B - k_s, \ldots, \)
\[ x_{i_1}(k_s + q) \leq \frac{\eta^q}{2} h(k_s) + (1 - \frac{\eta^q}{2}) H(k_s). \] (20)

Therefore, we conclude that
\[ \mathbb{P}\left( x_{i_1}(k_s + q) \leq \frac{\eta^q}{2} h(k_s) + (1 - \frac{\eta^q}{2}) H(k_s), \quad l = 0, 1; q = (\tau_1 + 1) B - k_s, \ldots \right) \geq p_{\hat{k}_1} q^{(n-1)^2}. \]

We proceed the analysis on time interval \([\tau_2 B, (\tau_2 + 1) B - 1]\). When \( i_0 \) is a center of \( \mathcal{G}_{[\tau_2 B; (\tau_2 + 1) B - 1]} \), there will be a node \( i_2 \in \mathcal{V} \) different with \( i_0 \) and \( i_1 \) and a time instance \( \hat{k}_2 \in [\tau_2 B, (\tau_2 + 1) B - 1] \) such that either \( (i_0, i_2) \in \mathcal{E}_{\hat{k}_2} \) or \( (i_0, i_2) \in \mathcal{E}_{\hat{k}_2} \). By similar analysis, we obtain that
\[ \mathbb{P}\left( x_{i_1}(k_s + q) \leq \frac{\eta^q}{2} h(k_s) + (1 - \frac{\eta^q}{2}) H(k_s), \quad l = 0, 1, 2; q = (\tau_2 + 1) B - k_s, \ldots \right) \geq p_{\hat{k}_2} p_{\hat{k}_2} q^{(n-1)^2}. \]

Repeating the estimations on time intervals \([\tau_j B, (\tau_j + 1) B - 1]\) for \( j = 3, \ldots, n - 1, \hat{k}_3, \ldots, \hat{k}_{n-1} \) can be defined respectively; and bounds for \( i_3, \ldots, i_{n-1} \) can be similarly given by
\[ \mathbb{P}\left( x_{i_1}(k_s + q) \leq \frac{\eta^q}{2} h(k_s) + (1 - \frac{\eta^q}{2}) H(k_s), \quad l = 0, \ldots, n - 1; q = (\tau_{n-1} + 1) B - k_s, \ldots \right) \geq \prod_{l=1}^{n-1} p_{k_l} q^{(n-1)^2}, \]
which implies
\[ \mathbb{P}\left( H(k_{s+1}) \leq \frac{\eta^{(n-1)^2}}{2} h(k_s) + (1 - \frac{\eta^{(n-1)^2}}{2}) H(k_s) \right) \geq \prod_{l=1}^{n-1} p_{k_l} q^{(n-1)^2}. \] (21)
Noticing the fact that \( h(k) \) is non-decreasing, if
\[
H(k_{s+1}) \leq \frac{\eta(q-1)^2}{2} h(k_s) + \left( 1 - \frac{\eta(q-1)^2}{2} \right) H(k_s)
\]
holds, we have
\[
\mathcal{H}(k_{s+1}) \leq \frac{\eta(q-1)^2}{2} h(k_s) + \left( 1 - \frac{\eta(q-1)^2}{2} \right) \mathcal{H}(k_s) \quad (22)
\]
Thus, (21) leads to
\[
P\left( \mathcal{H}(k_{s+1}) \leq \left( 1 - \frac{\eta(q-1)^2}{2} \right) \mathcal{H}(k_s) \right) \geq \bar{p}_s q(q-1)^2. \quad (23)
\]
A symmetric analysis will show that (23) also holds under the contrary condition with (15) by establishing the lower bound of \( h(k_{s+1}) \).

With (23), we have
\[
E\left[ \mathcal{H}(k_{s+1}) \right] \leq \left( 1 - \frac{\eta(q-1)^2}{2} \right) E\left[ \mathcal{H}(k_s) \right]. \quad (24)
\]
which implies
\[
E\left[ \mathcal{H}(k_M) \right] \leq \prod_{s=0}^{M-1} \left( 1 - \frac{\eta(q-1)^2}{2} \right) \mathcal{H}(0) \quad (25)
\]
for all \( M \geq 1 \) according to the connectivity-independence. Thus, it follows from Lemma 1 that
\[
\lim_{M \to \infty} E\left[ \mathcal{H}(k_M) \right] = 0, \quad (26)
\]
which yields
\[
\lim_{k \to \infty} E\left[ \mathcal{H}(k) \right] = 0 \quad (27)
\]
since \( \mathcal{H}(k) \) is non-increasing. Using Fatou’s lemma, we further obtain
\[
0 \leq E\left[ \lim_{k \to \infty} \mathcal{H}(k) \right] \leq \lim_{k \to \infty} E\left[ \mathcal{H}(k) \right] = 0. \quad (28)
\]
Therefore, the convergence claim of the conclusion holds because (28) implies \( P\left( \lim_{k \to +\infty} \mathcal{H}(k) = 0 \right) = 1 \).

Moreover, (23) leads to
\[
P\left( \frac{\mathcal{H}(k_M)}{\mathcal{H}(0)} \geq \epsilon \right) \leq \frac{1}{\epsilon} \cdot E\left[ \frac{\mathcal{H}(k_M)}{\mathcal{H}(0)} \right] \leq \frac{1}{\epsilon} \prod_{s=0}^{M-1} \left( 1 - \frac{\eta(q-1)^2}{2} \right) \bar{p}_s. \quad (29)
\]
Consequently, we have
\[
T_{com}(\epsilon) \leq \inf \left\{ M : \sum_{s=0}^{M-1} \log \left( 1 - \frac{\eta(q-1)^2}{2} \right) \bar{p}_s \right\} -1 \geq \log \epsilon^{-2} \times (n-1)^2 B. \quad (30)
\]
The desired conclusion follows.

Suppose \( p_{k+1} \leq p_k \) for all \( k \). Then it is not hard to see that \( \sum_{s=0}^{\infty} \bar{p}_s = \infty \) if and only if \( \sum_{k=0}^{\infty} p_k^{n-1} = \infty \). Therefore, the following corollary holds immediately from Proposition 1.

**Corollary 1**: Suppose \( \{ G_k \}_0^\infty \) is stochastically uniformly quasi-strongly connected and \( p_{k+1} \leq p_k \) for all \( k \). Then algorithm (1) achieves a global a.s. consensus if \( \sum_{k=0}^{\infty} p_k^{n-1} = \infty \).

