Approximations of differentiable convex functions on arbitrary convex polytopes

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Abstract

Let $X_n := \{x_i\}_{i=0}^n$ be a given set of $(n + 1)$ pairwise distinct points in $\mathbb{R}^d$ (called nodes or sample points), let $P = \text{conv}(X_n)$, let $f$ be a convex function with Lipschitz continuous gradient on $P$ and $\lambda := \{\lambda_i\}_{i=0}^n$ be a set of barycentric coordinates with respect to the point set $X_n$. We analyze the error estimate between $f$ and its barycentric approximation:

$$B_n[f](x) = \sum_{i=0}^{n} \lambda_i(x)f(x_i), \quad (x \in P),$$

and present the best possible pointwise error estimates of $f$. Additionally, we describe the optimal barycentric coordinates that provide the best operator $B_n$ for approximating $f$ by $B_n[f]$. We show that the set of (linear finite element) barycentric coordinates generated by the Delaunay triangulation gives access to efficient algorithms for computing optimal approximations. Finally, numerical examples are used to show the success of the method.

Keywords: Barycentric approximation – Barycentric coordinates – Convex functions – Function approximation – Delaunay triangulation – Upper approximation operator – Special functions.

1. Introduction, motivation and theoretical justification

We begin by considering the one-dimensional case since its simplicity allows us to analyse all the necessary steps through very simple computation. In the univariate approximation, say on an interval $[a, b]$, a simple way of approximating a given real function $f : [a, b] \to \mathbb{R}$ is to choose a partition
\( P := \{x_0, x_1, \ldots, x_n\} \) of the interval \([a, b]\), such that \( a = x_0 < x_1 < \ldots < x_n = b \), and then to fit to \( f \) using a spline \( S_n \) of degree 1 at these points in such a way that:

1. The domain of \( S_n \) is the interval \([a, b]\);
2. \( S_n \) is a linear polynomial on each subinterval \([x_i, x_{i+1}]\);
3. \( S_n \) is continuous on \([a, b]\) and \( S \) interpolates the data, that is, \( S_n(x_i) = f(x_i), i = 0, \ldots, n \).

This is a convenient class of interpolants because every such interpolant can be written in a barycentric form

\[
S_n(x) = \sum_{i=0}^{n} \lambda_i(x) f(x_i), \quad (x \in [a, b]),
\]

where

\[
\lambda_i(x) = \begin{cases} 
\frac{x - x_{i-1}}{x_i - x_{i-1}}, & \text{if } x_{i-1} \leq x \leq x_i; \\
\frac{x_{i+1} - x}{x_{i+1} - x_i}, & \text{if } x_i \leq x \leq x_{i+1}; \\
0, & \text{for all other } x.
\end{cases}
\]

Here, by a little abuse of notation, we set \( x_{-1} := a \) and \( x_{n+1} := b \). One of the main features of the usual linear spline approximation, in its simplest form (1), is that \( \{\lambda_i\}_{i=0}^{n} \) form a (unique) set of (continuous) barycentric coordinates. This means that they satisfy, for all \( x \in [a, b] \), three important properties:

\[
\lambda_i(x) \geq 0, \quad i = 0, \ldots, n;
\]
\[
\sum_{i=0}^{n} \lambda_i(x) = 1;
\]
\[
\sum_{i=0}^{n} \lambda_i(x) x_i = x.
\]

This simple approach can be generalized to general polytopes. Throughout, we will assume all of the polytopes we work with are convex. Indeed, consider a given finite set of pairwise distinct points \( X_n = \{x_i\}_{i=0}^{n} \) in \( P \subset \mathbb{R}^d \), with \( P = \text{conv}(X_n) \) denoting the convex hull of the point set \( X_n \). We are interested in approximating an unknown scalar-valued continuous convex function \( f : P \to \mathbb{R} \) from given function values \( f(x_0), \ldots, f(x_n) \) sampled at \( X_n \). In order to obtain a simple and stable global approximation of \( f \) on \( P \), we may consider a weighted average of the function values at data points of the following form:

\[
B_n[f](x) = \sum_{i=0}^{n} \lambda_i(x) f(x_i),
\]
or, equivalently, a convex combination of the data values \( f(x_0), \ldots, f(x_n) \).

This means that we require that the system of functions \( \lambda := \{\lambda_i\}_{i=0}^n \) forms a partition of unity, that is, for all \( x \in P \), we have

\[
\lambda_i(x) \geq 0, \quad i = 0, \ldots, n, \tag{3}
\]

\[
\sum_{i=0}^{n} \lambda_i(x) = 1. \tag{4}
\]

In addition, we shall also require the set of functions \( \lambda \) to satisfy the first-order consistency condition:

\[
x = \sum_{i=0}^{n} \lambda_i(x)x_i, \quad (\forall x \in P). \tag{5}
\]

We will call any set of functions \( \lambda_i : P \to \mathbb{R}, \ i = 0, \ldots, n \), barycentric coordinates if they satisfy the three properties (3), (4) and (5) for all \( x \in P \). In view of these properties, we shall refer to the approximation schemes \( B \) as barycentric approximation (schemes). Barycentric coordinates also exist for more general types of polytopes. The first result on their existence was due to Kalman \cite[Theorem 2]{kalman} (1961). It should be mentioned that one of the main difficulties in obtaining all barycentric approximations of functions, in dimensions higher than one, lies in the fact that their construction still remains a very difficult task in the general case. However, it should be emphasized, that as in the univariate case, one possible natural approach for constructing an interesting class of particular barycentric coordinates would be to simply construct a triangulation of the polytope \( P \) - the convex hull of the data set \( X_n \) - into simplices such that the vertices \( \mathbf{v}_i \) of the triangulation coincide with \( x_i \). After that, one can use the standard barycentric coordinates for these simplices. As a result, each triangulation of the data set \( X_n \) generates a set of barycentric coordinates. Hence, there exists at least one barycentric approximation of type (2) which is generated by a triangulation. Let us outline shortly how triangulations and barycentric approximations are connected. It is known that every convex polytope can be triangulated into simplices, and the triangulation of a polytope may not be unique. To better illustrate this phenomenon, let us consider the simple example of a two-dimensional square \( S \). Then two different triangulations are possible for \( S \). Now every convex combination of the two associated coordinates provides a set of barycentric coordinates. This allows us to generate new families of
barycentric approximations which are not generated by a triangulation. We refer to reference [4] for details.

