A dynamic model for advertising and pricing competition between national and store brands

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Received 1 March 2007; accepted 7 November 2007
Available online 24 November 2007

Abstract

We study the relationship between the pricing and advertising decisions in a channel where a national brand is competing with a private label. We consider a differential game that incorporates the carryover effects of brand advertising over time for both the manufacturer and the retailer and we account for the complementary and competitive roles of advertising. Analysis of the obtained equilibrium Markov strategies shows that the relationship between advertising and pricing decisions in the channel depends mainly on the nature of the advertising effects. In particular, the manufacturer reacts to higher competitive retailer’s advertising levels by offering price concessions and limiting his advertising expenditures. The retailer’s optimal reaction to competitive advertising effects in the channel depends on two factors: (1) the price competition level between the store and the national brands and (2) the strength of the competitive advertising effects. For example, in case of intense price competition between the two brands combined with a strong manufacturer’s competitive advertising effect, the retailer should lower both the store and the national brands’ prices as a reaction to higher manufacturer’s advertising levels. For the retailer, the main advantage from boosting his competitive advertising investments seems to be driven by increased revenues from the private label. The retailer should however limit his investments in advertising if the latter generates considerable competitive effects on the national brand’s sales.

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Keywords: Game theory; Marketing; Distribution; Pricing; Advertising

1. Introduction

The growing number and market share of private labels makes competition between national and store brands a hot topic for manufacturers and retailers. In the literature, many studies looked at the effects of store brands’ introductions in channels (see, e.g., Karray and Zaccour, 2006; Soberman and Parker, 2004). The main focus has been yet on understanding price competition between these brands (see, e.g., Raju et al., 1995; Narasimhan and Wilcox, 1998; Morton and Zettelmeyer, 2004; Sayman et al., 2002). However, many empirical studies suggest that marketing activities that reinforce the manufacturers’ brands equity are very important to win the battle against private labels (see, Richardson, 1997; Ailawadi, 2001). Advertising is one of the most utilized tools by marketing managers to build brands through improving perceived quality, creating positive brand associations and reinforcing the loyalty of customers (Simon and Sullivan, 1993; Yoo et al., 2000). Advertising is then a strategic decision for manufacturers in their battle against store brands. In order to analyze national and private labels competition considering the role of advertising, one must also take into account the retailer’s advertising decision. This is particularly true when the retailer’s private label carries its name. For example, Staples, an...
office supplies retailer in North America, offers a private label under its own name along with many national brands. It also invests in national advertising campaigns; Staples’ “the Easy button” campaign, diffused in North America and the UK, does not feature any particular product and its objective is to build preference for the store name (Sullivan, 2005). In this case, advertising can create counteracting effects in the channel. While manufacturer’s advertising increases the national brand’s equity and could lead to higher sales and profits for the retailer, it could also reduce demand for the private label. Similarly, the retailer’s brand advertising improves the store image and could create additional in-store traffic, which would benefit the manufacturers’ sales and profits. However, it could also reinforce the consumers’ confidence in the quality of the private label and steal market share from the manufacturer’s brands instead particularly when the retailer’s name or logo appears on the private label’s package1 (Dhar and Hoch, 1997; Ailawadi, 2001; Ailawadi and Keller, 2004).

These counteracting effects become even more complex when we consider that brand advertising could (1) generate competitive or complementary effects for the competing brand’s sales and (2) influence pricing and other members’ advertising decisions in the channel.

(1) The competitive and complementary (informative) roles of advertising: In a competitive market, the effect of brand advertising on the competitor’s sales and profits can be competitive or complementary. Dubé and Manchanda (2005) found that the nature of this effect is exogenous to the firms involved and that it depends on the availability of the media outlets in the market. Their empirical study showed that in small markets, the existence of only a few media outlets leads to competitive effects of advertising. In this case, an incremental investment in advertising by one firm helps differentiate the product and leads to stealing demand from competitors. However, in markets characterized by the availability of many media outlets, the effect of advertising tends to be complementary. In this case, advertising by one firm improves consumers’ awareness for the product category existence, quality and attributes. Consumers become then more attracted not only to the advertised product but also to its competitors. Roberts and Samuelson (1988) studied the effects of advertising on total market sales in an oligopolistic market of cigarette makers. They found also that the ultimate effect of advertising is exogenous to the firms involved and depends on the kind of cigarettes that are being advertised.

(2) The relationship between advertising and prices: The literature reports a controversial relationship between pricing and advertising decisions. Some studies support the theory that higher advertising expenditures result in higher prices because advertising is an indicator of the quality of the advertised products (Nelson, 1974; Bagwell and Ramey, 1994). However, other empirical studies showed that the relationship between prices and advertising depends on the role of advertising. For informative advertising, Gossman and Shapiro (1984) and Robert and Stahl (1993) found that advertising reduces differentiation between products that are physically similar and leads to lower prices and Soberman (2004) showed that advertising results into lower or higher prices depending on the level of differentiation between the competing firms. When advertising is competitive, higher advertising conveys positive messages to consumers about the advertised product which could lead to higher prices (see, Shankar and Bolton, 2004).

In this paper, we examine the relationship between pricing and advertising decisions in the context of store and national brands competition and when the store brand carries the retailer’s name. We aim at answering the following research questions:

- Considering the dynamic effects of advertising, what is the nature of the relationship between advertising and prices?
- In particular, how should a channel member adjust his prices to changes in his and the other member’s brand advertising expenditures?
- Would this relationship depend on the competitive or complementary effects of their advertising?
- Also, how are the retailer’s and the manufacturer’s advertising strategies related?

Our purpose is to improve our understanding of how manufacturers and retailers should manage their advertising and pricing decisions given different competitive interactions between national and private labels. While the previous literature focused mainly on pricing competition between store and national brands, we consider both the manufacturer’s and the retailer’s brand advertising decisions and we account for the different roles of advertising. We also provide a novel analysis of the relationship that exists between advertising and prices in a distribution channel by considering the retailer’s advertising decision. We study these issues by proposing a dynamic model that takes account of the carryover effects over time of both the retailer’s and the manufacturer’s advertising decisions. Many studies showed indeed that past campaigns influence the consumers’ present purchase behavior, which makes advertising an inherently dynamic variable (see, e.g., Lodish et al.,

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1 e.g., Carrefour in France, Tesco and Marks & Spencer in the UK and Kroger in the US.
We solve for stationary Markov perfect pricing and advertising equilibrium and show that the relationship between advertising and pricing decisions in the channel depends mainly on the nature of the advertising effects. In particular:

- The manufacturer offers price concessions and limits his advertising expenditures as the retailer invests more in competitive advertising.
- The retailer’s pricing reaction to changes in competitive advertising levels depends on both the price competition level between the store and the national brands and the strength of the competitive advertising effect. For example, when advertising (by either the manufacturer or the retailer) is highly competitive and the two brands are closely positioned, the retailer decreases consumer prices.
- Complementary advertising by both channel members leads to higher prices, advertising and revenues.

The rest of the paper is organized as follows. Section 2 describes the model and its assumptions. Section 3 characterizes the equilibrium pricing and advertising strategies. In Section 4, we discuss the relationship between the equilibrium pricing and advertising strategies. In Section 5, we analyze the relationship between the equilibrium advertising strategies of the manufacturer and the retailer. Section 6 concludes.

2. The model

We consider a distribution channel formed by one manufacturer and a single retailer. We assume that both the retailer and the manufacturer benefit from local monopoly and consider that the manufacturer supplies the national brand to the retailer (Soberman and Parker, 2004). The latter sells two competing products: the manufacturer’s national brand and a private label (or store brand). We make the usual assumption that the private label’s supplier is a dummy player, either because he is a very small unit in the industry or because the retailer is manufacturing the private label himself.