Now we see that Theorem 3 holds as a special case of Corollary 1 with \( B = 1 \) in the joint connectivity definition.
B. Bidirectional Connections

A diagraph $G$ is called to be bidirectional if for any two nodes $i$ and $j$, $i$ is a neighbor of $j$ if and only if $j$ is a neighbor of $i$. In this subsection, we will study the case when $G_k$ is always bidirectional.

Note that we do not impose an upper bound for the length of the intervals $[c_m, c_{m+1})$ in the definition of stochastically infinitely quasi-strong connectivity, which makes an essential difference from the bounded joint connections. The main result with for bidirectional communications under stochastically joint connectivity is stated as follows. Note that in this case quasi-strong connectivity is simplified as connectivity for bidirectional graphs.

**Proposition 2:** Let $G_k$ be bidirectional for all $k$. Suppose $\{G_k\}_0^\infty$ is stochastically infinitely connected. Algorithm (1) achieves a global a.s. consensus if $\sum_{s=0}^\infty \hat{p}_s = \infty$ with

$$\hat{p}_s = \inf \left\{ \prod_{l=1}^{n-1} p_{\alpha_l} : c_{s(n-1)} \leq \alpha_1 < \cdots < \alpha_{n-1} < c_{(s+1)(n-1)} \right\}.$$

and also

$$T_{\text{com}}(\epsilon) \leq \inf \left\{ c_{s(n-1)} : \sum_{i=0}^{s-1} \log \left(1 - (q\eta)^{(n-1)} \cdot \hat{p}_i\right)^{-1} \geq \log \epsilon^{-2} \right\}.$$

**Proof.** We will prove the conclusion by showing $P(\lim_{k \to +\infty} H(k) = 0) = 1$. Take a node $i_0 \in \mathcal{V}$ with $x_{i_0}(c_0) = h(c_0)$. With A2, we can assume that for every two nodes $i, j \in \mathcal{V}$, $i$ is a neighbor of $j$ if and only if $j$ is a neighbor of $i$.

Define $t_1 = \inf \{i_0\}$ has at least one neighbor in $G_k$ and $\mathcal{V}_1 = \{j | j$ is a neighbor of $i_0$ at $k = t_1\}$. Then we have $P(t_1 < c_1) \geq q$ because $\{G_k\}_0^\infty$ is stochastically infinitely connected. Moreover, noting the fact that

$$\sum_{j \in \mathcal{N}_{i_0}(t_1)} a_{i_0,j}(t_1) x_j(t_1) = a_{i_0,i_0}(t_1) x_{i_0}(t_1) + \sum_{j \in \mathcal{N}_{i_0}(t_1) \setminus \{i_0\}} a_{i_0,j}(t_1) x_j(t_1)$$

$$\leq a_{i_0,i_0}(t_1) h(c_0) + (1 - a_{i_0,i_0}(t_1)) H(c_0)$$

$$\leq \eta h(c_0) + (1 - \eta) H(c_0).$$

we have

$$P(x_{i_0}(t_1 + 1) \leq \eta h(c_0) + (1 - \eta) H(c_0)) = 1. \quad (32)$$

Similarly, for any $i_1 \in \mathcal{V}_1$, we have

$$\sum_{j \in \mathcal{N}_{i_1}(t_1)} a_{i_1,j}(t_1) x_j(t_1) = a_{i_1,i_0}(t_1) x_{i_0}(t_1) + \sum_{j \in \mathcal{N}_{i_1}(t_1) \setminus \{i_0\}} a_{i_1,j}(t_1) x_j(t_1)$$

$$\leq a_{i_1,i_0}(t_1) h(c_0) + (1 - a_{i_1,i_0}(t_1)) H(c_0)$$

$$\leq \eta h(c_0) + (1 - \eta) H(c_0).$$

which implies

$$P(x_{i_1}(t_1 + 1) \leq \eta h(c_0) + (1 - \eta) H(c_0)) \geq p_{i_1}. \quad (34)$$

Thus, we conclude that

$$P \left( x_l(t_1 + 1) \leq \eta h(c_0) + (1 - \eta) H(c_0), \ l \in \{i_0\} \cup \mathcal{V}_1 \right) \geq p_{i_1}^{\mathcal{V}_1}. \quad (35)$$

Furthermore, we define $t_2 = \inf_{k \geq k_1} \{\text{at least one other node is connected to} \ \{i_0\} \cup \mathcal{V}_1 \ \text{in} \ G_k \}$ and $\mathcal{V}_2 = \{j \in \mathcal{V} \setminus (\{i_0\} \cup \mathcal{V}_1) | j$ is connected to $\{i_0\} \cup \mathcal{V}_1$ at $k = t_2\}$. We see that $P\{t_2 < c_2\} \geq q^2$. Similar analysis will show that

$$P \left( x_l(t_2 + 1) \leq \eta^2 h(c_0) + (1 - \eta^2) H(c_0), \ l \in \{i_0\} \cup \mathcal{V}_1 \cup \mathcal{V}_2 \right) \geq p_{i_1}^{\mathcal{V}_1} p_{i_2}^{\mathcal{V}_2}. \quad (36)$$
Continuing the analysis, $t_3, \ldots, t_{\mu_0}$ and $\mathcal{V}_3, \ldots, \mathcal{V}_{\mu_0}$ can be defined until $\mathcal{V} = \{i_0\} \cup \bigcup_{l=1}^{\mu_0} \mathcal{V}_l$ for some $\mu_0 \leq n - 1$. Moreover, we also have $P\{t_l < c_l\} \geq q^l$ for $l = 1, \ldots, \mu_0$. Then we obtain

$$P\left(x_l(t_{\mu_0} + 1) \leq \eta t_{\mu_0} h(c_0) + (1 - \eta t_{\mu_0}) H(c_0), \; l \in \mathcal{V}\right) \geq \prod_{l=1}^{\mu_0} p_{t_l}^{\mathcal{V}_l},$$

which yields

$$P\left(\mathcal{H}(t_{\mu_0} + 1) \leq (1 - \eta t_{\mu_0}) \mathcal{H}(c_0)\right) \geq \prod_{l=1}^{\mu_0} p_{t_l}^{\mathcal{V}_l}. \tag{37}$$

