Linear fractional LPV model identification from local experiments: an $H_\infty$-based optimization technique

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Abstract—In this paper, a new identification technique is introduced to estimate a linear fractional representation of a linear parameter-varying (LPV) system from local experiments by using a dedicated non-smooth optimization procedure. More precisely, the developed approach consists in estimating the parameters of an LPV state-space model from local fully-parameterized identified state-space models through the non-smooth optimization of a specific $H_\infty$-based criterion. The method presented in this paper results directly in an LPV model whose parametric matrices can be rational functions of the scheduling variables without any interpolation step (required usually by the local approach) and without writing the local fully-parameterized LTI state-space models with respect to a coherent basis. A numerical example is used to illustrate the performance of the suggested technique.

I. INTRODUCTION

Although many identification techniques and solutions have been developed since the first attempts in 1995 [1] (see, e.g., [2], [3] and the references therein for a recent overview), identifying a linear parameter-varying (LPV) model of a system by using a local approach is still a problem which deserves attention. By local approach, it is meant that a multi-step procedure is undertaken where (1) local experiments are carried out in several operating points represented by fixed values of the scheduling signals while the inputs are excited, (2) local LTI models are estimated based on the locally gathered input-output data, (3) an LPV model is derived by interpolating of these locally-estimated models. Such an approach is considered among others in [4], [5], [6]. Notice that this approach differs from the global one which resorts to the simultaneous excitation of both the scheduling variables and the inputs of the system by performing one global experiment in order to capture the non-linear dynamics of the real system [7], [8], [9]. In this paper we consider the local approach because, in practice, mainly for safety and/or financial reasons, it is often difficult to perform one single experiment with a “rich” excitation of the control inputs and the scheduling variables simultaneously. On the contrary, applying small variations around particular operating points, as considered by the local approach, is more conceivable in many practical cases. Despite its practical simplicity, as pointed out first in [10], the interpolation step involved in the local approach can lead to a global LPV model with an inaccurate dynamic behavior even if the local LTI models are consistent. Thus, a number of issues deserve attention as far as the relation between the local models is concerned, as shown, e.g., in [2]. The main difficulty with the local approach for LPV state-space models identification is the determination of a suitable coherent basis for all of the local estimated models so that the interpolated LPV state-space representation is able to capture the dynamics of the process to identify. Many attempts to solve this problem are suggested in the literature. In the black-box framework, i.e., when no prior information about the system to identify is available, interesting solutions have been developed, e.g., in [11], [12], [13], [14] during the last decade. Unfortunately, as explained recently in [14], [15], whatever the black-box technique used to estimate an LPV model from local experiments, getting an LPV model able to picture the dynamic dependence on the scheduling variables is not an easy task even if the LPV system to identify satisfies a static dependence on the scheduling signal. In order to bypass this difficulty, optimization-based LPV model approximation solutions have been recently suggested in [16], [17], [18], [19], [15]. Notice however that (i) the solutions suggested in [17] are based on the $H_2$-norm and focus on system matrices which are linear combination of some basis functions of $p$, (ii) the $H_\infty$-based techniques available in [18], [19] are developed for LTI, linear time-periodic and/or affine LPV models the latter identified either via a global or a local approach, (iii) the technique used in [15] still requires an interpolation step.

In this paper, a solution, inspired by the developments available in [16], [19], is introduced to circumvent the aforementioned challenging problems. More precisely, this solution consists in estimating the unknown parameters of LPV models directly in the so-called linear fractional representation (LFR) [20] from fully-
parameterized (estimated) local LTI state-space models by using a dedicated $H_\infty$-based optimization algorithm. As shown hereafter, such a technique avoids the interpolation step usually required by the local approach by matching (in terms of the $H_\infty$-norm) a concatenation of the frequency responses of the local LTI models with the input-output response of a linear fractional LPV representation of the system, the structure of which is chosen by the user according to the available prior information. By construction, this procedure also bypasses the challenging problem of realizing all the local models with respect to the same state variables. Notice also that, by comparison with the solutions developed in [16], [19], the method suggested hereafter (i) can deal with black-box and gray-box (i.e. physically-parameterized) linear fractional LPV models equivalently, (ii) promotes the use of the $H_\infty$-norm (instead of the the $H_2$-norm used in [16]) because this norm allows the user to specify frequency ranges of interest (see Eq. (13)) and, as shown in [21], the $H_\infty$-norm framework provides a good compromise between identification for control and identification for simulation.