The consumers’ demand for the national brand ($Q_N$) and for the store brand ($Q_S$) in the retail store at any time ($t$) are linear in prices and in goodwill stocks (Dubé and Manchanda, 2005; Vilcassim et al., 1999; Kadiyali et al., 2001). At time $t$, they are given by

$$Q_N(t) = a + \psi G_N(t) + \theta R(t) - p_N(t) + \alpha [p_S(t) - p_N(t)],$$  \hspace{1cm} (1)

$$Q_S(t) = b + \psi_S G_R(t) + \phi G_N(t) - p_S(t) + \alpha [p_N(t) - p_S(t)],$$  \hspace{1cm} (2)

where $a$ and $b$ are positive parameters that represent the baseline sales for the national brand and for the private label, respectively, with $a > b$ to reflect stronger consumers’ preference for the national brand. $p_N(t)$ is the retail price of the national brand, $p_S(t)$ is the retail price of the store brand, $G_N(t)$ is the current stock of accumulated advertising goodwill for the national brand and $G_R(t)$ is the current stock of accumulated advertising goodwill for the retailer. Since the private label is associated with the store, it benefits directly from the retailer’s brand advertising whether it carries the retailer’s name or not.

The aggregate demand function decreases in own price and increases with the price of the competing brand. Since the own price effect should be larger than the cross-price competitive effect, we impose that $\alpha \in (0, 1)$. We consider symmetric price sensitivity parameters for the national and the store brand to obtain manageable results (Raju et al., 1995).

Advertising decisions affect demand through the accumulated goodwill stocks of the retailer and the manufacturer. The demand for the national brand ($Q_N$) increases with its own goodwill ($G_N$). It also varies with the retailer’s goodwill ($G_R$). The marginal effect of the retailer’s goodwill on the national brand’s demand is measured by the parameter $\theta$. As the retailer enjoys higher brand ratings by consumers for his store, higher store traffic could be generated, which could create additional unit sales for the national brand, in which case $\theta$ takes positive values (Doyle and Saunders, 1990). We do also consider the case where the increase in the retailer’s goodwill stock has a competitive effect on the national brand’s sales ($\theta < 0$). The retailer’s advertising could indeed reinforce the consumers’ confidence in the perceived quality of the private label, especially when the latter carries the retailer’s name. Therefore, it could drive sales away from the manufacturer’s brands (Dick et al., 1997; Hoch and Banerji, 1993).

The demand for the store brand ($Q_S$) increases with the retailer’s goodwill ($G_R$) and varies with the national brand’s goodwill ($G_N$). Based on empirical evidence, we assume that the effect of $G_N$ on $Q_S$, which is represented by the parameter $\phi$, is exogenous to the firms and depends mainly on the availability of media outlets in the marketplace (see, e.g., Dubé and Manchanda, 2005).\footnote{A possible extension would be to model $\phi$ as the manufacturer’s decision variable to reflect the fact that the content of the advertising message chosen by managers could affect both the sign and the value of $\phi$. The manufacturer can choose to invest in competitive advertising that promotes his brand and compares it to the competing brands in the market. He can also choose to invest in generic advertising that promotes the qualities of the product category with the goal of increasing demand for all sellers in the marketplace, including the private label (Bass et al., 2005). We thank an anonymous reviewer for his comments on this issue.}

We do not impose any specific sign on $\phi$ to account for different possible roles of advertising and their
effects on sales. When the manufacturer’s advertising is competitive, it drives unit sales away from the private label and \( \phi \) takes negative values. When it is complementary, it builds category demand for the private label, in which case we have positive values for \( \phi \).

Finally, the direct effects of own goodwill on demand are positive \((\psi, \psi_S > 0)\) and their values depend on the media choice, content and intensity.

The long-term effects of advertising are captured in the evolution of the goodwill stocks over time (the system dynamics). They are given by the state variables that correspond to the goodwill stocks for the national brand \((\frac{dG_N}{dt})\) and for the retail store \((\frac{dG_R}{dt})\). We consider that at any instant in time \((t)\), the goodwill stock of the national brand (retailer) is increased by the advertising expenditures of the manufacturer (retailer) and depreciated over time as consumers forget some of the past advertising campaigns (Nerlove and Arrow, 1962). Since we focus on brand advertising only, we consider that the retailer’s brand advertising decision \((A_R)\) does not cover campaigns that use product featuring.

We follow Chintagunta (1993) and consider that advertising by a firm has a decreasing marginal return on its goodwill evolution. We use the same depreciation rate \((\lambda)\) (also called decay rate) of the goodwill stock for the national brand and for the retail store:

\[
\frac{dG_N}{dt}(t) = \delta \sqrt{A_N(t)} - \lambda G_N(t), \quad G_N(0) = G_{N0} > 0, \\
\frac{dG_R}{dt}(t) = \delta \sqrt{A_R(t)} - \lambda G_R(t), \quad G_R(0) = G_{R0} > 0,
\]

where \(\lambda\) and \(\delta\) are positive parameters and \(G_{N0}\) and \(G_{R0}\) denote the initial goodwill stocks of the national brand and the retail store, respectively.

The manufacturer and the retailer choose the transfer \((w_N)\) and retail prices \((p_N, p_S)\) and advertising \((A_N, A_R)\) strategies that maximize their discounted profit streams over an infinite planning horizon. The manufacturer’s objective functional is given by

\[
\Pi_N = \int_0^{+\infty} \exp(-rt) \left[ w_N(t) Q_N(t) - \frac{\mu}{2} A_N(t) \right] dt.
\]

The retailer’s instantaneous profits are given by his revenues from selling the national brand and his private label diminished by his advertising expenditures. Therefore, the retailer’s objective functional is given by

\[
\Pi_R = \int_0^{+\infty} \exp(-rt) \left\{ \left[ p_N(t) - w_N(t) \right] Q_N(t) + p_S(t) Q_S(t) - \frac{\mu}{2} A_R(t) \right\} dt,
\]

where \(\mu\) is a cost parameter and \(r \in (0, 1)\) is the common discount rate. We assume that the unit production costs of both brands are identical and taken equal to zero for tractability and without loss of generality (for the same assumption see, Raju et al., 1995; Narasimhan and Wilcox, 1998; Sayman et al., 2002).³

The differential game is played à la Stackelberg where the manufacturer is the leader and the retailer is the follower. The manufacturer announces his pricing and advertising strategies to the retailer who reacts to this information by choosing the retail prices and advertising expenditures that maximize his profits. The manufacturer chooses next the optimal transfer price and advertising investment. The game is played under a feedback information structure. Therefore, channel members make their decisions based on their current observations of the goodwill stocks for the manufacturer and for the retailer.⁴

As it is usually the case in infinite time horizon differential games, we consider that channel members’ strategies are stationary feedback, which means that pricing and advertising strategies are time-independent and only vary with the current levels of the goodwill stocks for the manufacturer and the retailer. We consider that at any instant in time, each channel member observes his goodwill stock and the other channel member’s goodwill stock level (e.g., through market research studies). As in Roberts and Samuelson (1988) and Martín-Herráñ and Taboubi (2005), we assume however that each channel member observes only the evolution over time of his goodwill dynamics but does not consider the evolution over time of the other player’s goodwill stock because such information would be expensive to obtain or because he does not have such information.⁵

We next solve the model and discuss the equilibrium strategies.

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³ Note that the primary difference between the store and the national brands in our model is that while the retailer captures the entire margin for the private label, the retailer only takes a fraction of the margin on the national brand.

⁴ Our feedback Stackelberg equilibrium is time consistent and is therefore preferred to an open-loop Stackelberg equilibrium that is time inconsistent, given our model’s formulation.

⁵ For further discussion of this assumption and its implications on channel members’ strategies, see Jørgensen et al. (2003) and Roberts and Samuelson (1988).
3. Equilibria

To obtain the equilibrium pricing and advertising strategies, we solve the following optimization problems for the manufacturer and for retailer, respectively,\(^6\)

\[
\max_{w, d} \int_0^\infty \exp(-rt) \left[w_N Q_N - \frac{\mu}{2} A_N\right] dt,
\]
\[
s.t.: \quad \frac{dG_N}{dr} = \delta \sqrt{A_N} - \lambda G_N, \quad G_N(0) = G_{N0} > 0,
\]
\[
\max_{dR, pN, pS} \int_0^\infty \exp(-rt) \left[(p_N - w_N) Q_N + p_S Q_S - \frac{\mu}{2} A_R\right] dt,
\]
\[
s.t.: \quad \frac{dG_R}{dr} = \delta \sqrt{A_R} - \lambda G_R, \quad G_R(0) = G_{R0} > 0.
\]

(5)

(6)

In order to solve for the Stackelberg equilibrium, we first characterize the retailer’s reaction functions. Once the latter are known, the manufacturer injects this information into his objective function in order to determine his equilibrium transfer price and advertising strategies. Finally, the equilibrium retail prices for the national and private labels and the retailer’s advertising are obtained by replacing the equilibrium transfer price and manufacturer’s advertising into the retailer’s reaction functions.