Therefore, because $\mu_0 \leq n - 1$ and $P\{t_l < c_l\} \geq q$ for $l = 1, \ldots, \mu_0$, we have

$$P\left(\mathcal{H}(c_{n-1}) \leq (1 - \eta^{n-1}) \mathcal{H}(c_0)\right) \geq q^{n-1}. \tag{38}$$

Similar to (40), estimations for $\mathcal{H}(c_{s(n-1)})$ can be obtained for $s = 1, 2, \ldots$ by

$$P\left(\mathcal{H}(c_{s+1})(n-1) \leq (1 - \eta^{n-1}) \mathcal{H}(c_{s(n-1)})\right) \geq q^{n-1}. \tag{40}$$

Therefore, the convergence part of Theorem 2 holds by the same argument as the proof of Theorem 1 and we also have

$$T_{con}(\epsilon) \leq \inf \left\{c_{s(n-1)} : \sum_{i=0}^{s-1} \log \left(1 - (q\eta)^{(n-1)} \cdot \hat{p}_i\right)^{-1} \geq \log \epsilon^{-2}\right\}. \tag{41}$$

The proof is completed.

Similarly we have the following corollary.

Corollary 2: Assume that $G_k$ is bidirectional for all $k$ and $p_{k+1} \leq p_k$ for all $k$. Suppose $G_k$ is stochastically jointly connected. Then Algorithm 1 achieves a global a.s. consensus if $\sum_{s=0}^{\infty} P_{s(n-1)} = \infty.$

C. Acyclic Connections

A digraph $G = (\mathcal{V}, \mathcal{E})$ is called to be acyclic if it contains no directed cycle. This subsection focuses on acyclic communications.

Let $G$ be an acyclic, quasi-strongly connected digraph. Then it is not hard to see that $G$ has one and only one center. Denote this center as $v_0$. We can define a function on $G$ in the way that $h(v_0) = 0$ and $h(j) = \max\{|v_0 \to j| \mid v_0 \to j\}$ for any $j \neq v_0$. Let $h_* = \max_{i \in \mathcal{V}} h(i).$ Then we establish the following lemma indicating that this function $h$ is actually surjective onto $\{0, \ldots, d_*\}$.

Lemma 4: Let $G$ be an acyclic, quasi-strongly connected digraph. Then $h^{-1}(m) = \{i \mid h(i) = m\}$ is nonempty for any $m = 0, \ldots, h_*$.

Proof: The conclusion holds for $m = 0$ trivially.

Let us prove the conclusion holds for $m = 1$ by contradiction. Assume that $h^{-1}(1) = \emptyset.$ Then $m_0 = \inf_{i \neq v_0} h(i) > 1.$ Take a node $j_0$ with $h(j_0) = m_0.$ There exists a (simple) path $v_0 \to j_0$ in $G$ with length $m_0 > 1.$ Let $v_*$ be the node for which arc $(v_*, j_0)$ is included in $v_0 \to j_0.$ According to the definition of $m_0$, we have $h(v_*) \geq m_0.$ Suppose $v_0 \to v_*$ is a path with length $h(v_*).$ Note that, $j_0$ cannot be included in $v_0 \to v_*$ because otherwise it will generate a cycle $j_0 \to v_* \to j_0.$ Consequently, another path $v_0 \to v_* \to j_0$ is obtained whose length is $h(v_*) + 1 > m_0.$ This contradicts the selection rule of $j_0.$ Therefore, the conclusion holds for $m = 1.$

Next, we construct another graph $\hat{G}$ from $G$ by viewing node set $\{v_0\} \cup h^{-1}(1)$ as a single node in the new graph without changing the links. We see that $\hat{h}^{-1}(2)$ of $\hat{G}$ is exactly the same as node set $h^{-1}(1)$ of $G$, while the latter is nonempty via previous analysis. Continuing the argument, the conclusion follows.
Following Definition 3 we can similarly define $G_k$ to be stochastically jointly quasi-strongly connected if there exist a sequence $0 = c_0 < \cdots < c_m < \ldots$ and a constant $0 < q < 1$ such that $\{G_{(c_m,c_{m+1})}\}_{m=0}^{\infty}$ is connectivity-independent and $P(G_{(c_m,c_{m+1})})$ is quasi-strongly connected for all $m$. Here comes our main result for acyclic graphs.

**Proposition 3:** Assume that $P(G_{(0,\infty)})$ is acyclic and $\{G_k\}_{0}^{\infty}$ is stochastically infinitely quasi-strongly connected. Algorithm (1) achieves a global a.s. consensus if there exist a sequence $\{v_0, v_1, \ldots\}$ with $\tilde{p}_s = \inf_{x \leq \alpha < x_+} p_{x_+}$, $s = 0, 1, \ldots$. 

**Proof:** Let $v_0$ be the unique node of $G_{(0,\infty)}$. Based on Lemma 4, $V_i = h^{-1}(i)$ for $i = 0, \ldots, h_0$ can be defined with $V_0 = \{v_0\}$ and $V = \bigcup_{i=0}^{h_0} V_i$.

Obviously it holds that $P(x_{v_0}(k) = x_{v_0}(0), k = 0, 1, \ldots) = 1$ because with probability one, $v_0$ has no neighbor except itself for all $k$. We turn to $V_1$. The following claim holds.

**Claim.** $P\left(\lim_{k \to \infty} |x_{\ell}(k) - x_{v_0}(0)| = 0\right) = 1$ for all $\ell \in V_1$.

Take $v_1 \in V_1$. With probability one, $v_0$ is the only neighbor of $v_1$ excluding itself in $G_k$ for all $k$. Define $t_1 = \inf_{k \geq 0}(\{v_0, v_1\} \in E_k)$. Then $P\{t_1 < c_1\} \geq q$ because $\{G_k\}_{0}^{\infty}$ is SJQSC. Noting the fact that we may assume

$$\sum_{j \in N_{v_1}(t_1)} a_{v_1j}(t_1) x_j(t_1) - x_{v_0}(0) = \left| a_{v_1v_0}(t_1) x_{v_0}(t_1) + a_{v_1v_1}(t_1) x_{v_1}(t_1) - x_{v_0}(0) \right| = (1 - a_{v_1v_0}(t_1)) |x_{v_1}(0) - x_{v_0}(0)| \leq (1 - \eta) |x_{v_1}(0) - x_{v_0}(0)|,$$

which yields

$$P\left(|x_{v_1}(t_1 + 1) - x_{v_0}(0)| \leq (1 - \eta) |x_{v_1}(0) - x_{v_0}(0)| \right) \geq p_{t_1}, \quad (43)$$

