Error analysis for a non-standard class of differential quasi-interpolants

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Abstract

Given a B-spline \( M \) on \( \mathbb{R}^s \), \( s \geq 1 \) we consider a classical discrete quasi-interpolant \( Q_d \) written in the form

\[
Q_d f = \sum_{i \in \mathbb{Z}^s} f(i) L(\cdot - i),
\]

where \( L(x) := \sum_{j \in J} c_j M(x - j) \) for some finite subset \( J \subset \mathbb{Z}^s \) and \( c_j \in \mathbb{R} \). This fundamental function is determined to produce a quasi-interpolation operator exact on the space of polynomials of maximal total degree included in the space spanned by the integer translates of \( M \), say \( \mathbb{P}_m \). By replacing \( f(i) \) in the expression defining \( Q_d f \) by a modified Taylor polynomial of degree \( r \) at \( i \), we derive non-standard differential quasi-interpolants \( Q_{D_r} f \) of \( f \) satisfying the reproduction property

\[
Q_{D_r} p = p, \quad \text{for all } p \in \mathbb{P}_{m+r}.
\]

We fully analyze the quasi-interpolation error \( Q_{D_r} f - f \) for \( f \in C^{m+2} (\mathbb{R}^s) \), and we get a two term expression for the error. The leading part of that expression involves a function on the sequence \( c := (c_j)_{j \in J} \) defining the discrete and the differential quasi-interpolation operators. It measures how well the non-reproduced monomials are approximated, and then we propose a minimization problem based on this function.

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1. Introduction

Let \( M \) be a \( s \)-variate B-spline, i.e. a compactly supported non-negative polynomial piecewise function defined on \( \mathbb{R}^s \), \( s \geq 1 \), normalized by \( \sum_{i \in \mathbb{Z}^s} M(\cdot - i) = 1 \). Let \( S(M) := \text{span}(M(\cdot - i))_{i \in \mathbb{Z}^s} \) be the cardinal spline space spanned by...
the shifts of $M$. A quasi-interpolant for $S(M)$ is a linear map into $S(M)$ which is local, bounded, and reproduces some (nontrivial) polynomial space (see [11], p. 63). The standard structure for a quasi-interpolant is given by the expression

$$Qf := Q(f) := \sum_{i \in \mathbb{Z}^s} \lambda f(\cdot + i) M(\cdot - i),$$

(1)

$\lambda$ being some suitable linear functional.

Standard quasi-interpolants are very useful schemes for approximating multivariate functions. They are defined without solving any linear or non-linear system of equations, and provide the approximation power of $S(M)$ for smooth functions.

We refer to [11,10,16] for the various methods in the literature for constructing such quasi-interpolants.

In [1–3,5,7,15], the construction of new quasi-interpolation operators $Q$ exact on some space of polynomials $P_m$ and having small infinity norms is considered (see [1,4,15] for the non-uniform univariate case). Then, an estimate for the quasi-interpolation error $Qf - f$ is obtained taking into account the Lebesgue inequality

$$\|f - Qf\|_\infty \leq (1 + \|Q\|_\infty) \operatorname{dist}(f, P_m)$$

In [6], new discrete bivariate quasi-interpolants based on box splines are constructed by minimizing an expression involved in a particular estimate of the quasi-interpolation error defined from the errors for some non-reproduced monomials.

In a recent paper [9], we have proposed a general method to construct new differential quasi-interpolation operators from discrete ones in such a way that the new operators reproduce polynomials to the highest possible degree. That method has been used in [8] to define $C^1$ cubic quasi-interpolating splines on a type-2 triangulation starting from a particular discrete quasi-interpolant.

The aim of this paper is to combine both methods to derive new explicit differential spline quasi-interpolants, based on uniform type triangulation approximating regularly distributed data. They only use the values of the function to be approximated as well as their derivatives up to some prescribed order at the grid points.

In Section 2, we give some notations and recall the construction of the modified differential quasi-interpolants as well as some related results. In Section 3, we establish an integral representation for the error of approximation, from which we obtain a minimization problem leading to the non-standard class of differential quasi-interpolant. Finally, we give an example to show that the propose method can be produce good operators.