The paper is organized as follows. The problem formulation and important definitions are given in Section II. The $H_\infty$-based solution as well as the non-smooth technique used to compute the LPV model parameters are introduced in Section III. Section IV, dedicated to a simulation example, demonstrates the effectiveness of the method described in this article. Conclusions and possible future research directions end this article.

II. PROBLEM FORMULATION

A. Model description

Consider a continuous-time LPV system with the following state-space representation

$$\begin{align*}
\dot{x}(t) &= A(p(t), \Theta)x(t) + B(p(t), \Theta)u(t) \\
y(t) &= C(p(t), \Theta)x(t) + D(p(t), \Theta)u(t)
\end{align*}$$

(1)

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the input vector, $y(t) \in \mathbb{R}^{n_y}$ is the output vector and the matrices $(A, B, C, D)$ are functions of a measurable time-varying signal $p(t) \in \mathbb{R}^{n_p}$ used hereafter as the scheduling variable vector. As is apparent from Eq. (1), the system matrices have the so-called static dependence on the scheduling variable vector [2]. The vector $\Theta \in \mathbb{R}^N$ contains the unknown parameters to identify.

In this paper, a specific attention is paid to state-space LPV models satisfying a rational dependence of the system matrices with respect to $p(t)$. Such a parameter dependence is indeed more general than the affine dependence usually encountered in system identification and is often used in the robust control literature [20]. More precisely, we focus on LPV models which can be written into the so-called linear fractional representation (LFR) [20]. The block-diagram of the linear fractional transformation (LFT) description can be seen in Fig. 1. The $\Delta$ matrix is a $p(t)$-dependent matrix defined as

$$\Delta(p(t)) = \text{diag}(p_1(t)I_{r_1}, \ldots, p_{n_p}(t)I_{r_{n_p}}).$$

(2)

The signals $w(t) \in \mathbb{R}^r$ and $z(t) \in \mathbb{R}^r$ are inner auxiliary signals with $r = \sum_{k=1}^{n_p} r_k$ linked by the equation $w(t) = \Delta(p)z(t)$. $M$ is an LTI system with the following state-space representation

$$\begin{bmatrix}
\dot{x}(t) \\
z(t) \\
y(t)
\end{bmatrix} =
\begin{bmatrix}
A_0(\Theta) & B_w(\Theta) & B_0(\Theta) \\
C_z(\Theta) & D_{zw}(\Theta) & D_{zu}(\Theta) \\
C_0(\Theta) & D_{yw}(\Theta) & D_{yu}(\Theta)
\end{bmatrix}
\begin{bmatrix}
x(t) \\
w(t)
\end{bmatrix}$$

(3)

where the matrices $(A_0(\Theta), B_w(\Theta), \ldots, D_{yu}(\Theta))$ are time-invariant matrices with appropriate dimensions.

$$\begin{array}{c}
w \\
\Delta \\
z \\
M \\
y
\end{array}$$

Fig. 1. LFT description of the system to identify as an LTI dynamic system with a partial feedback connection $\Delta(p(t))$.

From this LFR, the direct state-space representation (1) can be obtained with the following state matrices

$$\begin{align*}
A(p) &= A_0 + B_w \Delta(p)(I - D_{zw}\Delta(p))^{-1}C_z \\
B(p) &= B_0 + B_w \Delta(p)(I - D_{zw}\Delta(p))^{-1}D_{zu} \\
C(p) &= C_0 + D_{yw}\Delta(p)(I - D_{zw}\Delta(p))^{-1}C_z \\
D(p) &= D_0 + D_{yw}\Delta(p)(I - D_{zw}\Delta(p))^{-1}D_{yu}
\end{align*}$$

(4)

provided that the inverse $(I - D_{zw}\Delta(p))^{-1}$ exists [20, Chapter 10] for all the possible trajectories of $p(t)$. Notice that the elements of the system matrices in Eq. (4) (i) are rational functions of the scheduling variable vector when $D_{zw} \neq 0$, (ii) are affine functions when $D_{zw} = 0$. The LFR of the LPV model can also be compactly written as

$$\mathcal{G}(s, \Delta(p), \Theta) = \mathcal{F}(M(s, \Theta), \Delta(p))$$

(5)

where $M(s, \Theta)$ represents the LTI system and $\mathcal{F}$ stands for the upper linear fraction transformation as presented in Fig. 1 (see [20, Chapter 10] for details).

B. Identification problem and model structure determination

The identification of the LPV model defined by Eq. (2)-(3) (see Fig. 1) consists in (i) determining the structure of the $\Delta$ matrix in (2) and (ii) identifying the matrices composing the LTI model $M(s, \Theta)$. Such an LF structure can be obtained in different manners according to the prior knowledge available on the system behavior.