Thus, we obtain

$$P\left(|x_{v_1}(c_1) - x_{v_0}(0)| \leq (1 - \eta) |x_{v_1}(0) - x_{v_0}(0)| \right) \geq \tilde{p}_0 q, \quad (44)$$

where $\tilde{p}_0 = \inf_{c_0 < \alpha < c_1} p_{\alpha}$. Repeating the analysis on time interval $[c_m, c_{m+1})$, $m = 1, 2, \ldots$, we have

$$P\left(|x_{v_1}(c_{m+1}) - x_{v_0}(0)| \leq (1 - \eta) |x_{v_1}(c_m) - x_{v_0}(0)| \right) \geq \tilde{p}_m q, \quad m = 1, 2, \ldots, \quad (45)$$

Similar to the proof of Theorem 1, the connectivity independence of $\{G_{(c_m,c_{m+1})}\}_{m=0}^{\infty}$ and (45) lead to $P\left(\lim_{k \to \infty} |x_{\ell}(k) - x_{v_0}(0)| = 0\right) = 1$ immediately. The claim is proved.

Further, we turn to $V_2$ and prove $P\left(\lim_{k \to \infty} |x_{\ell}(k) - x_{v_0}(0)| = 0\right) = 1, \ell \in V_2$. Take $\varepsilon = 1/\ell$ for some $\ell \geq 1$. The previous claim implies that there exists $T(\ell) > 0$ such that $P\left(|x_{\ell}(k) - x_{v_0}(0)| \leq 1/\ell, k \geq T, \ell \in V_1\right) = 1$. Suppose $c_{\ell_1} \geq T$. Let $v_2$ be an arbitrary node in $V_2$. We next show $P\left(\limsup_{k \to \infty} |x_{v_2}(k) - x_{v_0}(0)| \leq 1/\ell \right) = 1$.

Note that, if there exists $q_0 \geq 0$ such that $|x_{v_2}(q_0) - x_{v_0}(0)| \leq 1/\ell$, it will hold that $|x_{v_2}(k) - x_{v_0}(0)| \leq 1/\ell$ for all $k \geq q_0$. Thus, denoting $\mathcal{L} = \{|x_{v_2}(k) - x_{v_0}(0)| > 1/\ell \text{ for all } k\}$ as an event, we have $P\left(\limsup_{k \to \infty} |x_{v_2}(k) - x_{v_0}(0)| \leq 1/\ell \right) = 1$.

The fact that $\mathcal{H}(k)$ is non-increasing implies $|x_{v_2}(c_{\ell_1}) - x_{v_0}(0)| \leq \mathcal{H}(0)$. Moreover, with probability one, all the neighbors of $v_1$ excluding itself in $G_k$ are within $\{v_0\} \cup V_1$ for all $k$, i.e., $P\left(\{N_{v_2}(k) \setminus \{v_2\} \subseteq \{v_0\} \cup V_1, k = 0, 1, \ldots\right) = 1$. Define $t_1 = \inf_{k \geq c_{\ell_1}}$ (there is at least one are leaving from $\{v_0\} \cup V_1$ entering $v_2$ in $E_k$). Then we obtain

$$\left| \sum_{j \in N_{v_2}(t_1)} a_{v_2j}(t_1) x_j(t_1) - x_{v_0}(0) \right| = \sum_{j \in (v_0) \cup V_1} a_{v_2j}(\tilde{t}_1) |x_j(\tilde{t}_1) - x_{v_0}(0)| + a_{v_2v_2}(\tilde{t}_1) |x_{v_2}(\tilde{t}_1) - x_{v_0}(0)| \leq \frac{1 - a_{v_2v_2}(\tilde{t}_1) \frac{1}{\ell} + a_{v_2v_2}(\tilde{t}_1) |x_{v_2}(c_{\ell_1}) - x_{v_0}(0)| \leq \frac{1 - \eta}{\ell} + \eta |x_{v_2}(c_{\ell_1}) - x_{v_0}(0)|, \quad (46)$$
which yields
\[ | \sum_{j \in \mathcal{N}_{v_2}(t_i)} a_{v_2j}(t_i) x_j(t_i) - x_{v_2}(0) | - \frac{1}{\ell} \leq \eta \left[ | x_{v_2}(c_{t_i}) - x_{v_2}(0) | - \frac{1}{\ell} \right]. \]

(47)

Consequently, denoting
\[ y_m = | x_{v_2}(c_m) - x_{v_2}(0) | - \frac{1}{\ell}, \ m = 0, 1, \ldots \]

we have
\[ \mathbb{P} \left( y_{\tau_1 + 1} \leq \eta \cdot y_{\tau_1} \mid \mathcal{L} \right) \geq \tilde{p}_{\tau_1} q. \]

(48)

Proceeding the analysis will show that
\[ \mathbb{P} \left( y_{m+1} \leq \eta \cdot y_m \mid \mathcal{L} \right) \geq \tilde{p}_m q, \ m = \tau_1 + 1, \ldots \]

(49)

Similarly, (49) leads to
\[ \mathbb{P} \left( \lim_{m \to \infty} y_m = 0 \mid \mathcal{L} \right) = 1. \]

(50)

Then we have reached that \( \mathbb{P} \left( \limsup_{k \to \infty} | x_{v_2}(k) - x_{v_2}(0) | \leq 1/\ell \right) = 1. \) Since \( \ell \) is chosen arbitrarily, we finally obtain
\[ \mathbb{P} \left( \lim_{k \to \infty} | x_l(k) - x_{v_2}(0) | = 0 \right) = 1, \ l \in \mathcal{V}_2. \]

Therefore, continuing the estimations on node set \( \mathcal{V}_3, \ldots, \mathcal{V}_{h_0}, \) eventually we have \( \mathbb{P} \left( \lim_{k \to \infty} | x_l(k) - x_{v_2}(0) | = 0 \right) = 1 \) for all \( l \in \mathcal{V}. \) The desired conclusion thus follows.

We see that Theorem 3 leads to the following corollary immediately.