A difficulty in minimizing the error estimate using the barycentric approximations arises from the possible existence of many barycentric coordinates. This yields the problem of selecting the barycentric coordinates as to minimize the approximation error. It will be interesting to have a way of selecting favourable ones among all barycentric approximations associated with the data set $X_n$.

Convex functions appear naturally in many disciplines of science such as physics, biology, medicine and economics, and they constitute an important part of mathematics. A natural question is: can these functions be well approximated by simpler functions and how?

While there are several papers investigating various methods to approximate arbitrary functions, very little research has been done subject to the usual convexity. For instance, if some smoothness is allowed for the function $f$ which is to be approximated, say $C^2(P)$, this will play a crucial role in the determination of the "best" (or "optimal") cubature formulas, see [6, 10, 11, 9, 13, 12, 4]. This article builds on the previous work [4], where a theoretical framework for approximating $C^2(P)$—convex functions was developed.

An important part of this paper is finding a barycentric approximation $B_n[f]$ of the form (2), which approximates $f$ well at the points $x \in P$, distinct from the data, given that $f$ is a convex function with a Lipschitz continuous gradient. Error bounds and quality measures are provided, which estimate the influence of the barycentric coordinates on accuracy of the approximants $B_n$.

When defining the set of barycentric approximants, there are two main issues to be considered. These issues are very natural and also necessary for an approximation of a given convex function $f$ defined on an arbitrary convex polytope:

1. Since a barycentric approximation is not unique in general, it is of great interest to have a general method of constructing possible barycentric
coordinates, in hope of finding the "best" barycentric approximation for a given convex function.

2. The resultant approximant, generated by this method, should not be "complicated" to implement numerically.

Our contribution in this paper consists mainly of the following aspects. Firstly, under the assumption of convexity and the standard Lipschitz continuity of the gradient, we prove some results that pertain to sharp estimates of the error arising from such approximations. The most important property of barycentric approximations is that they fit into the framework of operators, since they approximate any convex function from above. Indeed, let \( f : P \to \mathbb{R} \) be a convex function. Then, for all \( x \in P \), the Jensen’s inequality implies

\[
  f(x) \leq B_n[f](x).
\]

Hence, secondly, our results also provide new upper bounds for the Jensen’s inequality on an arbitrary convex polytope.

The remainder of this paper is organized as follows. In Section 2, we first fix notation and provide a simple and elegant characterization of upper approximation operators. In Section 3, we show that it is possible to apply the error estimate given in Section 2 (Theorem 2.3) to a function \( f \), differentiable on \( P \) and such that its gradient \( \nabla f \) is Hölder continuous. Section 4 is dedicated to finding an optimal barycentric approximation (among all possible barycentric approximations - both those which are generated by a triangulation and those which are not). Indeed, we show that for a given set of sample points \( X_n \) in \( \mathbb{R}^d \), every set of barycentric coordinates associated to \( X_n \), which is generated by a Delaunay triangulation, is optimal. This means that those which are generated by a Delaunay triangulation provide the smallest error to the barycentric approximation of the quadratic function \( \| . \|^2 \) the square of the Euclidean norm. This will lead to a good approximation in most cases. In Section 5, we present two numerical examples which illustrate the proposed methodology.

2. A fundamental error estimate

In this section we show a simple and elegant characterization of upper approximation operators. To this end, we first list the necessary notation and clarify some terminology used throughout the rest of the paper.
For any $x, y \in \mathbb{R}^d$, we will denote the standard inner product of $x$ and $y$ by $\langle x, y \rangle$ and the Euclidean vector norm of $x \in \mathbb{R}^d$ by $\|x\| := \sqrt{\langle x, x \rangle}$.

We say that $f$ is \textit{continuously differentiable} on $P$ if it is continuously differentiable on an open set containing $P$. A function is \textit{Lipschitz continuous} (with Lipschitz constant) $L_f$ if, for all $x, y$ in $P$, there holds

$$
\|f(y) - f(x)\| \leq L_f \|y - x\|,
$$

with the constant $L_f$ which cannot be replaced by a smaller one. By $C^{1,1}(P)$ we denote the subclass of all functions $f$ which are continuously differentiable on $P$ with Lipschitz continuous gradients.

The Lipschitz continuity of $\nabla f$ will play a crucial role in our analysis. Hence, we state here a very important result derived from this property.

\textbf{Lemma 2.1.} Let $f \in C^{1,1}(P)$ with $L_f > 0$ and, in addition, let $f$ be convex. Then $\nabla f$ satisfies the following property:

$$
\frac{1}{L_f} \|\nabla f(y) - \nabla f(x)\|^2 \leq \langle \nabla f(y) - \nabla f(x), y - x \rangle, \forall x, y \in P.
$$

\textbf{Proof.} This proof is classical and it can be given in only a few lines. Indeed, by the mean value theorem, we have, for all $x, y \in P$,

$$
f(y) = f(x) + \int_0^1 \langle \nabla f(x + t(y - x)), y - x \rangle dt
$$

$$
= f(x) + \langle \nabla f(x), y - x \rangle + \int_0^1 \langle \nabla f(x + t(y - x)) - \nabla f(x), y - x \rangle dt.
$$

Using the Cauchy-Schwarz inequality and since $f \in C^{1,1}(P)$ with $L_f > 0$, it follows that

$$
f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \int_0^1 \|\nabla f(x + t(y - x)) - \nabla f(x)\| \|y - x\| dt
$$

$$
\leq f(x) + \langle \nabla f(x), y - x \rangle + \int_0^1 L_f \|y - x\|^2 t dt
$$

$$
\leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L_f}{2} \|y - x\|^2. \quad (6)
$$

Assume now that $f$ is, in addition, convex. For a fixed $u \in P$, let the function $g$ be defined by

$$
g(v) := f(v) - f(u) - \langle \nabla f(u), v - u \rangle, \quad (v \in P). \quad (7)
$$
It is immediately obvious from the definition that \( g \) is convex. Also, it is easy to see that \( \nabla g(v) = \nabla f(v) - \nabla f(u) \), so it follows that \( g \in C^{1,1}(P) \) with \( L_g = L_f \). Since \( f \) is convex, we have \( g(v) \geq 0, \forall v \in P \) and obviously we have \( g(u) = \nabla g(u) = 0 \).