It is important to note that due to the structure of our model,\(^7\) we obtain the same equilibrium pricing and advertising strategies whether each channel member decides first of his pricing and then of his advertising or vice versa or chooses his advertising and pricing strategies simultaneously.

Since in our model prices do not affect the evolution over time of the goodwill stocks, we can solve for the pricing strategies statically.\(^8\) For that reason, we will first present the equilibrium pricing strategies, and afterwards, the equilibrium advertising strategies. Note however that equilibrium prices will be influenced indirectly by advertising through the goodwill stocks. The pricing problems that the channel members are facing are

\[
\max_{w} \left[w_N Q_N - \frac{\mu}{2} A_N\right],
\]
\[
\max_{pN, pS} \left[(p_N - w_N) Q_N + p_S Q_S - \frac{\mu}{2} A_R\right].
\]

The equilibrium advertising strategies require the resolution of the Hamilton–Jacobi–Bellman equations associated with the manufacturer’s and the retailer’s optimization problems. In the rest of the paper, we restrict our analysis to the situations where equilibrium pricing and advertising are only positive.

3.1. Equilibrium pricing strategies

The next proposition characterizes the retailer’s pricing reaction functions.

**Proposition 1.** The retailer’s pricing reaction functions read

\[
p_N(w_N, G_N, G_R) = \frac{w_N}{2} + \frac{ax + b(1 + x) + [x^2 + (1 + x)\phi + (1 + x)\theta]G_N + [(1 + x)\psi_G + x\theta]G_R}{2(1 + 2x)},
\]
\[
p_S(w_N, G_N, G_R) = \frac{ax + b(1 + x) + [(1 + x)\phi + x\psi_G + (1 + x)\psi_S + x\theta]G_R}{2(1 + 2x)},
\]

if these are positive expressions, and zero otherwise.

**Proof.** From the necessary conditions for optimality, taking the partial derivatives of the objective function in (6), once the demand function in (2) has been replaced, with respect to \(p_N\) and \(p_S\) and equating to zero, we obtain

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\(^6\) From now on the time argument is eliminated when no confusion can arise.

\(^7\) Prices and advertising decisions are related indirectly through the goodwill stocks and there are no direct interactions between these variables neither in the payoff functionals nor in the goodwill stocks dynamics. Since we consider a continuous-time framework, variations of the goodwill stocks are instantaneous. Therefore, the interdependence that appears in the game through this variation cannot be captured in the first-order conditions obtained from the maximization of the right-hand side of the HJB equations.

\(^8\) See e.g., Dubé and Manchanda (2005) and Jørgensen and Zaccour (1999) for other pricing and advertising games where this occurs.
At equilibrium, the demand functions for the national brand and the private label, respectively, are given by

\[ \begin{align*}
1 & + 2(1 + x) p_N + 2 x p_S + \psi N + \theta G_R = 0, \\
b - a w_N - 2 x p_N - 2(1 + x) p_S + \phi N + \psi S G_R = 0.
\end{align*} \]

Solving the above system of equations and taking into account that the maximum in (6) is concave in \( p_N \) and \( p_S \), the retailer’s reaction functions in (7) and (8) are obtained.

The next proposition characterizes the manufacturer’s pricing strategy at the equilibrium.

**Proposition 2.** The manufacturer’s optimal pricing strategy is given by

\[
w_N^*(G_N, G_R) = \frac{a + \psi G_N + \theta G_R}{2(1 + x)}. \tag{9}
\]

**Proof.** Replacing the reaction functions given in (7) and (8) in the manufacturer’s problem in (6), once the demand functions in (1) and (2) have been substituted, we obtain

\[
\frac{1}{2} [(a + \psi G_N + \theta G_R) w_N - (1 + x) w_N^2 - \mu A_N].
\]

From the necessary conditions for optimality, taking the partial derivative of the above expression with respect to \( w_N \) and equating to zero, we obtain

\[
\frac{1}{2} [a + \psi G_N + \theta G_R - 2(1 + x) w_N] = 0,
\]

and taking into account that (10) is concave in \( w_N \), the result in the proposition follows.

The retailer’s optimal pricing strategies can now be completely characterized.

**Corollary 1.** The retailer’s optimal pricing strategies are given by

\[
\begin{align*}
p_S^*(G_N, G_R) &= \frac{a \alpha + b(1 + x) + [(1 + x) \phi + x \psi] G_N + [(1 + x) \psi S + x \theta] G_R}{2(1 + 2x)}, \\
p_N^*(G_N, G_R) &= \frac{1}{2} w_N^*(G_N, G_R) + p_S^*(G_N, G_R) + \frac{a - b + (\psi - \phi) G_N + G_R (\theta - \psi S)}{2(1 + 2x)} G_N \\
&= \frac{a(1 + 2x) + 2(1 + x) [a(1 + x) + b \psi]}{4(1 + x)(1 + 2x)} + \frac{(3 + 6x + 2 \psi^2 + 2 \psi(1 + x) \phi + 2 \alpha(1 + x) \phi)}{4(1 + x)(1 + 2x)} G_N \\
&+ \frac{(3 + 6x + 2 \psi^2) \theta + 2 \alpha(1 + x) \psi S}{4(1 + x)(1 + 2x)} G_R,
\end{align*}
\]

if these are positive expressions, and zero otherwise.

**Proof.** The retailer’s optimal pricing strategies are computed after replacing the manufacturer’s optimal strategy given in (9) into the retailer’s reaction functions given by (7) and (8).

These results show that equilibrium prices depend on both the manufacturer’s and the retailer’s goodwill stocks meaning that brand advertising changes the amounts consumers pay for these products. We study in the following section the sensitivity of these equilibrium prices to changes in both goodwill stocks.

Since the demand functions depend on the goodwill stocks and the prices for the manufacturer and the retailer, we can now derive results for the equilibrium demand functions for the national and the private labels.

**Corollary 2.** At equilibrium, the demand functions for the national brand and the private label, respectively, are given by

\[
\begin{align*}
Q_N^*(G_N, G_R) &= \frac{1}{4} (a + \psi G_N + \theta G_R), \\
Q_S^*(G_N, G_R) &= \frac{2 \beta (x + 1) + a \alpha + [2 \phi (x + 1) + x \psi] G_N + [2 \psi S (x + 1) + \theta] G_R}{4(x + 1)}.
\end{align*}
\]

**Proof.** The result is obtained by writing the retailer’s optimal pricing strategies given in Corollary 1 in the demand functions in (1) and (2).

This result shows that the manufacturer’s equilibrium demand is positively related to his goodwill. However \( Q_N \) increases (decreases) with \( G_R \) only for complementary (competitive) retailer’s advertising effects.
The demand for the private label increases with higher levels of the retailer’s goodwill stock for \( \frac{\partial w}{\partial G} > -\frac{2(\alpha+1)}{a} \). This means
that when the retailer’s advertising is complementary or slightly competitive, \( Q_s \) increases with higher values of \( G_R \). However, when the retailer’s advertising is competitive enough \( \left( \frac{\partial w}{\partial G} > -\frac{2(\alpha+1)}{a} \right) \), an increase in the retailer’s goodwill stock leads to lower unit sales for the store brand.

The effect of \( G_N \) on \( Q_s \) depends on the role of the manufacturer’s advertising. It is positive if and only if \( \frac{\partial w}{\partial G} > -\frac{2(\alpha+1)}{a} \). Hence, when the manufacturer’s advertising is complementary or slightly competitive, this effect is positive. However, higher levels of the manufacturer’s goodwill stocks lead to lower sales for the store brand for highly competitive manufacturer’s advertising.

### 3.2. Equilibrium advertising strategies

We now discuss the results for the manufacturer’s and the retailer’s equilibrium advertising strategies (\( A_N \) and \( A_R \)).