**Corollary 3:** Assume that \( \mathbb{P}(G_{[0, \infty)} \text{ is acyclic}) = 1. \)

(i) Suppose \( \{G_k\}_{k=0}^\infty \) is stochastically infinitely quasi-strongly connected and \( p_{k+1} \leq p_k \) for all \( k. \) Then Algorithm (1) achieves a global a.s. consensus if \( \sum_{m=0}^\infty p_{cm} = \infty. \)

(ii) Suppose either \( \{G_k\}_{k=0}^\infty \) is stochastically uniformly quasi-strongly connected with \( B = 1 \) or \( p_{k+1} \leq p_k \) for all \( k. \) Then Algorithm (1) achieves a global a.s. consensus if and only if \( \sum_{k=0}^\infty p_k = \infty. \)

IV. **Arc-independent Graphs**

In this section, we turn to the convergence analysis for the arc-independent graph processes. Different from previous discussions, we will prove Theorem 4 using a stochastic matrix argument.

Let \( e_i = (0 \ldots 1 \ldots 0)^T \) be an \( n \times 1 \) unit vector with the \( i \)th component equal to 1. Denote \( r_i(k) = (r_{i1} \ldots r_{in})^T \) as an \( n \times 1 \) unit vector with \( r_{ij}(k) = a_{ij}(k) \) if \( j \in \mathcal{N}_i(k), \) and \( r_{ij}(k) = 0 \) otherwise for \( j = 1, \ldots, n. \) Let \( W(k) = (w_1(k) \ldots w_n(k))^T \in \mathbb{R}^{n \times n} \) be a random matrix with
\[ w_i(k) = \begin{cases} r_i(k), & \text{with probability } p_k \\ e_i, & \text{with probability } 1 - p_k \end{cases} \]
for \( i = 1, \ldots, n. \) Algorithm (1) is transformed into a compact form:
\[ x(k+1) = W(k)x(k). \]

(52)

In what follows of this section, we will first establish several useful lemmas on the product of stochastic matrices, and then the consensus analysis for Theorem 4 will be presented.
A. Stochastic Matrices

A finite square matrix \( M = \{m_{ij}\} \in \mathbb{R}^{n \times n} \) is called stochastic if \( m_{ij} \geq 0 \) for all \( i, j \) and \( \sum_j m_{ij} = 1 \) for all \( i \).

For a stochastic matrix \( M \), introduce
\[
\delta(M) = \max_j \max_{\alpha,\beta} |m_{\alpha j} - m_{\beta j}|, \quad \lambda(M) = 1 - \min_{\alpha,\beta} \sum_j \min\{m_{\alpha j}, m_{\beta j}\}. \tag{53}
\]

If \( \lambda(M) < 1 \) we call \( M \) a scrambling matrix. The following lemma can be found in [10].

**Lemma 5:** For any \( k \) (\( k \geq 1 \)) stochastic matrices \( M_1, \ldots, M_k \),
\[
\delta(M_1 M_2 \ldots M_k) \leq \prod_{i=1}^{k} \lambda(M_i). \tag{54}
\]

We can associate a unique digraph \( G_M = \{V, E_M\} \) with node set \( V = \{1, \ldots, n\} \) to a stochastic matrix \( M = \{m_{ij}\} \in \mathbb{R}^{n \times n} \) in the way that \( (j, i) \in E_M \) if and only if \( m_{ij} > 0 \), and vice versa.

We first establish several lemmas. The following lemma is given on the induced graphs of products of stochastic matrices.

**Lemma 6:** For any \( k \) (\( k \geq 1 \)) stochastic matrices \( M_1, \ldots, M_k \) with positive diagonal elements, we have \( \left( \bigcup_{i=1}^{k} G_{M_i} \right) \subseteq \hat{G}_{M_1 \ldots M_k} \).

**Proof:** We prove the case for \( k = 2 \), and the conclusion will follow by induction for other cases.

Denote \( \bar{a}_{ij}, \hat{a}_{ij} \) and \( a_{ij}^* \) as the \( ij \)-entries of \( M_1, M_2 \) and \( M_1 M_2 \), respectively. Note that, we have
\[
a_{i_1 i_2}^* = \sum_{j=1}^{n} \bar{a}_{i_1 j} \hat{a}_{j i_2} \geq \bar{a}_{i_1 i_2} \hat{a}_{i_1 i_2} + \bar{a}_{i_1 i_2} \hat{a}_{i_1 i_2}. \tag{55}
\]

Then the conclusion follows immediately since \( \bar{a}_{i_1 i_2}, \hat{a}_{i_1 i_2} > 0 \). \( \square \)

Another lemma holds for determining whether a product of several stochastic matrices is a scrambling matrix.

**Lemma 7:** Let \( M_1, \ldots, M_{n-1} \) be \( n-1 \) stochastic matrices with positive diagonal elements. Assume that \( G_{M_\tau}, \tau = 1, \ldots, n-1 \) are all quasi-strongly connected sharing a common center. Then \( M_{n-1} \ldots M_1 \) is a scrambling matrix.

**Proof:** Let \( v_0 \) be a center node of \( G_{M_\tau}, \tau = 1, \ldots, n-1 \). Denote the \( ij \)-entry of \( M_\tau \) as \( m_{ij}^{(\tau)} \), \( \tau = 1, \ldots, n-1 \). Since \( v_0 \) is a center of \( G_{M_\tau} \), at least one node \( v_1 \) exists different with \( v_0 \) such that \( (v_0, v_1) \in E_{M_\tau} \). Thus, we have
\[
m_{v_1 v_0}^{(1)} > 0. \tag{56}
\]

Further, we denote the \( ij \)-entry of \( M_2 M_1 \) as \( m_{ij}^{(2/1)} \). According to Lemma 6, it holds that \( m_{v_1 v_0}^{(2/1)} > 0 \). Since \( v_0 \) is also a center of \( G_{M_2} \), there must be a node \( v_2 \) different with \( v_0 \) and \( v_1 \) such that there is at least one arc leaving from \( \{v_0, v_1\} \) entering \( v_2 \) in \( G_{M_2} \). As a result, there will be two cases.