Setting \( f = g, y = v - \frac{\nabla g(v)}{L_f} \) and \( x = v \) in (6) gives

\[
0 \leq g \left( v - \frac{\nabla g(v)}{L_f} \right) \\
\leq g(v) + \left\langle \nabla g(v), -\frac{\nabla g(v)}{L_f} \right\rangle + \frac{L_f}{2} \left\| \frac{\nabla g(v)}{L_f} \right\|^2 \\
= g(v) - \frac{1}{2L_f} \left\| \nabla g(v) \right\|^2.
\]

Thus, it follows from (8) that, for all \( u \) and \( v \) in \( P \), we have

\[
0 \leq f(v) - f(u) - \left\langle \nabla f(u), v - u \right\rangle - \frac{1}{2L_f} \left\| \nabla f(v) - \nabla f(u) \right\|^2.
\]

Similarly, but with the roles of \( v \) and \( u \) interchanged, we have

\[
0 \leq f(u) - f(v) - \left\langle \nabla f(v), u - v \right\rangle - \frac{1}{2L_f} \left\| \nabla f(v) - \nabla f(u) \right\|^2.
\]

By adding inequalities (9) and (10), we finally obtain the desired result.

Lemma 2.1 implies the following result:

**Proposition 2.2.** If \( f \in C^{1,1}(P) \) with \( L_f > 0 \), then the functions defined by

\[
g_\pm := \frac{L_f}{2} \left\| . \right\|^2 \pm f
\]

are both convex and belong to \( C^{1,1}(P) \). If in addition \( f \) is convex, then \( L_{g_-} \leq L_f \).

**Proof.** We need to show that the functions \( g_\pm \) also belong to \( C^{1,1}(P) \). Indeed, they are obviously differentiable and it is easy to check that

\[
\left\| \nabla g_\pm(y) - \nabla g_\pm(x) \right\| = \left\| L_f(y - x) \pm (\nabla f(y) - \nabla f(x)) \right\|
\]

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which implies, using the triangle inequality,
\[
\|\nabla g_+(y) - \nabla g_+(x)\| \leq L_f \|y - x\| + \|\nabla f(y) - \nabla f(x)\|
\leq 2L_f \|y - x\|.
\]

Hence, we have \(L_{g_+} \leq 2L_f\). Moreover, since \(f \in C^{1,1}(P)\), then by the Cauchy-Schwartz inequality we have
\[
\pm \langle \nabla f(y) - \nabla f(x), y - x \rangle \leq L_f \|y - x\|^2, \quad (12)
\]
and so
\[
L_f \|y - x\|^2 \pm \langle \nabla f(y) - \nabla f(x), y - x \rangle \geq 0. \quad (13)
\]

From this, it immediately follows
\[
\langle \nabla g_+(y) - \nabla g_+(x), y - x \rangle = L_f \|y - x\|^2 \pm \langle \nabla f(y) - \nabla f(x), y - x \rangle \geq 0,
\]
which means that \(g_\pm\) are both convex.

What remains to be shown is that if \(f\) is in addition convex, we have \(L_{g_-} \leq L_f\). Since function \(g_-\) has a Lipschitz continuous gradient, then from the convexity of \(f\) together with Lemma 2.1, it follows
\[
\|\nabla g_-(y) - \nabla g_-(x)\|^2 \leq L_{g_-} \langle \nabla g_-(y) - \nabla g_-(x), y - x \rangle \\
= L_{g_-} \langle L_f(y - x) - (\nabla f(y) - \nabla f(x)), y - x \rangle \\
= L_{g_-} \cdot L_f \|y - x\|^2 - L_{g_-} \langle \nabla f(y) - \nabla f(x), y - x \rangle \\
\leq L_{g_-} \cdot L_f \|y - x\|^2.
\]

This allows us to conclude that \(L_{g_-} \leq L_f\), since \(L_{g_-}\) is the smallest possible Lipschitz constant. This completes the proof of Proposition 2.2.

We are now in a position to state and prove our announced characterization of all upper approximation operators.

**Theorem 2.3.** Let \(A : C^1(P) \to C(P)\) be a linear operator. The following statements are equivalent:

(i) For every convex function \(g \in C^{1,1}(P)\), we have
\[
g(x) \leq A[g](x), \quad (x \in P). \quad (14)
\]

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(ii) For every \( f \in C^{1,1}(P) \) with a Lipschitz constant \( L_f \), we have
\[
|f(x) - A[f](x)| \leq \frac{L_f}{2} (A[\|\cdot\|^2](x) - \|x\|^2).
\]
(15)

Equality is attained for all functions of the form
\[
f(x) = a(x) + c\|x\|^2,
\]
(16)
where \( c \in \mathbb{R} \) and \( a(\cdot) \) is any affine function.

\textbf{Proof.} Let \( f \in C^{1,1}(P) \) with a Lipschitz constant \( L_f \) and suppose that \((i)\) holds. Define the two following functions
\[
g_\pm := \frac{L_f}{2} \|\cdot\|^2 \pm f.
\]
Due to Proposition 2.2, we know that both of these functions are convex and belong to \( C^{1,1}(P) \). Therefore, since \( A \) is linear, statement \((i)\) implies that
\[
\frac{L_f}{2} A[\|\cdot\|^2] \pm A[f] \geq \frac{L_f}{2} \|\cdot\|^2 \pm f,
\]
which gives the error estimate in statement \((ii)\). The case of equality is easily verified.