2. Notations and preliminaries

For a real valued function $f$ and $k \in \mathbb{N}$, we say $f \in C^k(\mathbb{R}^s)$ if $f$ is $k$ times continuously differentiable in the following sense: the directional derivatives of order $l$, $l = 0, \ldots, k$, at $x \in \mathbb{R}^s$ along the direction $y \in \mathbb{R}^s$ defined as

$$D^l_y f(x) = \frac{d^l}{dt^l} f(x + ty)|_{t=0}$$

exist and depend continuously on $x$. When the directional derivative exists for $y$, it can be extended to multiples by defining

$$D^l_{\alpha y} f(x) = \alpha^l D^l_y f(x), \alpha \in \mathbb{R}.$$ 

For $f \in C^k(\mathbb{R}^s)$, we introduce

$$|D^k f| = \sup_{x \in \mathbb{R}^s} \sup_{\|y\|=1} \left\{ \left| D^k_y f(x) \right| : y \in \mathbb{R}^s, \|y\| = 1 \right\},$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^s$ and $|f|_k = \sum_{|\alpha|=k} |D^\alpha f|$. We use the notations $|\alpha| := \sum_{k=1}^s \alpha_k$ for the length of the multi-integer $\alpha := (\alpha_1, \ldots, \alpha_s) \in \mathbb{Z}^s$ with non-negative components, $\alpha! := \prod_{k=1}^s \alpha_k!$, $\lambda^\alpha := \prod_{k=1}^s \lambda_k^{\alpha_k}$, and $m_\alpha(x) := x^\alpha/\alpha!$. 

A standard discrete quasi-interpolant (dQIO for short) based on the B-spline \( M \) is a local and bounded linear map \( Q_d \) into \( S(M) \) reproducing a space of polynomials and given by the expression

\[
Q_d f(x) := \sum_{i \in \mathbb{Z}} \lambda f(\cdot + i) M(x - i),
\]

(2)

where \( \lambda \) is the linear functional defined as

\[
\lambda f := \sum_{j \in J} c_j f(-j),
\]

(3)

for a finite subset \( J \subset \mathbb{Z} \) and \( c := (c_j)_{j \in J} \in \mathbb{R}^J \) (cf. [11], p. 63). We will suppose that the parameters \( c_j \) of the functional \( \lambda \) are determined in such a way that \( Q_d \) is exact on \( P_m \).

Then \( Q_df \) can be expressed as

\[
Q_d f(x) = \sum_{i \in \mathbb{Z}} f(i) L(x - i),
\]

(4)

where

\[
L(x) := \lambda M(x + \cdot) = \sum_{j \in J} c_j M(x - j)
\]

is the fundamental function of \( Q_d \), which is a compactly supported function due to the finiteness of \( J \).

In general, for approximation, relatively high order polynomials have more flexibility than low order ones. For this reason, based on the extension in [13] of the univariate result given in [18] (see also [14]) to get better approximation, in [9] has been developed new approximation schemes based on dQIOs with a high order of polynomial reproduction.

The value \( f(i) \) in (4) is replaced by its modified Taylor polynomial approximation of degree \( r \) at \( i \), then resulting the differential operator

\[
Q_{D,r} f(x) := \sum_{i \in \mathbb{Z}} \left( \sum_{l=0}^{r} \frac{a_{mrl}}{l!} D_{x-i}^l f(i) \right) L(x - i),
\]

(5)

where

\[
a_{mrl} := \frac{(m + r - l)! r!}{(m + r)! (r - l)!}.
\]

The differential operator \( Q_{D,r} \) reproduces polynomials up to degree \( m + r \). This result is an important implication of the integral error representation obtained in [13].

**Theorem 1.** Let \( f \in C^{m+r+1}(\mathbb{R}^s) \). Then, for all \( x \in \mathbb{R}^s \), we have

\[
f(x) - Q_{D,r} f(x) = \sum_{i \in \mathbb{Z}} \left( \int_0^1 H_{m,r}(t) D_{x-i}^{m+r+1} f(i + t(x - i)) dt \right) L(x - i),
\]

where

\[
H_{m,r}(t) := (-1)^m t^m \frac{(1 - t)^r}{(m + r)!}.
\]

As another immediate corollary of Theorem 1, we obtain the following error estimate.