(i) When \( (v_0, v_2) \in E_{M_2} \), we have \( m_{v_2 v_0}^{(2)} > 0 \). With Lemma 6 we obtain \( m_{v_2 v_1}^{(1)} > 0 \).
(ii) When \( (v_1, v_2) \in E_{M_2} \), we have \( m_{v_2 v_1}^{(2)} > 0 \). Thus, we obtain
\[
m_{v_2 v_0}^{(2)} = \sum_{\tau=1}^{n} m_{v_2 v_0}^{(2)} m_{v_0 v_1}^{(1)} = m_{v_2 v_1}^{(2)} m_{v_1 v_0}^{(1)} > 0. \tag{57}
\]

Therefore, both the two cases lead to
\[
m_{v_1 v_0}^{(2/1)} > 0 \quad \text{and} \quad m_{v_2 v_0}^{(2/1)} > 0. \tag{58}
\]

Proceeding the analysis, \( v_3, \ldots, v_{n-1} \) can be found and we can finally arrives at
\[
m_{v_i v_{i+1}^{(n-1)}}^{(n-1)} > 0, \quad \tau = 0, 1, \ldots, n-1. \tag{59}
\]
Thus, one has

The sufficiency part therefore follows.

\[ \lambda(M_{n-1} \ldots M_1) \leq 1 - \min_{\tau=0,\ldots,n-1} m_{v, v_0}^{(n-1) \ldots (1)} < 1. \]

The desired conclusion follows immediately. \(\square\)

### B. Proof of Theorem 4: Convergence

This subsection presents the proof of the convergence claim in Theorem 4. We only need to prove the sufficiency part. Note that, a global almost sure consensus of (1) is equivalent with

- **Necessity.** Assume that

\[ \sum_{n=0}^{\infty} p_n < \infty. \]

We will provide a simple intuitive proof in the following.

\[ \text{Proof.} \]

We call a node \( i \) succeeds at time \( k \) if it chooses to follow the averaging dynamics at time \( k \). Then we define

\[ \Psi_k = \begin{cases} 1, & \text{if at least one node succeeds at time } k; \\ 0, & \text{otherwise}. \end{cases} \]

Then, we have \( \Psi_k = 1 \) with probability \( 1 - (1 - p_k)^n \) and \( \Psi_k = 0 \) with probability \( (1 - p_k)^n \). Moreover, \( \Psi_0, \Psi_1, \ldots \) are independent. The following lemma holds on \( \Psi_k \).

**Lemma 8:** \( \mathbb{P}(\Psi_k = 1 \text{ for infinitely many } k) = 1 \) if and only if \( \sum_{k=0}^{\infty} p_k = \infty. \)

**Proof.** The proof can actually be obtained similarly by Borel-Cantelli lemma based on a similar argument as (12). Nevertheless, we will provide a simple intuitive proof in the following.

- **Necessity.** Assume that \( \sum_{k=0}^{\infty} p_k < \infty. \) Lemma 1 implies that \( \prod_{k=0}^{\infty} (1 - p_k) > 0 \), which yields \( \prod_{k=0}^{\infty} (1 - p_k)^n > 0. \) Therefore, we have \( \mathbb{P}(\Psi_k = 0, k = 0, \ldots) > 0. \) The necessity claim holds.

- **Sufficiency.** Assume that \( \sum_{k=0}^{\infty} p_k = \infty. \) Then Lemma 1 leads to \( \prod_{k=0}^{\infty} (1 - p_k)^n = 0 \) for all \( T \geq 0. \) Thus,

\[ \mathbb{P}(\Psi_k = 1 \text{ for finitely many } k) = \mathbb{P}(\exists T \geq 0, \text{i.e., } \Psi_k = 0 \text{ for all } k \geq T) \leq \sum_{T=0}^{\infty} \prod_{k=T}^{\infty} (1 - p_k)^n = 0. \]

The sufficiency part therefore follows. \(\square\)

Noting the fact that

\[ 1 - ny \leq (1 - y)^n, \quad y \in [0, 1], n \geq 1, \]

we obtain

\[ 1 - (1 - p_k)^n \leq np_k, \quad k = 0, \ldots. \]

Thus, one has

\[ \mathbb{P}(\text{node } i \text{ succeeds at time } k \mid \Psi_k = 1) = \frac{p_k}{1 - (1 - p_k)^n} \geq \frac{p_k}{np_k} = \frac{1}{n} \]

for all \( i = 1, \ldots, n \) and \( k = 0, \ldots. \)

According to Lemma 8 with probability one, we can well define the (Bernoulli) sequence of \( \Psi_k \),

\[ \zeta_1 < \cdots < \zeta_m < \zeta_{m+1} < \cdots \]

where \( \zeta_m \) is the \( m \)th time when \( \Psi_k = 1. \)

According to (61), for any fixed \( (i, j) \in E^a \), we have

\[ \mathbb{P}( (i, j) \in G_{W(\zeta_m)} ) = \mathbb{P}( j \text{ succeeds at time } \zeta_m ) \cdot \mathbb{P}( (i, j) \in G_{\zeta_m} ) \geq \frac{\theta_0}{n}, \quad m = 1, \ldots. \]

(63)
Denote \( Q_1 = W(\zeta_1 | E^*|) \ldots W(\zeta_2)W(\zeta_1) \), where \(|E^*|\) represents the number of elements in \( E^* \). We see from (6.3) and the definition of arc-independence that

\[
P\left( (i_\tau, j_\tau) \in G_{W(\zeta_\tau)}, \ \tau = 1, \ldots, |E^*| \right) \geq \left( \frac{\theta_0}{n} \right)^{|E^*|}, \tag{64}
\]

where \((i_\tau, j_\tau)\) denotes an element in \( E^* \). As a result, Lemma 6 yields that

\[
P\left( G^* \subseteq G_{Q_1} \right) \geq P\left( G^* \subseteq \bigcup_{\tau=1}^{|E^*|} G_{W(\zeta_\tau)} \right) \geq \left( \frac{\theta_0}{n} \right)^{|E^*|}. \tag{65}
\]

We continue to define \( Q_s = W(\zeta_s | E^*|) \ldots W(\zeta_{s-1} | E^*|+1) \) for \( s = 2, 3, \ldots \), and similar analysis leads to

\[
P\left( G^* \subseteq G_{Q_s} \right) \geq \left( \frac{\theta_0}{n} \right)^{|E^*|} \tag{66}
\]

for all \( s \). Because \( G^* \) is quasi-strongly connected, \( G_{Q_s}, s = 1, \ldots, n-1 \) have a common center. Thus, Lemma 7 yields

\[
P\left( \lambda(Q_{n-1} \ldots Q_1) < 1 \right) \geq \left( \frac{\theta_0}{n} \right)^{(n-1)|E^*|}. \tag{67}
\]