Conversely, let \( g \in C^{1,1}(P) \) be a convex function, and suppose that statement \((ii)\) holds. Let the function \( f \) be defined by
\[
f := \frac{L_g}{2} \|\cdot\|^2 - g.
\]
and set \( E := A - I \), where \( I \) is the identity on \( C^{1,1}(P) \). Applying Proposition 2.2 again, we have \( f \in C^{1,1}(P) \) with \( L_f \leq L_g \). Now, the error estimate in statement \((ii)\), applied to \( f \), implies that
\[
E \left[ \frac{L_g}{2} \|\cdot\|^2 - g \right] \leq \frac{L_f}{2} E[\|\cdot\|^2] \leq \frac{L_g}{2} E[\|\cdot\|^2].
\]
This shows that \( E[|g|] \geq 0 \), as was to be proved.
Note that in the error estimate established in Theorem 2.3, it is not required that the function \( f \) is convex as long as statement (i) holds. The latter condition, as mentioned in the introduction, is always satisfied by our barycentric approximation operator \( B_n \). Hence, Jensen’s inequality and Theorem 2.3 imply the following error estimate.

**Corollary 2.4.** Let \( B_n \) be the barycentric approximation given by (2). Then for every function \( f \in C^{1,1}(P) \) with a Lipschitz constant \( L_f \), we have

\[
|f(x) - B_n[f](x)| \leq \frac{L_f}{2} (B_n[\|\cdot\|^2](x) - \|x\|^2).
\]

(17)

Equality is attained for all functions of the form (16).

### 3. Pointwise error estimations

In this section, we show that it is possible to extend the error estimate given in Corollary 2.4 for a class of functions which are differentiable on \( P \) and such that their derivative \( \nabla f \) has a Hölder continuous gradient, that is, there exists \( r \in (0, 1] \) and \( L_f > 0 \) such that

\[
\|\nabla f(y) - \nabla f(x)\| \leq L_f \|y - x\|^r, \forall x, y \in P.
\]

(18)

The set of all such \( f \) will be denoted by \( C^{1,r}(P) \). For a convex function \( f \in C^{1,r}(P) \), the symbol

\[
E_n[f](x) := E_n[f, \lambda](x) = \sum_{i=0}^{n} \lambda_i(x) f(x_i) - f(x)
\]

will be reserved exclusively to denote the incurred approximation error between \( f \) and its barycentric approximation \( B_n[f] \).

Before we state and prove the main error estimate for functions with Hölder continuous gradients, we first prove some useful identities which are needed in the rest of the paper.

**Lemma 3.1.** Let \( f \) be a differentiable function on \( P \) and let \( x, x_i \in P, i = 0, \ldots, n \). Then the following identity holds:

\[
\sum_{i=0}^{n} \sum_{j=0}^{n} \lambda_i(x) \lambda_j(x) \langle \nabla f(x_j), x_i - x \rangle = 0,
\]

(19)
and the error function can be represented as:

\[ E_n[f](x) = \frac{1}{2} \sum_{i=0}^{n} \sum_{j=0}^{n} \lambda_i(x) \lambda_j(x) \int_{0}^{1} \left\langle e_{i,j}[f](t,x), x_i - x_j \right\rangle dt \]  \hspace{1cm} (20)

\[ = \sum_{i=0}^{n} \lambda_i(x) \int_{0}^{1} \left\langle e_i[f](t,x), x_i - x \right\rangle dt, \]  \hspace{1cm} (21)

where

\[ e_{i,j}[f](t,x) : = \nabla f(x + t(x_i - x)) - \nabla f(x + t(x_j - x)) \]  \hspace{1cm} (22)

\[ e_i[f](t,x) : = \nabla f(x + t(x_i - x)). \]  \hspace{1cm} (23)

Moreover, the error \( E_n[\|\cdot\|]^2 \) can be expressed in terms of the barycentric coordinates as:

\[ E_n[\|\cdot\|]^2(x) = \frac{1}{2} \sum_{i=0}^{n} \sum_{j=0}^{n} \lambda_i(x) \lambda_j(x) \|x_i - x_j\|^2 \]  \hspace{1cm} (24)

\[ = \sum_{i=0}^{n} \lambda_i(x) \|x - x_i\|^2. \]  \hspace{1cm} (25)

**Proof.** The first identity is an immediate consequence of the partition of unity property (4) and linear precision (5) of the barycentric coordinates \( \{\lambda_i\}_{i=0}^{n} \). Indeed, they imply

\[ \sum_{i=0}^{n} \lambda_i(x) (x_i - x) = 0, \]  \hspace{1cm} (26)

and consequently we get

\[ \sum_{i=0}^{n} \sum_{j=0}^{n} \lambda_i(x) \lambda_j(x) \langle \nabla f(x_j), x_i - x \rangle \]

\[ = \sum_{j=0}^{n} \lambda_j(x) \left( \nabla f(x_j), \sum_{i=0}^{n} \lambda_i(x)(x_i - x) \right) = 0, \]