The first case corresponds to the black-box models where $M(s, \Theta) = M_1(s, \Theta_1)$ is a fully-parameterized state-space representation. This framework is shared in [16] where only the input-output behavior needs to be
captured by the identified model. In this case, the LFR can be defined as follows
\[ G_1(s, \Delta(p), \Theta_1) = F(M_1(s, \Theta_1), \Delta(p)). \] (6)

The second case \( M(s, \Theta) = M_2(s, \Theta_2) \) corresponds to gray-box models\(^1\), \textit{i.e.} when some prior information can be used. For instance, a structure of the state-space model is often determined from the laws of Physics, resulting in some fixed entries of the state-space representation (basically, some entries are equal to 0 and 1). Only the unknown parameters of the state-space representation, stacked in \( \Theta_2 \), then need to be estimated and the LFR is given by
\[ G_2(s, \Delta(p), \Theta_2) = F(M_2(s, \Theta_2), \Delta(p)). \] (7)

From the previous case, one refinement can be derived when the modeling allows us to write the parameter vector \( \Theta_2 \) as functions of the physical parameters. Denoting \( \theta \) the vector of the physical parameters, then the notation \( \Theta_2 = \Theta_2(\theta) \) is introduced to emphasize this dependence. The LPV model of the system is then determined under the form \( M_2(s, \Theta_2(\theta)) = F(M_2(s), \Delta(\theta)) \) where \( \Delta(\theta) \) is a matrix that contains the physical parameters with possibly several occurrences [22]. Note that \( M_3(s) \) is an LTI system containing known parameters. Then, the LPV model can be written as a new LFR where the \( \Delta \) matrix includes both the scheduling variables and the parameters to be estimated, \textit{i.e.}
\[ G_3(s, \Delta(p), \Theta_2(\theta)) = F(M_4(s), \text{blockdiag}(\Delta(p), \Delta(\theta))) \] (8)
where \( M_4(s) \) is an LTI model containing known parameters.

The different model structures being explained and fixed according to the prior knowledge about the equations governing the behavior of the system, the method dedicated to the estimation of the parameter vectors \( \Theta_1 \), \( \Theta_2 \) or \( \theta \) from local LTI models can be now described.

III. Linear fractional LPV model identification by using an \( H_\infty \) optimization algorithm

As a method belonging to the local techniques [2], the first step of the method developed in this paper consists in estimating local LTI models for a set of operating points represented by the elements of the scheduling variable vector \( p_i \), \( i \in \{1, \cdots, N_{op}\} \). These local black-box LTI models can be obtained by using standard techniques which can be found, \textit{e.g.}, in [23]. By denoting these \( i \) local black-box LTI state-space models with the quadruplet \( (A_i, B_i, C_i, D_i) \), \( i \in \{1, \cdots, N_{op}\} \), we can easily define \( i \) local LTI transfer functions as
\[ G_i(s) = C_i(sI - A_i)^{-1}B_i \] (9)
\[ \Theta) = \text{blockdiag}(\Delta(p_i), \Delta(\theta)) \] (10)
\[ \Theta) = \text{blockdiag}(\Delta(p_i), \Delta(\theta)) \] (11)

Different cost functions can be used to match the LPV model with the available local LTI realizations. For instance
\[ \min_{\Theta} \max_{i \in \{1, \cdots, N_{op}\}} \| G_i(s) - G(s, \Delta(p_i), \Theta) \|_\infty, \] (12)
Because the \( H_\infty \)-norm is the maximal singular values of the complex gain matrix over all the frequencies, it holds by definition that, if the maximal value of the \( H_\infty \)-norm found in Eq. (12) is small enough \( (\text{e.g., much smaller than 1}) \), then the maximum amplification of the difference between the involved frequency responses is small, so the distance between the considered systems is small as well. Said differently, if the maximal value of the \( H_\infty \)-norm found in Eq. (12) is small enough, the LPV model is a decent approximation of all the local LTI state-space models for all the considered values of the scheduling variables. Moreover, it should provide a good approximation of the system for any trajectory of the scheduling variables.