Moreover, note that \( Q_{n-1} \ldots Q_1 \) represents a product of \( (n-1)|E^*| \) stochastic matrices, each of which satisfies the weights rule. Therefore, each nonzero entry, \( \bar{q}_{ij} \) of \( Q_{n-1} \ldots Q_1 \), satisfies

\[
\bar{q}_{ij} \geq \eta^{(n-1)|E^*|}. \tag{68}
\]

As a result, (70) and (68) lead to

\[
P\left( \lambda(Q_{n-1} \ldots Q_1) \leq 1 - \eta^{(n-1)|E^*|} \right) \geq \left( \frac{\theta_0}{n} \right)^{(n-1)|E^*|}. \tag{69}
\]

We can now further conclude

\[
P\left( \lambda(Q_{\tau(n-1)} \ldots Q_{(\tau-1)(n-1)+1}) \leq 1 - \eta^{(n-1)|E^*|} \right) \geq \left( \frac{\theta_0}{n} \right)^{(n-1)|E^*|}, \ \tau = 1, 2, \ldots. \tag{70}
\]

Thus, based on Lemmas 11 and 5, we have

\[
\lim_{m \to \infty} E\left[ \delta(Q_{m(n-1)} \ldots Q_1) \right] \leq \lim_{m \to \infty} E\left[ \prod_{\tau=1}^m \lambda(Q_{\tau(n-1)} \ldots Q_{(\tau-1)(n-1)+1}) \right] = 0, \tag{71}
\]

which implies

\[
P\left( \lim_{m \to \infty} \delta(Q_{m(n-1)} \ldots Q_1) = 0 \right) = 1. \tag{72}
\]

This leads to

\[
P\left( \lim_{k \to \infty} \delta(W(k) \ldots W(0)) = 0 \right) = 1
\]

since \( W(k) = I_n \) for any \( k \notin \{ \zeta_1, \zeta_2, \ldots \} \), where \( I_n \) is the identical matrix. This completes the proof of the convergence statement in Theorem 4.
C. Proof of Theorem 4: Computation Time

In this subsection, we establish the upper bound of \( T_{\text{com}}(\varepsilon) \) given in Theorem 4.

Denote the \( ij \)-entry of \( W(k-1) \ldots W(0) \) as \( A_{ij} \). Then for all \( i, j \) and \( \tau \), we have

\[
|x_i(k) - x_j(k)| = \left| \sum_{\alpha=1}^{n} A_{i\alpha}x_{\alpha}(0) - \sum_{\alpha=1}^{n} A_{j\alpha}x_{\alpha}(0) \right|
\]

\[
= \left| \sum_{\alpha=1}^{n} A_{i\alpha}(x_{\alpha}(0) - x_{\alpha}(0)) - \sum_{\alpha=1}^{n} A_{j\alpha}(x_{\alpha}(0) - x_{\alpha}(0)) \right|
\]

\[
\leq \sum_{\alpha=1}^{n} |A_{i\alpha} - A_{j\alpha}| \cdot \max_{\alpha} |x_{\alpha}(0) - x_{\alpha}(0)|
\]

\[
\leq n\delta(W(k-1) \ldots W(0)) \cdot \max_{\alpha} |x_{\alpha}(0) - x_{\alpha}(0)|,
\]

which implies

\[
\mathcal{H}(k) \leq n\delta(W(k-1) \ldots W(0)) \mathcal{H}(0).
\]

Let us define a random variable for any \( k = 0, \ldots \) as

\[
\xi_k = \max \{ m : \zeta_m \leq k - 1 \} = \sum_{i=0}^{k-1} \Psi_i
\]

where \( \zeta_m \) is the Bernoulli sequence defined in (62) and \( \Psi_i \) follows (60).

Then according to Lemma 5 (74) implies

\[
P\left( \frac{\mathcal{H}(k)}{\mathcal{H}(0)} \geq \varepsilon \right) \leq P \left( \delta(Q_{\xi_k} \ldots Q_1) \geq \frac{\varepsilon}{n} \right)
\]

\[
\leq \frac{n}{\varepsilon} E \left[ \delta(Q_{\xi_k} \ldots Q_1) \right]
\]

\[
\leq \frac{n}{\varepsilon} E \left[ \lambda_{\xi_k} \ldots \lambda_1 \right],
\]

(75)

where by definition \( \lambda_{\tau} = \lambda(Q_{\tau(n-1)} \ldots Q_{(\tau-1)(n-1)+1}) \) for \( \tau = 1, \ldots \) and \( \lfloor z \rfloor \) represents the largest integer no greater than \( z \).

Further, (70) leads to

\[
E[\lambda_{\tau}] \leq 1 - \left( \frac{\eta\theta_0}{n} \right)^{(n-1)|\mathcal{E}^*|}, \quad \tau = 1, 2 \ldots
\]

(76)

which yields

\[
E \left[ \lambda_{\frac{\xi_k}{n-1}} \ldots \lambda_1 \right] = E \left[ E \left[ \lambda_{\frac{\xi_k}{n-1}} \ldots \lambda_1 | \xi_k \right] \right]
\]

\[
= \left( 1 - \left( \frac{\eta\theta_0}{n} \right)^{(n-1)|\mathcal{E}^*|} \right) E \left[ \xi_k \frac{\Psi_i}{n-1} \right]
\]

\[
\leq \left( 1 - \left( \frac{\eta\theta_0}{n} \right)^{(n-1)|\mathcal{E}^*|} \right) \sum_{i=0}^{k-1} E \left[ \xi_k \right] - 1
\]

\[
= \left( 1 - \left( \frac{\eta\theta_0}{n} \right)^{(n-1)|\mathcal{E}^*|} \right) \sum_{i=0}^{k-1} \left[ 1 - (1 - \frac{p_i}{n})^n \right] - 1
\]

(77)

since the nodes’ decisions are independent with the graph process.

Therefore, with (75) and (77), we have

\[
P \left( \frac{\mathcal{H}(k)}{\mathcal{H}(0)} \geq \varepsilon \right) \leq \frac{n}{\varepsilon} \left( 1 - \left( \frac{\eta\theta_0}{n} \right)^{(n-1)|\mathcal{E}^*|} \right) \sum_{i=0}^{k-1} \left[ 1 - (1 - \frac{p_i}{n})^n \right] - 1
\]

(78)
and thus (4) can be obtained immediately.