which proves (19). Identity (21) follows from a similar argument.
Next, since $f$ is differentiable on $P$ for all $x_i, x$ in $P$, we have the identity
\[ f(x_i) - f(x) = \int_0^1 \langle \nabla f(x + t(x_i - x)), x_i - x \rangle dt, \]
Hence, for all $x \in P$, $E_n[f](x)$ can be rewritten as follows:
\[
E_n[f](x) := \sum_{i=0}^n \lambda_i(x) f(x_i) - f(x)
\]
\[
= \sum_{i=0}^n \lambda_i(x) \int_0^1 \langle \nabla f(x + t(x_i - x)), x_i - x \rangle dt
\]
\[
= \sum_{i=0}^n \sum_{j=0}^n \lambda_i(x) \lambda_j(x) \int_0^1 \langle \nabla f(x + t(x_i - x)), x_i - x \rangle dt.
\]
Now, combining the last relation and identity (19), we get
\[
E_n[f](x) = \sum_{i=0}^n \sum_{j=0}^n \lambda_i(x) \lambda_j(x) \int_0^1 \langle \nabla f(x + t(x_j - x)), x_i - x \rangle dt
\]
\[
- \sum_{i=0}^n \sum_{j=0}^n \lambda_i(x) \lambda_j(x) \int_0^1 \langle \nabla f(x + t(x_i - x)), x_j - x \rangle dt,
\]
(27)
where $e_{i,j}$ is defined as in (22). Further, interchanging the indices $i$ and $j$, we get
\[
E_n[f](x) = \sum_{i=0}^n \sum_{j=0}^n \lambda_i(x) \lambda_j(x) \int_0^1 \langle e_{j,i}[f](t, x), x_j - x \rangle dt
\]
\[
- \sum_{i=0}^n \sum_{j=0}^n \lambda_i(x) \lambda_j(x) \int_0^1 \langle -e_{j,i}[f](t, x), x - x_j \rangle dt
\]
\[
= \sum_{i=0}^n \sum_{j=0}^n \lambda_i(x) \lambda_j(x) \int_0^1 \langle e_{i,j}[f](t, x), x - x_j \rangle dt.
\]
(28)
By adding identities (27) and (28), we obtain the desired formula (20).
In order to show (24), we apply (4) and (5) to deduce
\[
\sum_{i=0}^{n} \sum_{j=0}^{n} \lambda_i(x) \lambda_j(x) \|x_i - x_j\|^2
\]
\[
= \sum_{i=0}^{n} \sum_{j=0}^{n} \lambda_i(x) \lambda_j(x) \left( \|x_i\|^2 - 2 \langle x_i, x_j \rangle + \|x_j\|^2 \right)
\]
\[
= \sum_{i=0}^{n} \lambda_i(x) \|x_i\|^2 - 2 \left( \sum_{i=0}^{n} \lambda_i(x) x_i \right) \left( \sum_{j=0}^{n} \lambda_j(x) x_j \right) + \sum_{j=0}^{n} \lambda_j(x) \|x_j\|^2
\]
\[
= 2 \left( \sum_{i=0}^{n} \lambda_i(x) \|x_i\|^2 - \|x\|^2 \right) = 2E_n[\|.\|^2](x).
\]
Moreover, it is easily verified that
\[
\|x_i - x_j\|^2 = \|x - x_i\|^2 + 2 \langle x_i - x, x - x_j \rangle + \|x - x_j\|^2.
\]
Applying this, yields (25) and completes the proof of the lemma.

Now everything is set for giving an error estimate for convex functions with Hölder continuous gradients, related to the error estimate (17).

**Theorem 3.2.** For every convex function \( f \in C^{1,r}(P) \) with a constant \( L_f \), we have
\[
0 \leq E_n[f](x) \leq \frac{L_f}{2(r + 1)} \sum_{i=0}^{n} \sum_{j=0}^{n} \lambda_i(x) \lambda_j(x) \|x_i - x_j\|^{r+1}.
\]
(29)
This inequality is sharp in the sense that the equality is attained for all affine functions.

**Proof.** Let us fix a convex function \( f \in C^{1,r}(P) \). Starting from the representation formula (20) and after first applying the Cauchy-Schwartz inequality, it follows:
\[
0 \leq E_n[f](x) \leq \frac{L_f}{2} \sum_{i=0}^{n} \sum_{j=0}^{n} \lambda_i(x) \lambda_j(x) \|x_i - x_j\|^{r+1} \int_0^1 t^r dt
\]
\[
= \frac{L_f}{2(r + 1)} \sum_{i=0}^{n} \sum_{j=0}^{n} \lambda_i(x) \lambda_j(x) \|x_i - x_j\|^{r+1}.
\]
4. A practical error bound

In this section we derive a practical error estimate, which will lead to a computationally attractive barycentric approximation. As a consequence of Theorem 3.2 we immediately have:

**Theorem 4.1.** For every convex function $f \in C^{1,r}(P)$ with a constant $L_f$, we have

$$0 \leq E_n[f](x) \leq \frac{L_f}{(r+1)^{1/r}} E_n^{1/r} \|\cdot\|^2(x), \ (x \in P).$$

This inequality is sharp in the sense that the equality is attained for all affine functions.

**Proof.** This follows immediately from the error estimate established in (29) of Theorem 3.2. Indeed, it is clear that in the special case $r = 1$, the conclusion of Theorem 4.1 is an immediate consequence of identity (24). Also, this case was covered by Corollary 2.4. For $r \in (0,1)$, the claim follows after applying the discrete weighted Hölder’s inequality to the right-hand side of (29) with parameters $p = \frac{2}{1+r}$ and $q = \frac{2}{1-r}$ (which are admissible values), and then using identity (24) again.

In order to use a well-known numerical technique to construct a ”good” barycentric approximation, we will now derive a novel upper bound which is somewhat poorer than (30), but it may still be good enough for practical applications. Indeed, the error bound which is discussed in what follows, will be formulated in terms of the smallest enclosing ball containing $P$. We denote it by $SEB(P)$. $SEB(P)$ is the ball with the smallest radius which contains all the points in $P$, (see Figure 1). Let

$$SEB(P) := \{ x \in \mathbb{R}^d : \|x - c_{seb}\| \leq r_{seb} \}.$$  

Clearly, $x_i \in SEB(P)$ if and only if $\|x_i - c_{seb}\|^2 \leq (r_{seb})^2$. Using this observation, the $SEB(P)$ problem can be cast as the following optimization problem:

$$\min_{r \in \mathbb{R}} r$$

subject to $\|c - x_i\|^2 \leq r^2, \ \forall i = 0, \ldots, n,$

$c \in \mathbb{R}^d,$
which in turn can be reformulated as the following min max problem:

$$\min_{c \in \mathbb{R}^d} \max_{x_i \in P} \| c - x_i \|^2.$$  

It has been shown that $SEB(P)$ always exists and is unique [20]. The $SEB(P)$ problem can be dated back as early as 1857, when Sylvester first investigated the smallest radius disk enclosing $m$ points on the plane. It has found applications in diverse areas such as biological swarms [14], [17]; robot communication [20], [2]; data mining, learning, statistics, computer graphics, and computational geometry [1], etc. Numerical algorithms for the construction of the $SEB(P)$ have been developed in [18], [21], [23] and the references therein.