Let us now show how the identification problem can be recast into a structured \( H_\infty \) synthesis problem. Denoting
\[ G_i(s, \Theta) = W_0(s)(\Theta_i(s) - G(s, \Delta(p_i), \Theta))W_1(s) \] (13)
where \( W_0(s), W_1(s) \) are weighting functions that enables the identification to be more accurate over some frequency range, let us introduce the block-diagonal term \( \tilde{G}(s, \overline{\theta}, \Theta) \) having the form
\[ \tilde{G}(s, \overline{\theta}, \Theta) = \text{blockdiag}(G_1(s, \Theta), \cdots, G_{N_{op}}(s, \Theta)) \] (14)
where \( \overline{\theta} = [p_1, \cdots, p_{N_{op}}]^T \) is a vector containing the scheduling variables. As a result of the block diagonal structure, it can be written that
\[ \| \tilde{G}(s, \overline{\theta}, \Theta) \|_\infty = \max_i \| G_i(s, \Theta) \|_\infty. \] (15)

\(^1\)This framework, as well as the third one, is not considered in the aforementioned references. This is one of the main contributions of this paper.
Hence, the cost function given by Eq. (12) can be recast as
\[
\min_{\Theta} \| \tilde{G}(s, \bar{p}, \Theta) \|_{\infty}. \quad (16)
\]
In order to minimize the $H_{\infty}$-norm of $\tilde{G}(s, \bar{p}, \Theta)$, it is necessary to compute the value of this function efficiently. Clearly,
\[
\| \tilde{G}(j\omega, \bar{p}, \Theta) \|_{2}^{2} = \sup_{\omega \in [0, \infty]} \lambda_1(\tilde{G}^{H}(j\omega, \bar{p}, \Theta)\tilde{G}(j\omega, \bar{p}, \Theta)).
\]
Thus, solving Eq. (16) is equivalent to the following optimization problem
\[
\min_{\Theta} \sup_{\omega \in [0, \infty]} \lambda_1(\tilde{G}^{H}(j\omega, \bar{p}, \Theta)\tilde{G}(j\omega, \bar{p}, \Theta)) \quad (17)
\]
where $\lambda_1(\cdot)$ is the maximum eigenvalue function which is a convex but a non-smooth function [24]. From Eq. (17), the computation of the objective function seems to be a hard task because, by construction, the global optimum of a non-smooth and non-convex function over $[0, \infty]$ must be estimated. Fortunately, when $\tilde{G}$ has a state-space representation (which is the case herein), this step can be performed efficiently by using a bisection algorithm based on a hamiltonian calculus [25].

In the present work, the hinstruct function has been used. Developed by Apkarian et al., this tool is available in the Robust Control Toolbox of MATLAB. A description of the optimization algorithm is available in [26]. Briefly, a descent direction is generated from a local model, then a line-search technique is used in order to decrease the value of the cost function. The local model is built mainly from the set of active frequencies [26], and gives first order information about the local geometry of the cost function.

Let us summarize the main advantages of the suggested technique: (i) the linear fractional representation used in this paper is more general than the LPV-affine representation [18] (ii) and can then be directly estimated avoiding the standard interpolation step. Furthermore, (iii) black-box local LTI models do neither have to be written in coherent basis (iv) nor to have the same orders. Last but not least, (v) physical parameters can be estimated by taking the advantage of the available prior information. The generality of this method presented in this paper is the main interest of the proposed method.

IV. SIMULATION EXAMPLE

A. System description

\[
\begin{align*}
F & \quad m_1 \quad \tilde{\delta} \quad m_2 \\
& \quad k \quad f_1 \quad f_2
\end{align*}
\]

Fig. 2. System used for demonstration.

The considered application example is a two-mass system in translation that can be seen in Fig. 2. The first vehicle of mass $m_1$ and linear speed $v_1$ is actuated with a force $F$. It suffers from a dissipative force $-f_1 v_1$ where $f_1$ is the friction ratio. The second vehicle of mass $m_2$ and linear speed $v_2$ suffers from a dissipative force $-f_2 v_2$. The two vehicles are linked with a flexible transmission of strength $k$ and friction ratio $f$. Denoting $\delta$ the distance between the two vehicles, the force produced on the first vehicle is $-k \delta - f(v_1 - v_2)$ and the opposite on the second vehicle. The driving vehicle is equipped with a linear speed controller: $F = K (u - v_1)$ where $u$ is the speed reference and $K$ is the control gain. The first vehicle speed is considered as the output ($y = v_1$). The second vehicle is loaded with a mass $m_c$ that is considered as variable. By considering $x = [m_1 v_1 \ (m_2 + m_c) v_2 \ \delta]$ as the state vector, an LPV model structure similar to the one described in Eq. (1) can be obtained with $p(t) = m_c(t)$ and where
\[
\begin{bmatrix}
A(p) & B(p) \\
C(p) & D(p)
\end{bmatrix} = \begin{bmatrix}
\frac{-f - f_1}{m_1} & \frac{f}{m_1} & -k & K \\
\frac{1}{m_1} & \frac{1}{m_1} & 0 & 0 \\
\end{bmatrix}
\]