Note that the obtained equilibrium advertising strategies are the same whether the manufacturer and the retailer decide simultaneously or sequentially of their advertising investments. In fact, the reaction functions for the advertising game only depend on the state variables, that is why they can be qualified as stationary feedback Nash strategies. When this happens, the first movement disadvantage disappears yielding the coincidence between the stationary feedback Nash equilibrium (simultaneous play) and the stationary feedback Stackelberg equilibrium (hierarchical play). This means that the equilibria are identical independently of which player is the leader of the game.\(^9\) Moreover, as we have previously established, the equilibrium advertising strategies are the same whether each channel member decides first of his pricing and then of his advertising or vice-versa or chooses his advertising and pricing strategies simultaneously.

To compute these equilibria, we assume that the manufacturer’s and the retailer’s optimal pricing decisions are known. Therefore, we first replace the transfer price by \( w^*_N(G_N, G_R) \) and the retail prices by \( p^*_N(G_N, G_R) \) and \( p^*_S(G_N, G_R) \) in the manufacturer’s and the retailer’s objective functionals. Then, we maximize the obtained objective functional for each channel member relative to his advertising decision taking into account the evolution over time of his goodwill stock.

The Hamilton–Jacobi–Bellman (HJB) equations for the retailer and the manufacturer are given by

\[
\begin{align*}
    rV_N(G_N, G_R) &= \max_{A_N} \left\{ w^*_N(G_N, G_R)Q^*_N(G_N, G_R) - \frac{\mu}{2} A_N + \frac{\partial V_N}{\partial G_N}(G_N, G_R) \left[ \sqrt{A_N} - \frac{\lambda}{\mu} \right] G_N \right\}, \\
    rV_R(G_N, G_R) &= \max_{A_R} \left\{ p^*_N(G_N, G_R) - w^*_N(G_N, G_R)Q^*_N(G_N, G_R) + p^*_S(G_N, G_R)Q^*_S(G_N, G_R) - \frac{\mu}{2} A_R + \frac{\partial V_R}{\partial G_R}(G_N, G_R) \left[ \sqrt{A_R} - \frac{\lambda}{\mu} \right] G_R \right\},
\end{align*}
\]

where \( Q^*_N(G_N, G_R) \) and \( Q^*_S(G_N, G_R) \) denote the optimal consumers’ demands for the national brand and for the private label, respectively. \( V_N(G_N, G_R) \) and \( V_R(G_N, G_R) \) represent the manufacturer’s and retailer’s value functions, which give the optimal profit for each channel member when the goodwill stocks for the national and the store brands are, respectively, \( G_N \) and \( G_R \).

After solving the HJB equations for the manufacturer and the retailer, we get two possible expressions for each of the equilibrium advertising strategies and we choose the ones that lead to asymptotically stable steady states with the least restrictive conditions (see Appendices 1 and 2 for detailed computations of results).

The obtained equilibrium advertising strategies are functions of the goodwill stocks \( G_N \) and \( G_R \) and are given by

\[
\begin{align*}
    A_N^*(G_N, G_R) &= \left[ \frac{\delta}{\mu} \frac{\partial V_N}{\partial G_N}(G_N, G_R) \right]^2 = \left[ \frac{\delta}{\mu} (M_1 G_N + M_2 G_R + M_3) \right]^2, \\
    A_R^*(G_N, G_R) &= \left[ \frac{\delta}{\mu} \frac{\partial V_R}{\partial G_R}(G_N, G_R) \right]^2 = \left[ \frac{\delta}{\mu} (R_1 G_N + R_2 G_R + R_3) \right]^2.
\end{align*}
\]

For clarity of presentation, we include the expressions of the coefficients \( M_i \) and \( R_i \) (\( i = 1, \ldots, 5 \)) in Appendix 1.

We next analyze the obtained equilibrium pricing and advertising strategies in order to answer our research questions, i.e., to study the relationship between the pricing and advertising strategies and to examine the relationship between the advertising strategies of channel members.

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\(^9\) See Rubio (2006) for a characterization of the conditions needed to obtain the coincidence between the stationary feedback Nash equilibrium and the stationary feedback Stackelberg equilibrium and for a list of papers where this structure has been used.
4. Relationship between equilibrium advertising and pricing strategies

From the equilibrium strategies, we can see that both the manufacturer’s and the retailer’s advertising strategies have an impact on equilibrium transfer and retail prices through the goodwill stocks. To study this impact, we first analyze the effects of own goodwill on the advertising strategies of each channel member in the next proposition.

Proposition 3. At equilibrium, the manufacturer’s (retailer’s) advertising is positively related to his goodwill stock.

Proof. See Proposition 8 in Appendices 2 and 3.

This proposition shows that at equilibrium, the manufacturer and the retailer should invest higher amounts in advertising when an increase in own goodwill stock is observed. This means that highly rated brands should be advertised more heavily. A similar result has been obtained by e.g., Deal (1979), Erickson (1991) and Espinosa and Mariel (2001).

Note that the sign of the relationship between the equilibrium prices \((w^*_N, p^*_N, p^*_S)\) and advertising strategies \((A^*_N, A^*_R)\) is the same as the sign of the relationship between the equilibrium prices and the goodwill stocks \((G_N, G_R)\). Indeed, the relationship between say, the transfer price and the equilibrium advertising can be derived from the following expressions:

\[
\frac{\partial w^*_N}{\partial G_N} = \frac{\partial w^*_N}{\partial A^*_N} \frac{\partial A^*_N}{\partial G_N} \quad \text{and} \quad \frac{\partial w^*_N}{\partial G_R} = \frac{\partial w^*_N}{\partial A^*_R} \frac{\partial A^*_R}{\partial G_R}.
\]

From Proposition 3, we know that \(\frac{\partial A^*_N}{\partial G_N} > 0\) and \(\frac{\partial A^*_R}{\partial G_R} > 0\). Hence,

\[
\text{sign} \left( \frac{\partial w^*_N}{\partial G_N} \right) = \text{sign} \left( \frac{\partial w^*_N}{\partial A^*_N} \right), \quad \text{sign} \left( \frac{\partial w^*_N}{\partial G_R} \right) = \text{sign} \left( \frac{\partial w^*_N}{\partial A^*_R} \right).
\]

The same reasoning applies to the sensitivity analysis of the retail prices to changes in the advertising strategies. The relationship between advertising and prices can then be studied through the effects of the goodwill stocks on the equilibrium prices.

We propose first to investigate the sensitivity of \(p^*_N, p^*_S, w^*_N\) to changes in the manufacturer’s goodwill stock. The obtained results are summarized in the next proposition.

Proposition 4. At equilibrium, the relationship between prices and the manufacturer’s advertising is as follows:

(4.a) The manufacturer charges higher transfer prices as his advertising increases.

(4.b) For \(\frac{\partial}{\partial A^*_N} > - \frac{1}{\frac{1}{\partial G_N}}\), the retailer should increase the prices of both the national and the private labels with higher manufacturer’s advertising levels.

(4.c) For \(-\frac{2(2+x)}{2x(1+x)} < \frac{\partial}{\partial A^*_N} < -\frac{1}{\partial G_N}\), the retailer should increase the price of the national brand and decrease the private label’s price with higher manufacturer’s advertising levels.

(4.d) For \(\frac{\partial}{\partial A^*_N} < -\frac{2(2+x)}{2x(1+x)}\), the retailer should lower the prices of both the national brand and the private label with higher manufacturer’s advertising levels.

Proof. Derive the equilibrium prices with respect to \(G_N\) to get

\[
\frac{\partial w^*_N}{\partial G_N} = \frac{\psi}{2(1+x)}, \quad \frac{\partial p^*_N}{\partial G_N} = \frac{(3+6x+2x^2)\psi + 2x(1+x)\phi}{4(1+x)(1+2x)}, \quad \frac{\partial p^*_S}{\partial G_N} = \frac{(1+x)\psi + x\psi}{2(1+2x)}.
\]

Note that

\[
\left( -\frac{2x^2 + 6x + 3}{2x(1+x)} \right) - \left( -\frac{x}{1+x} \right) = -\frac{3(2x+1)}{2(x+1)x},
\]

which is negative since \(x\) is positive.

This proposition shows first the intuitive result that the manufacturer should charge higher prices as he invests more in advertising his brand. This confirms the empirical result of Dubé and Manchanda (2005) who found that firms increase prices with higher own goodwill in a dynamic setting.