For the sake of simplicity, in the rest of this paper, we denote by

$$R_n := \text{span} \left( \{ \lambda_i \}_{i=0}^n \right)$$

the subspace of $C(P)$ spanned by $\{ \lambda_i \}_{i=0}^n$, which contains affine functions as a subspace.

Note that the barycentric approximation $B_n$, as defined in (2), is in general non-interpolatory on the point set $X_n$. However, at the vertices of the polytope $P$, the function $\lambda_i$ associated to the vertex $v_i$ has to equal 1 at $v_i$ and 0 at all other points in $X_n \setminus \{ v_i \}$, i.e. $\lambda_i(v_j) = \delta_{ij}$ (where $\delta$ is the Kronecker delta), see [4, Corollary 3.4]. This property - called the delta property - is the foundation of using barycentric coordinates for interpolation purposes. Indeed, it immediately implies vertex interpolation for any continuous function: $B[f](v_i) = f(v_i)$.

The reader can easily verify that the delta property of barycentric coordinates, as defined in (3)-(5), may simply be characterized as follows:

**Lemma 4.2.** Let $\{ \lambda_i \}_{i=0}^n$ be a set of barycentric coordinates with respect to the point set $\{ x_i \}_{i=0}^n$. Then the following assertions are equivalent:

(i) $\{ \lambda_i \}_{i=0}^n$ satisfies the delta property.

(ii) Every function $h$ in $R_n$ has the unique representation $B_n[h]$ with respect to the basis $\{ \lambda_i \}_{i=0}^n$.

(iii) The error $E_n[h]$ of every function $h$ in $C(P)$ vanishes at the nodes $\{ x_i \}_{i=0}^n$.

(iv) The error $E_n[\| . \|^2]$ vanishes at the nodes $\{ x_i \}_{i=0}^n$.  

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Under the delta property, the next result is a simple consequence of Theorem 4.1. At the end of this section, a similar result will be given, with relaxed assumptions.

**Theorem 4.3.** Under the delta property of \( \{\lambda_i\}_{i=0}^n \), for every convex function \( f \in C^{1,r}(P) \) with a constant \( L_f \), we have

\[
0 \leq E_n[f](x) \leq \frac{L_f}{(r+1)^{2-\frac{1}{2}}} \left( \left( r^{seb} \right)^2 - \|x - c^{seb}\|^2 \right)^{\frac{1+r}{2}}, \quad (x \in P). \tag{31}
\]

This inequality is sharp in the sense that the equality is attained for all affine functions.

**Proof.** Indeed, by Theorem 4.1, it remains to show that

\[
E_n[\|\cdot\|^2](x) \leq h^{seb}(x) := \left( r^{seb} \right)^2 - \|x - c^{seb}\|^2. \tag{32}
\]

First, note that:

\[
h^{seb}(x) - E_n[\|\cdot\|^2](x) = \left( r^{seb} \right)^2 + 2 \langle x, c^{seb} \rangle - \|c^{seb}\|^2 - \sum_{i=0}^n \lambda_i(x) \|x_i\|^2. \tag{33}
\]

Function \( h^{seb}(x) - E_n[\|\cdot\|^2](x) \) clearly belongs to the subspace \( R_n \), since affine functions are elements of \( R_n \). Hence, by Lemma 4.2 (ii), we have

\[
h^{seb}(x) - E_n[\|\cdot\|^2](x) = \sum_{i=0}^n \lambda_i(x) \left( h^{seb}(x_i) - E_n[\|\cdot\|^2](x_i) \right). \tag{34}
\]

Since \( h^{seb} \) is nonnegative on \( P \), while \( E_n[\|\cdot\|^2] \) vanishes at all points of \( X_n \), we clearly have

\[
h^{seb}(x_i) - E_n[\|\cdot\|^2](x_i) \geq 0, \quad i = 0, \ldots, n.
\]

This, combined with identity (34), allows us to conclude that (32) holds for all \( x \in P \).

We shall later make use of the following simple, but useful technical lemma:
Lemma 4.4. Let $B$ be any ball in $\mathbb{R}^d$ with radius $r$ and center $c$ which contains $P$. Then, for all $x$ in $P$, there holds
\[ 0 \leq E_n[\|\cdot\|^2](x) \leq h(x) := r^2 - \|x - c\|^2. \]  
\[(35)\]

Proof. The proof is very similar to that of Theorem 4.3, only here we start from the fact that any affine function $g$ can be represented as
\[ g(x) = \sum_{i=0}^{n} \lambda_i(x) g(x_i). \]

Therefore, since
\[ l(x) := \|c\|^2 - 2 \langle x, c \rangle \]
is an affine function, it follows
\[ l(x) = \sum_{i=0}^{n} \lambda_i(x) (\|c\|^2 - 2 \langle x_i, c \rangle). \]

Consequently, we obtain
\[ h(x) - E_n[\|\cdot\|^2](x) = r^2 + 2 \langle x, c \rangle - \|x\|^2 - \sum_{i=0}^{n} \lambda_i(x) \|x_i\|^2 \]
\[ = r^2 - \sum_{i=0}^{n} \lambda_i(x) (\|c\|^2 - 2 \langle x_i, c \rangle) - \sum_{i=0}^{n} \lambda_i(x) \|x_i\|^2 \]
\[ = r^2 - \sum_{i=0}^{n} \lambda_i(x) \|x_i - c\|^2 \]
\[ = \sum_{i=0}^{n} \lambda_i(x) \left( r^2 - \|x_i - c\|^2 \right) \]
\[ = \sum_{i=0}^{n} \lambda_i(x) h(x_i). \]

Since the right hand side of the above expression is nonnegative on $P$, this implies that inequality (35) is valid for all $x$ in $P$.