Then, the LFR given in Eq. (3) is obtained with $\Delta(p) = p = m_c$ and
\[
\begin{bmatrix}
A_0 & B_w & B_0 \\
C_z & D_z & D_z u \\
C_0 & D_y & D_y u
\end{bmatrix} = \begin{bmatrix}
\frac{-f - f_1}{m_1} & \frac{f}{m_2} & -k & m_c \\
\frac{1}{m_1} & \frac{1}{m_2} & 0 & 0 \\
\end{bmatrix}
\]

B. Performing the optimization

The first step of the method used in this paper consists in selecting $N_{op} = 17$ local operating points, then determining $N_{op}$ local fully-parameterized LTI state-space models $\Theta_i(s) = (\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i, \mathcal{D}_i)$, $i \in \{1, \ldots, N_{op}\}$, by using the PI-MOESP subspace based identification method [23]. In this example, the evolution of $p$ is $[0 : 0.05 : 0.8]$ with a 0.05 step between the operating points. These initial fully-parameterized LTI models are easily validated by comparing the frequency responses of these estimated models see Fig. 3 and the analytic local models given in Fig. 4.

In order to illustrate the effectiveness of the proposed method, two cases are presented: a fully-parameterized and a structured LFR identification. First, a black-box approach is considered. In this case $\Theta_i \in \mathbb{R}^{25}$, 25 different and uniformly distributed random values generated in the range $[0,1]$ are used as initial values of the 25 unknown matrix entries. The final value of the cost function given by Eq. (11) is $4.53c - 06$. Second, the prior information about the system under study is taken into account in order to determine the physical parameters, namely, $[m_1 \ m_2 \ k_r \ K \ f_1 \ f_2] (\theta \in \mathbb{R}^{7})$. By assuming
that we have access to prior information about the unknown parameters such as a range of evolution, the initialization is performed herein randomly and independently for each parameter $\theta_i$, $i \in \{1, \cdots, 7\}$, by using the following form

$$\theta_{\text{init}} = \theta_{\text{real}}(1 + 4(r_i - 0.5))$$  \hspace{1cm} (18)

where $i = \{1, \cdots, 7\}$, $\theta_{\text{real}}$ is the real value of the sought parameter and $r_i$ is a random number uniformly distributed on the open interval [0, 1]. At the end of the optimization, the final value of the cost-function is 1.3702e – 04. This figure proves the efficiency of the developed technique quantitatively. Notice here that two identical band pass filters are used during optimization as weighting functions defined by Eq. (13). The block-matrix $\hat{M}_2(s, \Theta_2(\theta))$ has the following numerical form

$$
\begin{bmatrix}
-21.46 & 0.611 & -9.979 & 0 & 3.995 & 0 & 0 \\
0.630 & -0.666 & 9.979 & 0 & 0.611 & 0 & 0 \\
5.147 & -4.996 & 0 & 4.996 & 0 & 0 & 0 \\
0 & 4.996 & 0 & -4.996 & 0 & 0 & 0 \\
5.147 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Fig. 3. Bode diagram of the black box fully-parameterized local LTI state-space models.

Fig. 4. Bode diagram of the analytical models w.r.t. $p$.

In Table I, the physical parameters and the corresponding estimated values can be seen. The physical parameters are well estimated except for the frictions.

### Table I

<table>
<thead>
<tr>
<th>Physical param.</th>
<th>Real Value</th>
<th>Est. value</th>
<th>Error [%]</th>
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<tr>
<td>$m_1$ (kg)</td>
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<td>$m_2$ (kg)</td>
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<td>$k_c$ (N/m)</td>
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**V. CONCLUSIONS**

In this paper, through the study of LPV state-space representations, some problems and solutions to the standard local approach have been illustrated. More specifically, by starting from reliable locally-estimated fully-parameterized state-space forms, a new algorithm based on a specific $H_{\infty}$ model matching cost function has been developed to give access to a consistent final linear fractional LPV state-space model without requiring any interpolation step. The efficiency of the aforementioned technique has been illustrated via a specific numerical example built so that black-box and gray-box LPV state-space model identification can be tested. Such a method can be considered as a promising solution to the challenging problem of coordinate bases coherence encountered by most of the local techniques.
Fig. 5. Comparison of the outputs of the analytical model and fully-parameterized LPV/LFR model.

Fig. 6. Comparison of the outputs of the analytical model and the structured LPV/LFR model.

REFERENCES