At the retail level, we show that although the retailer pays more for a national brand that is more heavily advertised, it is not always optimal to pass on the price increase to consumers. In particular, the retailer’s price reaction to a change in the manufacturer’s advertising level depends on both the intensity of price competition between the national and the store brands and on the strength of the manufacturer’s competitive advertising effect.
As the result in (4.b) shows, the retailer should charge higher prices not only for the national brand but also for the private label if the manufacturer’s advertising is complementary \( \left( \frac{\phi}{\psi} > 0 \right) \) \( \Rightarrow \frac{1}{1+\alpha} \). Hence, when the manufacturer’s advertising expands demand for the product category, e.g., by informing consumers about the quality and the product’s attributes, higher prices are charged to the retailer and to consumers. In this case, the manufacturer’s advertising is adding value to both brands, and the retailer is better off charging higher prices for this added value. This result differs from those reported in Gossman and Shapiro (1984) and Robert and Stahl (1993) who found that complementary advertising leads to lower prices. Also, Soberman (2004) showed that complementary advertising could result into lower or higher prices depending on the level of differentiation between the competing firms. These differences in results are mainly due to the facts that we account for retail advertising and store brand competition, incorporate advertising dynamics and consider the effects generated by non-price advertising.

Interestingly, the same positive relationship between prices and the manufacturer’s advertising still exists in case the latter has a slightly competitive effect on the private label’s sales \( \left( \frac{\phi}{\psi} < 0 \right) \) \( \Rightarrow \frac{1}{1+\alpha} \). As we can see in Fig. 1, this situation is represented by Region 1, which corresponds to (1) low levels of price competition between the national and the store brands and low competitive advertising effect \( (\text{as measured by} \frac{\psi}{\phi}) \) and (2) intense price competition levels between the national and the store brands combined with a competitive advertising effect that is lower than the direct advertising effect \( \left( \frac{\phi}{\psi} \text{ close to } 1 \right) \).

When the price competition between the national and the store brands is low, and the competitive advertising effect is high such as \(-\frac{\psi}{\phi} \text{ close to } 1 \Rightarrow \frac{1}{1+\alpha} \) (Region 2 in Fig. 1), we can see that the optimal strategy for the retailer in this case is to lower the private label’s price but increase the national brand’s price. A possible explanation of this result is that the retailer has to make the store brand more attractive to consumers by offering price reductions.

Finally, for high levels of both price competition and competitive advertising effect (Region 3 in Fig. 1), the retailer should react to an increase in the manufacturer’s advertising by offering price concessions to consumers on both brands. Note that in this case, the retailer should charge lower prices for both the national and the store brands, even if he is buying the national brand at a higher price. The intense competition between the national and the store brands forces the retailer to lower his prices for the product category. Price wise, this situation is beneficial to the manufacturer and to final consumers but detrimental to the retailer’s margin.\(^{10}\)

These results show the importance of both the nature and the intensity of the advertising effects in understanding the relationship between the manufacturer’s advertising and the retail prices. For example, when the manufacturer’s advertising is competitive and for the same level of price competition between the national and the store brands (say \( x = 0.4 \)), we can see that as \( \frac{\psi}{\phi} \) increases, the retailer changes his pricing reaction to an increase in \( A_N^* \) from boosting both brands’ prices, to increasing \( p_N^* \) and decreasing \( p_S^* \), to decreasing both prices. These findings extend the results in Soberman (2004) and Shankar and Bolton (2004) by emphasizing the importance of considering both the competitive advertising effect and the degree of differentiation between the two brands to understand how prices are influenced by the manufacturer’s advertising.

\(^{10}\) Results do not change if we consider that own price effect is lower than the cross-price competitive effect (i.e., \( x > 1 \)).
Proposition 5. At equilibrium, the relationship between prices and the retailer’s advertising is as follows:

(5.a) The manufacturer charges higher (lower) transfer prices when the retailer’s advertising is complementary (competitive).

(5.b) For \( \frac{\partial p_N}{\partial a} > - \frac{2(1+\alpha)}{3 + 6\alpha + 2\alpha^2} \), the retailer should increase the prices of both the national and the private labels with higher retailer’s advertising levels.

(5.c) For \( - \frac{1+\alpha}{a} < \frac{\alpha}{\psi} < - \frac{2(1+\alpha)}{3 + 6\alpha + 2\alpha^2} \), the retailer should decrease the price of the national brand and increase the private label’s price with higher retailer’s advertising levels.

(5.d) For \( \frac{\partial p_S}{\partial a} < - \frac{1+\alpha}{a} \), the retailer should lower the prices of both the national brand and the private label with higher retailer’s advertising levels.

Proof. Derive the equilibrium prices with respect to \( G_R \) to get

\[
\frac{\partial w_N}{\partial G_R} = \frac{\theta}{2(1 + \alpha)}, \quad \frac{\partial p_N}{\partial G_R} = \frac{(3 + 6\alpha + 2\alpha^2)\theta + 2\alpha(1 + \alpha)\psi_S}{4(1 + \alpha)(1 + 2\alpha)}, \quad \frac{\partial p_S}{\partial G_R} = \frac{2\theta + (1 + \alpha)\psi_S}{2(1 + 2\alpha)}.
\]

Note that:

\[
\frac{1 + \alpha}{a} - \left( \frac{2\alpha(1 + \alpha)}{3 + 6\alpha + 2\alpha^2} \right) = -\frac{3(2\alpha + 1)(1 + \alpha)\psi_S}{(3 + 6\alpha + 2\alpha^2)\alpha},
\]

which is negative since \( \alpha \) and \( \psi_S \) are positive. \( \square \)

From this Proposition, we can see that when the retailer’s advertising has a complementary role \( (\theta > 0) \), there is a positive relationship between the equilibrium prices and the retailer’s advertising no matter the level of price competition between the national and the store brands. In this case, the higher is \( A_R \), the more the retailer pays for the manufacturer’s product and the more he charges consumers for both the national and the store brands. An increase in complementary retailer’s advertising levels leads then to higher prices.

When the retailer’s advertising has a competitive role, i.e., it drives some unit sales away from the national brand, an increase in the retailer’s advertising leads to lower transfer prices; the retailer gets a better deal on the national brand in this case. On the retail side, it could lead to lower or higher pricing strategies for both or either brands depending on the intensity of price competition between the national and the store brands and also on the strength of the retailer’s competitive advertising effect.

We note first that the retail prices for both the national and the store brands increase for very low competitive advertising effects \( \left( \frac{\partial p_S}{\partial a} > - \frac{2(1+\alpha)}{3 + 6\alpha + 2\alpha^2} \right) \). As price competition between the two brands intensifies, i.e., the two brands are more closely positioned, the retailer would still optimally increase prices for the national and the store brands even if the competitive advertising effect is a little higher (Region 1 in Fig. 2).

However, when the retailer’s competitive advertising effect is moderate \( \left( - \frac{1+\alpha}{a} < \frac{\alpha}{\psi} < - \frac{2(1+\alpha)}{3 + 6\alpha + 2\alpha^2} \right) \), the increase in the retailer’s advertising level would lead to a lower national brand’s retail price and to a higher private label’s price (Region 2 in Fig. 2). Finally, for high levels of price competition between the national and the store brands combined with higher

![Fig. 2. Relationship between \( p_N \), \( p_S \) and \( A_R \).](image-url)
competitive advertising effects (in absolute values), the retailer would optimally charge lower prices to consumers for both brands (Region 3 in Fig. 2).  

When \( A_R \) is competitive, the main advantage for the retailer from investing in higher advertising levels seems to be driven by the increased revenues generated from the store brand.  

However, this would not be the case if the manufacturer’s brand is competing closely with the store brand (high \( x \)) and the retailer’s advertising is highly competitive. Further, competitive advertising allows the retailer to charge higher prices for the product category and obtain higher margins on the national brand only when the advertising effects are very low. As \( A_R \) generates stronger competitive effects, the retailer has to pass on part of the manufacturer’s price concessions.

In the following section, we discuss the relationship between the advertising strategies of the retailer and the manufacturer.