Two important consequences of Lemma 4.4 are the following:
Corollary 4.5. For every function $f \in C^{1,1}(P)$, the error $E_n[f]$ vanishes at the vertices of $P$.

**Proof.** By Corollary 2.4, it suffices to show that the error $E_n$ of $\| . \|^2$ vanishes at the vertices of $P$. Let us now pick a vertex $v$ of $P$. Clearly, since $P$ is convex, we can find a ball $B$ in $\mathbb{R}^d$ such that it contains $P$ and $v$ lies on its boundary. The assertion now follows from inequality (35) of Lemma 4.4 applied for $x = v$.

We will now relax the assumptions on the set of barycentric coordinates $\{\lambda_i\}_{i=0}^n$. Namely, the requirement that they satisfy the delta property shall be removed. Without this condition, we have the following result.

**Theorem 4.6.** For every convex function $f \in C^{1,r}(P)$ with a constant $L_f$, we have

$$0 \leq E_n[f](x) \leq \frac{L_f}{(r + 1)2^{1/2}} \left( (r_{\text{seb}})^2 - \|x - c_{\text{seb}}\|^2 \right)^{1/r}, \quad (x \in P). \quad (36)$$

This inequality is sharp in the sense that the equality is attained for all affine functions.

**Proof.** Similarly as in the proof of Theorem 4.3, it needs to be shown that

$$E_n[\| . \|^2](x) \leq (r_{\text{seb}})^2 - \|x - c_{\text{seb}}\|^2. \quad (37)$$

Since $SEB(P)$ contains $P$, this is a simple consequence of Lemma 4.4.

It is interesting to note that, as in Lemma 4.4, the estimate (37) does not depend on the choice of the set of barycentric coordinates used in the barycentric approximation $B_n$.

### 4.1. Practical considerations

One prominent approach for the construction of barycentric coordinates is to use a decomposition of $P = \text{conv}(X_n) = \bigcup_{k=1}^m \Omega_k$ to construct a piecewise defined local barycentric approximation, composed of simple functions $f_k| \Omega_k$. The easiest and most practical methods, due to their low cost and simplicity, are triangulation schemes, where $\Omega_k$ are simplices and $f_k$ are piecewise affine.
functions. There exist several triangulations of a set of points. Some of them are not very interesting in practice, but one can try to compute those that optimize some desirable criterion. When approximating a function \( f \) by the "best" barycentric approximation, the resulting approximation quality depends heavily on the quality of the barycentric coordinates used in the approximation. For those which are generated by the use of triangulations, several quality measures have already been proposed, see e.g. [22].

A very natural triangulation \( DT(P) \) of \( P \) is the one which uses only the points of \( X_n \) as triangulation vertices and such that no point in \( X_n \) lies inside the circumscribing ball of any simplex of \( DT(P) \). Such a triangulation exists and is called a Delaunay triangulation of \( P \) with respect to \( X_n \). It can be obtained as the geometric dual of the Voronoi diagram of \( X_n \), see e.g.[3]. Associated with \( DT(P) \) is a collection of balls, called Delaunay balls. One for each simplex \( S^{DT} \) of \( DT(P) \), the Delaunay ball circumscribes \( S^{DT} \). We denote by \( DB(P) \) the set of all circumscribing balls, (see Figure 2 and 3).

Let \( T(P) \) be any triangulation of the point set \( X_n \). Then \( \lambda^{T(P)} := \{ \lambda_i^{T(P)} \}_{i=0}^n \) denotes the set of barycentric coordinates associated with each \( x_i \) of \( X_n \). We now list the basic properties of \( \lambda^{T(P)} \) of which the following are particularly relevant to us:

(1) They are well-defined, piecewise linear and nonnegative real-valued continuous functions.
(2) The function \( \lambda_i^{T(P)} \) has to equal 1 at \( x_i \) and 0 at all other points in \( X_n \setminus \{x_i\} \), that is, \( \lambda_i^{T(P)}(x_j) = \delta_{ij} \) (\( \delta \) is the Kronecker delta).

We will denote by

\[
E_n^{T(P)}[f](x) := \sum_{i=0}^n \lambda_i^{T(P)}(x)f(x_i) - f(x).
\]

Now, in view of the error estimate (30), the following problem arises naturally:

**Problem 4.7.** Given a set of points, find the barycentric coordinates of the point set which provide the smallest error of the barycentric approximation of the quadratic function \( \| \cdot \|^2 \).

The following lemma is crucial in the proof of the optimality theorem below.
Lemma 4.8. Let $S^{DT}(P)$ be any simplex of $DT(P)$ and let $c_{S^{DT}(P)}$ and $r_{S^{DT}(P)}$ be the center and the radius of its Delaunay ball. Then, for all $x \in S^{DT}(P)$, it holds

$$0 \leq h_{S^{DT}(P)}(x) := r_{S^{DT}(P)}^2 - \|x - c_{S^{DT}(P)}\|^2 \leq E_n[\|\cdot\|^2](x). \quad (38)$$

Proof. The idea of the proof is similar to the proof of Lemma 4.4. Indeed, it follows analogously that

$$h_{S^{DT}(P)}(x) - E_n[\|\cdot\|^2](x) = \sum_{i=0}^{n} \lambda_i(x) h_{S^{DT}(P)}(x_i). \quad (39)$$

Now, the empty circumscribed ball interior property of the Delaunay triangulation implies that the right hand side of the above expression is less than or equal to zero on $P$. Hence, the required inequality (38) is valid for all $x$ in $P$.