**Remark 1:** We see from the convergence analysis that Theorem 1 still holds if the definition of arc-independence is relaxed to arc-independence of different time instances by just assuming that for any finite subsequence of instances, \(0 \leq k_1 < \cdots < k_m, m \geq 1\), it holds that

\[
P\left((i_\tau, j_\tau) \in G_{k_\tau}, \tau = 1, \ldots, m\right) = \prod_{\tau=1}^{m} P\left((i_\tau, j_\tau) \in G_{k_\tau}\right)
\]  

(79)

for all \((i_\tau, j_\tau) \in \mathcal{E}^*, \tau = 1, \ldots, m\).

V. **APPLICATION: BELIEF AGREEMENT IN SOCIAL NETWORKS**

In this section, we discuss some application of previous results on an opinion dynamics model in social networks which was introduced in [40].

We restate the model of communication graphs in [31], [40] as follows.

**Gossip Communication**

(C1) At time \(k\), with equal probability \((1/n)\) among all the nodes, a node \(i\) is selected to meet another node;  
(C2) When node \(i\) is selected, node \(i\) will meet node \(j\) with probability \(p_{ij} \geq 0\). Then agents updates their states according to algorithm (1). Also, \(p_{ii} = 0\) for all \(i\) and \(\sum_{j=1}^{n} p_{ij} = 1\) for all \(i\). Denote \(G^0 = (\mathcal{V}, \mathcal{E}^0)\) as the digraph induced by the meeting probabilities, i.e., \(\mathcal{E}^0 = \{(i, j): p_{ij} > 0\}\).

In this gossip communication model, when node \(i\) is selected to meet node \(j\) at time \(k\), the arc set \(\mathcal{E}_k\) consists of two arcs, \((i, j)\) and \((j, i)\) in the communication graph \(G_k = (\mathcal{V}, \mathcal{E}_k)\). Then it is easy to verify that \(G^0 = (\mathcal{V}, \mathcal{E}^0)\) is a basic graph of this gossip graph process satisfying the relaxed case described in Remark 1.

In [40], a randomized consensus algorithm is presented to describe the opinion dynamics in social networks. Each node \(i\) is considered as a social member and its state \(x_i(k)\) represents its belief at time slot \(k\). Now we consider the case with time-varying decision probabilities. When node \(i\) is selected to meet \(j\) at time \(k\) under the gossip communications, their beliefs are updated by the following rule.

**Belief Evolution**

(E1) With probability \(\beta_{ij}(k)\), \(x_i(k+1) = x_j(k+1) = \frac{x_i(k) + x_j(k)}{2}\).

(E2) With probability \(\alpha_{ij}(k)\), \(x_i(k+1) = \varepsilon x_i(k) + (1-\varepsilon)x_j(k)\) and \(x_j(k) = x_i(k)\).

(E3) With probability \(\gamma_{ij}(k) = 1 - \alpha_{ij}(k) - \beta_{ij}(k)\), \(x_i(k+1) = x_i(k)\) and \(x_j(k+1) = x_j(k)\).

Here \(0 < \varepsilon \leq 1/2\) is a given constant. This belief evolution model is not entirely the same as algorithm (1) because when node \(i\) is selected to meet another node \(j\), \(j\) could also update its state, and their \((i\) and \(j)\) updating are not independent (because the condition that \(j\) changes its state but \(i\) does not will never happen).

However, Algorithm E1–E3 can be restated to the following equivalent form:

(E1’) Suppose node \(i\) is selected to meet another node \(j\) at time \(k\). Node \(i\) first updates its belief via the following rule.

\[
x_i(k+1) = \begin{cases} \frac{x_i(k) + x_j(k)}{2}, & \text{with probability } \beta_{ij}(k); \\
\varepsilon x_i(k) + (1-\varepsilon)x_j(k), & \text{with probability } \alpha_{ij}(k); \\
x_i(k), & \text{with probability } \gamma_{ij}(k).
\end{cases}
\]

(80)

(E2’) Node \(j\) then updates its belief via the following rule.

\[
x_j(k+1) = \begin{cases} \frac{x_i(k) + x_j(k)}{2}, & \text{if } x_i(k+1) = \frac{x_i(k) + x_j(k)}{2}; \\
x_j(k), & \text{otherwise}.
\end{cases}
\]

(81)
E1’–E2’ implies, node $i$ can be viewed as independent with $j$’s choice. Letting $\Psi_k$ follow the definition in (50), we have $\Psi_k = 1$ with probability $1 - \gamma_{ij}(k)$ and $\Psi_k = 0$ with probability $\gamma_{ij}(k)$. Moreover, we also have

$$
P(\text{node } i \text{ succeeds at time } k | \Psi_k = 1) \geq P(i \text{ is selected at time } k) = \frac{1}{n}. \quad (82)$$

Applying the same analysis of Theorem 4 along the sequence of $\Psi_k = 1$ and noting the fact that weights rule is kept by Algorithm E1–E3, one may see from E1’–E2’ that node $j$’s choices actually do not break the convergence property. We have the following conclusion.

**Proposition 4:** Suppose $G^0$ is quasi-strongly connected. The randomized belief evolution algorithm E1–E3 achieves a global a.s. consensus if and only if $\sum_{k=0}^{\infty} \left( \alpha_{ij}(k) + \beta_{ij}(k) \right) = \infty$.

VI. CONCLUSIONS

This paper investigated standard consensus algorithms coupled with randomized individual node decision-making over stochastically time-varying graphs. Each node determined its dynamics by a sequence of Bernoulli trials with time-varying probabilities. The central aim of this work was to investigate the relation between different levels of independence and the overall convergence. We introduced connectivity-independence and arc-independence for random graph processes. An impossibility theorem showed that an a.s. consensus could not be achieved unless the sum of the success probability sequence diverges. Then a serial of sufficiency conditions were given for the network to reach a global a.s. consensus under different connectivity assumptions.

Particularly, when either the graph was arc-independent or overall acyclic, the sum of the success probability sequence diverging was a sharp threshold condition for consensus under a simple self-confidence assumption. In other words, consensus appeared from probability zero to one as the sum of the probability sequence goes to infinity. Consistent with classical random graph theory, this so-called $0 - 1$ law was first established in the literature for dynamics on random graphs. Finally, using the result for arc-independent graph processes, a general information spread model over social networks was discussed, and sufficient and necessary condition was obtained for the social network to reach an a.s. belief agreement.

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