5. Relationship between the manufacturer’s and the retailer’s equilibrium advertising strategies

From Proposition 3, we know that each channel member’s advertising is positively related to his goodwill stock, that is,  

\[
\left( \frac{\partial A_N}{\partial G_N} > 0 \right) \quad \text{and} \quad \left( \frac{\partial A_R}{\partial G_R} > 0 \right). 
\]

The next proposition is then obtained using the following result:

\[
\text{sign} \left( \frac{\partial A_N}{\partial G_N} \right) = \text{sign} \left( \frac{\partial A_R}{\partial G_R} \right). 
\]

The next proposition announces the manufacturer’s (retailer’s) advertising reaction to changes in the retailer’s (manufacturer’s) advertising level.

**Proposition 6.** At equilibrium, the manufacturer’s advertising is positively (negatively) related to the retailer’s advertising when the latter has complementary (competitive) effects.

**Proof.** See Proposition 8 in Appendix 2. \( \square \)

This proposition shows that the manufacturer’s reaction to changes in the retailer’s advertising level depends on the latter’s complementary or competitive role.

When the retailer’s advertising expands demand for the national brand, the manufacturer should increase his advertising with higher retailer’s advertising levels. This is in part explained by the fact that the manufacturer gets higher revenues as the retailer’s complementary advertising increases (from Corollary 2 and Proposition 5).

Interestingly, when the retailer’s advertising reduces the national brand’s demand, it is optimal for the manufacturer to lower his advertising effort. In this case, we also showed in Corollary 2 and Proposition 5 that the manufacturer gains a lower margin and fewer unit sales for the national brand. The manufacturer obtains then lower revenues when the retailer increases his investments in national advertising, which could explain why he should cut his advertising expenses.

This result shows that it is important for the manufacturer to carefully study the retailer’s national advertising effects on the national brand’s sales. We find indeed that manufacturers cannot always use advertising to counter competition from a private label that is growing stronger because it carries the retailer’s advertised name. In particular, manufacturers should reduce their advertising expenditures in response to higher levels of the retailer’s advertising when the latter reinforces the perceived quality of the private label but drives some consumers away from purchasing the national brand. Under such conditions, the empirical result that manufacturers could win the battle against private labels by investing in marketing activities that reinforce the manufacturer’s brand equity does not hold (see, Richardson, 1997; Ailawadi, 2001). Furthermore, when the retailer’s advertising is competitive for the national brand, the best response of the retailer to increased manufacturer’s advertising is to advertise less but offer a higher price for the national brand.

6. Summary and conclusions

This paper provides a novel study of the relationship between the manufacturer’s and the retailer’s advertising and pricing strategies in a bilateral monopoly. We consider that the retailer sells a private label in addition to the manufacturer’s product. Brand advertising is undertaken by the manufacturer for his national brand and by the retailer in order to build preference for his store. The findings from our dynamic model suggest that the relationship between advertising and pricing decisions in the channel depends strongly on the competitive or complementary effects of both members’ advertising.

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11 Results do not change if we consider that own price effect is lower than the cross-price competitive effect (i.e., \( x > 1 \)).

12 From Proposition 5, \( \frac{\partial A_N}{\partial A_R} > 0 \iff \frac{\partial Q_N}{\partial A_R} > 0 \iff \frac{\partial G_N}{\partial A_R} > 0 \), and remember from Corollary 2 that \( \frac{\partial G_N}{\partial Q_R} > 0 \iff \frac{\partial A_N}{\partial A_R} > 0 \). Therefore, \( \frac{\partial G_N}{\partial Q_R} > 0 \) for \( \frac{\partial Q_R}{\partial A_R} > -\frac{1}{x} \).
We show that when advertising is complementary for both brands, an increase in advertising expenditures by one or both channel members leads to higher transfer and consumer prices. It also expands demand for the national and the private labels and therefore creates additional revenues at both channel’s levels. Increased investments in complementary advertising are then beneficial to both the retailer and the manufacturer, who pay cheaper prices and get higher sales. However, consumers end up paying higher prices for the store and national brands. Further, when the channel’s advertising is complementary, both members’ advertising reaction to increase in each other’s advertising is also positive. This means that the retailer should build higher preference for his store when the national brand’s advertising builds demand for the private label and vice versa.

When the advertising effects are competitive, the channel members’ pricing and advertising reactions to changes in own and each other’s advertising levels are more complex and depend mainly on the intensity of the competitive advertising effects.

At the upper stream of the channel, the manufacturer reacts to higher levels of the retailer’s competitive advertising by offering price concessions and by decreasing his advertising investments. This could be explained by the fact that when the retailer’s advertising is detrimental to the national brand’s sales, the manufacturer gets lower revenues and reduces his advertising investments consequently.

At the downstream of the channel, the retailer adjusts the prices of the national and the store brands with changes in the channel’s advertising levels differently depending on two factors: (1) the price competition level between the national and the store brands and (2) the intensity of the competitive advertising effects. In particular, when the private label’s position is very far from the manufacturer’s brand, the retailer should increase the store brand’s price and offer price cuts for the national brand as his advertising increases. However, as the manufacturer invests more in competitive advertising, the retailer should charge higher prices for the national brand and offer the private label at a cheaper price. When the store and the national brands are highly differentiated, an increase in the advertising spending by one channel member leads then to a higher consumer price for his product and to a lower price for the competing brand. For moderate and strong levels of price competition between the national and the store brands, the retailer’s pricing reaction to higher levels of advertising changes as the advertising competitive effects get stronger. For the retailer, the main advantage from investing in higher competitive advertising levels seems to be driven by increased revenues from the store brand unless the competitive effects start to be too detrimental for the national brand.

These findings have interesting implications for marketing managers involved in the battle between national brands and private labels. In order to better understand the retailer’s pricing decisions for these two brands, the manufacturer should estimate not only his goodwill level but also the retailer’s goodwill. He also should be informed about the complementary or competitive effects of his and the retailer’s advertising in order to better predict the retailer’s pricing and advertising decisions. In particular, the retailer’s advertising should be considered as a strategic decision in the channel not only by the retailer but also by the manufacturer.

Finally, future work can look at extending our analysis to the case where competing manufacturers are selling their national brands to the retailer. In particular, it would be interesting to see how our results change given the different degrees of competitiveness in the product market. Our research could also be extended by considering other marketing mix variables, e.g., the retailer’s non-price promotions for the national and the private labels and the manufacturer’s trade promotions.

Acknowledgements

We wish to thank two anonymous reviewers for their comments. The first author’s research is funded by NSERC, Canada. The second author’s research was partially supported by MEC under Project SEJ2005-03858/ECON and by JCYL under Project VA045A06, co-financed by FEDER funds.

Appendix 1. Computation of the equilibrium advertising strategies

The Hamilton–Jacobi–Bellman (HJB) equations for the retailer and the manufacturer are given by (11) and (12). Consider the following change of variables,

\[ u_N = \sqrt{A_N}, \quad u_R = \sqrt{A_R}, \]

and the following functional specifications for the national brand’s \( V_N(G_N, G_R) \) and for the retailer’s \( V_R(G_N, G_R) \) value functions.

\[ 13 \] We thank an anonymous reviewer for this suggestion.
where $M_i$ and $R_i, i = 1, \ldots, 6$ are constants to be determined in order for the proposed value functions to satisfy the HJB equations.

From the manufacturer's HJB equation we get

$$M_1 = \mu \frac{r + 2\lambda}{2\delta^2} - \frac{\sqrt{(2(1 + \alpha)(r + 2\lambda)\mu)^2 - (1 + \alpha)\mu\delta^2\psi^2}}{4\delta^2(1 + \alpha)},$$

$$M_3 = -\frac{\theta\psi}{4(1 + \alpha)(\delta^2 M_1 - (r + \lambda)\mu)}, \quad M_4 = -\frac{a\psi}{4(1 + \alpha)(\delta^2 M_1 - (r + \lambda)\mu)} = \frac{a M_3}{\theta^2}.$$  \hspace{1cm} (15)

Under the assumption that $4(1 + \alpha)(r + 2\lambda)\mu - \delta^2\psi^2 > 0$, i.e., the value inside the square root in (15) is positive, then $M_1$ is a positive real number.