The following expression for $E_n^{T}(P)[\|\cdot\|^2]$ is also useful:

Lemma 4.9. Let $T(P)$ be a triangulation, not necessarily Delaunay, of the point set $X_n$. Let $S^{T}(P)$ be any simplex of $T(P)$ and let $c_{S^{T}(P)}$ and $r_{S^{T}(P)}$ be the center and the radius of its circumscribed ball. Then, for all $x \in S^{T}(P)$, there holds

$$E_n^{T}(P)[\|\cdot\|^2](x) = r_{S^{T}(P)}^2 - \|x - c_{S^{T}(P)}\|^2. \quad (40)$$

Proof. Pick a simplex $S^{T}(P)$ in $T(P)$. From the definition of $\left\{\lambda_i^{T(P)}\right\}_{i=0}^{n}$, it is easy to see that the restriction of $E_n^{T}(P)[\|\cdot\|^2]$ to $S^{T}(P)$ is a quadratic polynomial. Since $\left\{\lambda_i^{T(P)}\right\}_{i=0}^{n}$ satisfy the delta property, by Lemma 4.2, $E_n^{T}(P)[\|\cdot\|^2]$ vanishes at all the vertices of $S^{T}(P)$. The quadratic polynomial $r_{S^{T}(P)}^2 - \|x - c_{S^{T}(P)}\|^2$ also vanishes at all the vertices of $S^{T}(P)$, so the result follows from the fact that the difference of these polynomials is an affine function.

We are now prepared to show that every set of barycentric coordinates generated by a Delaunay triangulation is optimal, in the sense that for all possible barycentric coordinates, Delaunay triangulation provides the minimal barycentric approximation error $E_n[\|\cdot\|^2]$. Indeed, in light of the above two lemmas the following theorem follows.
Theorem 4.10. Let $T(P)$ be a triangulation of the point set $X_n$. Then the following statements are equivalent.

(i) $T(P)$ is a Delaunay triangulation.

(ii) For any set of barycentric coordinates $\lambda := \{\lambda_i\}_{i=0}^n$ and for all $x \in P$, there holds

$$0 \leq E_n^T(P)[\|\cdot \|^2](x) \leq E_n[\|\cdot \|^2, \lambda](x).$$

Proof. The implication $(i) \Rightarrow (ii)$ is an immediate consequence of Lemmas 4.8 and 4.9. The proof of the reverse implication $(ii) \Rightarrow (i)$ will be given through contradiction. Assume that $T(P)$ is not a Delaunay triangulation. This implies that there exists at least one simplex $S$ of $T(P)$ for which there exists a point $p \in X_n$ such that $p$ lies in the interior of the circumscribed ball $B(c_S, r_S)$ of $S$. Let $S'$ be a Delaunay simplex of $DT(P)$ such that $p$ is its vertex and $F$ be a common lower dimensional face of $S$ and $S'$. Further, let $B(c_{S'}, r_{S'})$ be its circumscribed ball, see Figure 4. Define, for all $x \in S'$,

$$h(x) = r_S^2 - \|x - c_s\|^2$$
$$g(x) = r_{S'}^2 - \|x - c_{S'}\|^2.$$

These polynomials vanish at all the vertices of the face $F$, while $h(p) > 0$ and $g(p) = 0$. Taking into account that $h - g$ is an affine function and arguing as in the proof of Lemma 4.4, we conclude that $g(x) < h(x)$ on Int $S'$. Now, Lemmas 4.8 and 4.9 imply that $E_n^{DT(P)}[\|\cdot \|^2](x) < E_n^T(P)[\|\cdot \|^2](x)$ on Int $S'$. This contradicts our assumption (ii) and completes the proof of the theorem.

5. Numerical experiments

Suppose a set of scattered data $\{(x_i, y_i, f_i)\}_{i=1}^N$ sampled from a convex function $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, is given. Taking the $N$ scattered points as nodes, an optimal triangulation mesh, $T$, is constructed in domain $\Omega$ using Delaunay triangulation. In this section, we present two numerical examples which illustrate the proposed methodology. We approximate two test convex functions using randomly scattered data points in the indicated domain.

Example 5.1. We take the following convex function:

$$f(x, y) := 500 \exp((x - 0.5)^2 + (y - 0.5)^2),$$

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with the restriction of domain $D := [0, 1] \times [0, 1]$. The data is generated from
the above function and it is based on 21 equally spaced nodes on each edge of
the boundary of square $D$ and 216 nodes in the square $D$. The nodes in the
domain are placed randomly selected from $D$ while the nodes on the boundary
is equally spaced. Figure 5 on the left presents the graph of $f$. Figure 5
on the right describes the graph for the linear interpolation of scattered data
generated from the function $f$.

**Example 5.2.** In the second example the data points is generated from the
following test function:

$$g(x, y) = x^3 + 5(y^2 - 0.6)^2 + 1,$$

with the restriction of domain $D := [0, 1] \times [0, 1]$. As it is mentioned in
[16], the data points in this example belong to a surface that models part of
a car. Here, the data is generated from the above function and it is based on
21 equally spaced nodes on each edge of the boundary of square $D$ and 216
nodes randomly selected in the square. Figure 6 on the left is the graph of
$g$ while Figure 6 on the right shows the linear interpolation of scattered data
generated from the function $g$.

From Figures 5 and 6 it is clear that the convexity of $f$ and $g$ has been
preserved and there are no visual differences between the test functions and
their linear interpolants.

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**Figure Legends and Figures**

Figure 1:
Circumscribed ball and small enclosing ball of a simplex.

Figure 2:
Convex hull and Delaunay triangulation for a 2-D set of points.

Figure 3:
The empty circumscribed ball interior property of Delaunay triangulations.

Figure 4:
A triangulation that is not a Delaunay.

Figure 5:
The figure on the left shows the graph of $f$ and the graph on the right for the linear interpolation of the data generated from $f$.

Figure 6:
The figure on the left shows the graph of $g$ and the graph on the right for the linear interpolation of the data generated from $g$. 
Fig. 6