Condition for interior solution for $A_N$ is then

$$\sqrt{A_N(G_N, G_R)} = u_N(G_N, G_R) = \frac{\delta}{\mu} \frac{\partial V_N}{\partial G_N}(G_N, G_R) = \frac{\delta}{\mu} (M_1 G_N + M_3 G_R + M_4).$$

To guarantee a positive advertising investment we need to verify that $(M_1 G_N + M_3 G_R + M_4) > 0$.

The manufacturer's equilibrium advertising strategy is then

$$A_N(G_N, G_R) = \left(\frac{\delta}{\mu} \frac{\partial V_N}{\partial G_N}(G_N, G_R)\right)^2 = \left[\frac{\delta}{\mu} (M_1 G_N + M_3 G_R + M_4)\right]^2.$$  \hspace{1cm} (16)

From the retailer’s HJB equation we get

$$R_2 = \frac{r + 2\lambda}{2\delta^2} - \frac{\sqrt{8(1 + \alpha)(1 + 2\alpha)(r + 2\lambda)\mu^2 - 32(1 + \alpha)(1 + 2\alpha)\delta^2 X}}{16\delta^2(1 + \alpha)(1 + 2\alpha)} > 0,$$

where

$$X = \mu(1 + 2\alpha)\phi^2 + 4(\alpha + (1 + \alpha)\psi_s)^2. \hspace{1cm} (17)$$

Under the assumption that $2(1 + \alpha)(1 + 2\alpha)(r + 2\lambda)\mu^2 - \delta^2 X > 0$, $R_2$ is a positive real number. Coefficients $R_3$ and $R_5$ are given by

$$R_3 = -\mu \frac{[(4\alpha(1 + \alpha) + 1 - 2\alpha)\theta + 4\alpha(1 + \alpha)\psi_s] \psi + 4(1 + \alpha)[\alpha + (1 + \alpha)\psi_s] \phi}{8(1 + \alpha)(1 + 2\alpha)(R_2 \delta^2 - (r + \lambda)\mu)},$$

$$R_5 = -\mu \frac{a[(1 + 2\alpha + 4\alpha^2)\theta + 4\alpha(1 + \alpha)\psi_s] + 4(1 + \alpha)b[(1 + \alpha)\psi_s + \alpha\theta]}{8(1 + \alpha)(1 + 2\alpha)(R_3 \delta^2 - (r + \lambda)\mu)}.$$  \hspace{1cm} (18)

The condition for interior solution for $A_R$ is then

$$\sqrt{A_R(G_N, G_R)} = u_R(G_N, G_R) = \frac{\delta}{\mu} \frac{\partial V_R}{\partial G_R}(G_N, G_R) = \frac{\delta}{\mu} (R_2 G_R + R_3 G_N + R_5).$$

To guarantee a positive retailer's advertising investment, we need to verify that the expression $(R_2 G_R + R_3 G_N + R_5)$ is positive.

The retailer’s equilibrium advertising strategy is

$$A_R'(G_N, G_R) = \left(\frac{\delta}{\mu} \frac{\partial V_R}{\partial G_R}(G_N, G_R)\right)^2 = \left[\frac{\delta}{\mu} (R_2 G_R + R_3 G_N + R_5)\right]^2.$$  \hspace{1cm} (19)
Appendix 2. Stability of the steady states for the goodwill stocks

Denote by $M_1^\pm$ and $R_2^\pm$ the following expressions:

$$M_1^\pm = \frac{r + 2\lambda}{2\delta^2} \pm \frac{\sqrt{(2(1 + \alpha)(r + 2\lambda)\mu^2 - (1 + \alpha)\mu^2 \hat{\psi}^2}}{4\delta^2(1 + \alpha)},$$

$$R_2^\pm = \frac{r + 2\lambda}{2\delta^2} \pm \frac{\sqrt{(8(1 + \alpha)(1 + 2\lambda)\mu^2 - 32(1 + \alpha)(1 + 2\lambda)\delta^2X)}}{16\delta^2(1 + \alpha)(1 + 2\alpha)},$$

where constant $X$ is given in (16).

A sufficient condition guaranteeing that the expressions in (13), (14), and

$$A_N(G_N, G_R) = \left[ \frac{\delta}{\mu} (M_1^+ G_N + M_1^+ G_R + M_4^+) \right]^2, \quad A_R(G_N, G_R) = \left[ \frac{\delta}{\mu} (R_2^+ G_R + R_3^+ G_N + R_3^+) \right]^2,$$

are the national and store brands’ value functions and advertising strategies is given by

$$\lim_{t \to -\infty} \exp(-rt)V_N(G_N(t), G_R(t)) = 0, \quad \lim_{t \to -\infty} \exp(-rt)V_R(G_N(t), G_R(t)) = 0,$$

where $(G_N(t), G_R(t))$ is the solution of the closed-loop dynamics obtained after substitution of the advertising strategies in (17) into the goodwill dynamics given by (3) and (4). This solution can be written as

$$G_N(t) = B_1 e^{\xi_1 t} + B_2 e^{\xi_2 t} + G_N^\infty,$$

$$G_R(t) = B_1 \frac{\xi_1 + \lambda - \frac{\hat{\psi}}{\mu} M_1^+}{\delta^2 M_1^+} e^{\xi_1 t} + B_2 \frac{\xi_2 + \lambda - \frac{\hat{\psi}}{\mu} M_1^+}{\delta^2 M_3^+} e^{\xi_2 t} + G_R^\infty,$$

where $B_1, B_2$ are constants, $G_N^\infty, G_R^\infty$ are the steady-state values or long run values of the goodwill stocks, and $\xi_1, \xi_2$ are the real Eigen values associated to the system of linear differential equations given by

$$\xi_1 = \frac{C_1 + \sqrt{C_1^2 - 4C_2}}{2}, \quad \xi_2 = \frac{C_1 - \sqrt{C_1^2 - 4C_2}}{2},$$

where we have assumed $C_1^2 - 4C_2 > 0$ and constants $C_1$ and $C_2$ are given as follows:

$$C_1 = \frac{\delta^2}{\mu} (M_1^+ + R_2^+) - 2\lambda, \quad C_2 = \left( \frac{\delta^2}{\mu} M_1^+ - \lambda \right) \left( \frac{\delta^2}{\mu} R_2^+ - \lambda \right) - \frac{\delta^4}{\mu^2} M_1^+ R_1^+.$$

The steady-state values of the goodwill stocks, denoted by $G_N^\infty$ and $G_R^\infty$ for the national and store brands, are

$$G_N^\infty = \frac{\delta^2 \left[ \delta^2 M_1^+ R_1^+ - M_1^+ (R_2^+ \delta^2 - \lambda \mu) \right]}{(M_1^+ \delta^2 - \lambda \mu) (R_2^+ \delta^2 - \lambda \mu) - M_1^+ R_1^+ \delta^4},$$

$$G_R^\infty = \frac{\delta^2 \left[ \delta^2 M_2^+ R_1^+ - M_2^+ (M_1^+ \delta^2 - \lambda \mu) \right]}{(M_1^+ \delta^2 - \lambda \mu) (R_2^+ \delta^2 - \lambda \mu) - M_1^+ R_1^+ \delta^4}.$$

The initial conditions for the goodwill stocks allow us to determine constants $B_1$ and $B_2$:

$$B_1 = \frac{(G_N^\infty - G_{N0}) M_1^+ \delta^2 + (G_N^\infty - G_{N0}) (M_1^+ \delta^2 - \mu (\lambda + \xi_2))}{\mu (\xi_2 - \xi_1)},$$

$$B_2 = -\frac{(G_R^\infty - G_{R0}) M_1^+ \delta^2 + (G_R^\infty - G_{R0}) (M_1^+ \delta^2 - \mu (\lambda + \xi_1))}{\mu (\xi_2 - \xi_1)}.$$

The fulfillment of conditions in (18) is guaranteed when the goodwill stocks are bounded. Therefore, as usual in infinite horizon problems, we focus on optimal time paths converging to the steady-state values. The following proposition states the conditions for $G_N$, $G_R$ to be bounded for the different cases identified for the coefficients of the optimal advertising strategies.
Proposition 7. The goodwill stocks are bounded if the conditions listed below apply:

(i) $C_2$ is negative and the initial goodwill stocks are related according to the following expression:

$$\left( G_R^\infty - G_{r0} \right) M_t^R \delta^2 + \left( G_N^\infty - G_{n0} \right) (M_t^N \delta^2 - \mu (\lambda + \xi_2)) = 0. \tag{24}$$

(ii) $C_1$ is negative and $C_2$ is positive.

Proof. The eigen values $\xi_1$ and $\xi_2$ of the matrix associated to the closed-loop dynamics are given in (21).

It is straightforward to see that, firstly, if $C_2$ is negative and $C_1$ is positive, then $\xi_1 > 0$, $\xi_2 < 0$; secondly, if both $C_2$ and $C_1$ are negative, then $\xi_1 < 0$, $\xi_2 > 0$; Therefore, in these cases one eigen value is positive and the other one is negative and the steady-state $(G_N^\infty, G_R^\infty)$ is a saddle point. The initial conditions on the goodwill lying on the stable subspace associated to the negative eigen value ($\xi_2$ in the first case and $\xi_1$ in the second one), given by (24), allow the system to converge to the steady state as time approaches infinity.

If $C_1$ is negative and $C_2$ is positive then both eigen values $\xi_1$ and $\xi_2$ are negative. Under this assumption, the steady-state $(G_N^\infty, G_R^\infty)$ is globally asymptotically stable. \qed

Item (i) characterizes the situation where the steady state is stable in the saddle point sense. The saddle point property means that given the initial goodwill level $G_{r0}$, we can find values of $G_{n0}$ (that satisfy Eq. (24)) such that the closed-loop system converges to the steady-state $(G_N^\infty, G_R^\infty)$ as time approaches infinity;

Item (ii) guarantees a globally asymptotically stable equilibrium. In this case, any initial level of the goodwill stocks converges to the steady state as time approaches infinity and leads to a bounded time path.

From the previous proposition we can conclude that conditions in item (ii) characterize the most favorable case to guarantee optimal paths for the goodwill stock converging towards the steady state. Conditions in item (i) are more restricted since the relationship between the initial values of the goodwill stocks established in (24) must be satisfied. Finally, if $C_1$ and $C_2$ are both positive, it is easy to deduce that the steady state is completely unstable and therefore, it is impossible to be attained from any other different position. To eliminate this last possibility, we concentrate on the case $C_1$ negative, which always ensures at least an asymptotically convergent path.

It is straightforward to see that $C_1$ is positive if $M_1^N$ and $R_2^R$ are selected. Therefore, this choice for the reason explained above is removed. The other three choices can lead to bounded optimal paths approaching their steady states as time goes to infinity.

Assume that the manufacturer and the retailer choose as optimal advertising strategies

$$A_N(G_N, G_R) = \left[ \frac{\delta}{\mu} \left( M_1^N G_N + M_3^N G_R + M_4^N \right) \right]^2$$

and

$$A_R(G_N, G_R) = \left[ \frac{\delta}{\mu} \left( R_2^N G_N + R_3^N G_R + R_4^N \right) \right]^2,$$

respectively. This choice implies $\left( \frac{\delta}{\mu} M_1^N - \lambda \right)$ is positive and, therefore, to guarantee a negative value of $C_1$, which ensures the existence of optimal paths converging to the steady state, $\left( \frac{\delta}{\mu} R_2^N - \lambda \right)$ should be negative and lower than $\left( \lambda - \frac{\delta}{\mu} M_1^N \right)$.

The same reasoning applies when the optimal advertising strategies are

$$A_N(G_N, G_R) = \left[ \frac{\delta}{\mu} \left( M_1^N G_N + M_3^N G_R + M_4^N \right) \right]^2$$

and

$$A_R(G_N, G_R) = \left[ \frac{\delta}{\mu} \left( R_2^N G_N + R_3^N G_R + R_4^N \right) \right]^2,$$

for the national brands and for the retailer, respectively. In this case, $\left( \frac{\delta}{\mu} R_2^N - \lambda \right)$ is positive and $\left( \frac{\delta}{\mu} M_1^N - \lambda \right)$ has to be negative and lower than $\left( \lambda - \frac{\delta}{\mu} R_2^N \right)$.

Finally, consider that the optimal advertising strategies are given by

$$A_N(G_N, G_R) = \left[ \frac{\delta}{\mu} \left( M_1^N G_N + M_3^N G_R + M_4^N \right) \right]^2$$
and
\[ A_R(G_N, G_R) = \left( \frac{\delta}{\mu} (R_G^2 + R_G^3 + R_N^3) \right)^2, \]
respectively. In this case, the signs of \( \frac{\delta}{\mu} M_1 - \lambda \) and \( \frac{\delta}{\mu} R_N^3 - \lambda \) are unknown and the only restriction is that the sum of these two terms has to be negative to ensure a negative value of \( C_1 \).

Therefore, the first two possibilities imply stronger conditions than the third one. For that reason, from now on we focus on this last possibility. The next proposition shows the different scenarios associated with this choice.

**Proposition 8.** Assume that \( C_1 < 0 \) and that the national brand's and the retailer's optimal advertising strategies are given by
\[ A_N(G_N, G_R) = \left( \frac{\delta}{\mu} (M_1^2 G_N + M_3^2 G_R + M_4) \right)^2, \] \[ A_R(G_N, G_R) = \left( \frac{\delta}{\mu} (R_G^2 + R_G^3 + R_N^3) \right)^2. \]

Then,
1. \( M_1^2 \) and \( M_4^2 \) have the same sign as \( \theta \).
2. \( R_G^2 \) has the same sign as \( \psi(4x^2 + 2x + 1) + 4x(x + 1)p_k + 4p_k(x + 1)p_k \).
3. \( R_N^3 \) has the same sign as \( a(4x^2 + 2x + 1) + 4x(x + 1)p_k \).

**Proof.** Under the hypothesis that the channel members choose the optimal advertising strategies in the statement, it is straightforward to prove that \( M_1^2 \delta^2 - (r + \lambda)\mu < 0 \) and \( R_G^2 \delta^2 - (r + \lambda)\mu < 0 \). From the expressions of the other coefficients, \( M_1^2, M_4^2, R_G^2, R_N^3 \), the different behaviors about the signs can be easily derived. □

**Remark 1.** Let us note that to have positive steady-state values of the goodwill stocks given by (22) and (23), the sign of the following expressions: \( (\delta^2 M_1^2 R_G^2 + M_4^2 (R_G^2 \delta^2 - \lambda)\mu) \) and \( (\delta^2 M_4^2 R_N^3 - R_N^3 (M_1^2 \delta^2 - \lambda)\mu) \), must equal the sign of \( C_2 \).

**Appendix 3. Sensitivity analysis of the advertising strategies to the goodwill stocks**

The sensitivity of \( A_N \) and \( A_R \) to \( G_N \) and \( G_R \) is given by
\[ \frac{\partial A_N}{\partial G_N}(G_N, G_R) = \frac{2\delta^2}{\mu^2} M_1 (M_1 G_N + M_3 G_R + M_4), \] \[ \frac{\partial A_N}{\partial G_R}(G_N, G_R) = \frac{2\delta^2}{\mu^2} M_1 (M_1 G_N + M_3 G_R + M_4), \] \[ \frac{\partial A_R}{\partial G_R}(G_N, G_R) = \frac{2\delta^2}{\mu^2} R_G^2 (R_G^2 + R_G^3 + R_N^3), \] \[ \frac{\partial A_R}{\partial G_N}(G_N, G_R) = \frac{2\delta^2}{\mu^2} R_G^2 (R_G^2 + R_G^3 + R_N^3). \]

From Appendix 1, \( M_1 > 0, R_G > 0, (M_1 G_N + M_3 G_R + M_4) > 0 \) and \( (R_G^2 + R_G^3 + R_N^3) > 0 \), hence \( \frac{\partial A_N}{\partial G_N}(G_N, G_R) > 0 \) and \( \frac{\partial A_R}{\partial G_R}(G_N, G_R) > 0 \). Moreover, \( \frac{\partial A_N}{\partial G_R}(G_N, G_R) \) and \( \frac{\partial A_R}{\partial G_N}(G_N, G_R) \) have the same sign as \( M_3 \) and \( R_3 \), respectively.

**References**