**Corollary 2.** Suppose \( f \in C^{m+r+1}(\mathbb{R}^s) \). Then, for all \( x \in \mathbb{R}^s \), we have

\[
|f(x) - Q_{D,r} f(x)| \leq R(x) |D^{m+r+1} f|,
\]

where

\[
R(x) := \frac{m! r!}{(m + r)! (m + r + 1)!} \sum_{i \in \mathbb{Z}} \|x - i\|^{m+r+1} |L(x - i)|.
\]
3. A non-standard class of differential quasi-interpolants

In this section, we fully analyze the quasi-interpolation error $Q_{D,r}f - f$. We obtain a pointwise error estimate that permit us to propose a new construction of differential quasi-interpolation operators by minimizing a function appearing in the leading term of an appropriate error estimate that depends on the sequence $c$.

Since $Q_{D,r}$ reproduces $\mathbb{P}_{m+r}$, we need to consider $f$ in $C^{m+r+1}(\mathbb{R}^d)$ to derive a standard pointwise estimate. However we will suppose greater regularity for $f$.

Theorem 3. Let $f \in C^{m+r+2}(\mathbb{R}^d)$. Then, for all $x \in \mathbb{R}^d$, we have

$$Q_{D,r}f(x) = f(x) + T(x) - R(x),$$

where

$$T(x) = \sum_{l=0}^{r} (-1)^{m+r+1-l} \frac{a_{mrl}}{l!(m + r + 1 - l)!} \sum_{i \in \mathbb{Z}^d} D_{x-i}^{m+r+1} f(x) L(x - i)$$

and

$$R(x) = \sum_{i \in \mathbb{Z}^d} \sum_{l=0}^{r} \frac{a_{mrl}}{l!(m + r + 1 - l)!} \int_0^1 (-t)^{m+r+1-l} D_{x-i}^{m+r+2} f(i + t(x - i)) dt L(x - i).$$

Proof. The Taylor expansion of order $m + r + 1$ provides the expression

$$D_{x-i}^j f(i) = \sum_{k=l}^{m+r+1} (-1)^{k-l} \frac{1}{(k-l)!} D_{x-i}^k f(x) - \int_0^1 (-t)^{m+r+1-l} \frac{1}{(m + r + 1 - l)!} D_{x-i}^{m+r+2} f(i + t(x - i)) dt.$$

If we multiply both sides by $(a_{mrl})/(l!)$ and sum over $l$ from 0 to $r$, we get

$$\sum_{l=0}^{r} \frac{a_{mrl}}{l!} D_{x-i}^j f(i) = \sum_{l=0}^{r} \frac{a_{mrl}}{l!} \sum_{k=l}^{m+r+1} (-1)^{k-l} \frac{1}{(k-l)!} D_{x-i}^k f(x)
- \sum_{l=0}^{r} \frac{a_{mrl}}{l!} \int_0^1 (-t)^{m+r+1-l} \frac{1}{(m + r + 1 - l)!} D_{x-i}^{m+r+2} f(i + t(x - i)) dt
= \sum_{k=0}^{m+r+1} \left( \sum_{l=0}^{k} (-1)^{k-l} \frac{a_{mrl}}{l!(k-l)!} \right) D_{x-i}^k f(x)
- \sum_{l=0}^{r} \frac{a_{mrl}}{l!} \int_0^1 (-t)^{m+r+1-l} \frac{1}{(m + r + 1 - l)!} D_{x-i}^{m+r+2} f(i + t(x - i)) dt,$$

where $k' := \min \{k, r\}$. It is shown in Ref. [18] that

$$\sum_{l=0}^{k} (-1)^{k-l} \frac{a_{mrl}}{l!(k-l)!} = 0, k = m + 1, \ldots, m + r,$$

whence

$$\sum_{k=1}^{m+r} \left( \sum_{l=0}^{k} (-1)^{k-l} \frac{a_{mrl}}{l!(k-l)!} \right) = \sum_{k=1}^{m} \left( \sum_{l=0}^{k} (-1)^{k-l} \frac{a_{mrl}}{l!(k-l)!} \right).$$
Thus, we have
\[
\sum_{l=0}^{r} \frac{a_{mlr}}{l!} D_{x-i}^l f(i) = f(x) + \sum_{k=1}^{m} \left( \sum_{l=0}^{k'} (-1)^{k-l} \frac{a_{mlr}}{l! (k-l)!} \right) D_{x-i}^k f(x)
\]
\[
+ \sum_{l=0}^{r} (-1)^{m+r+1-l} \frac{a_{mlr}}{l! (m+r+1-l)!} D_{x-i}^{m+r+1} f(x)
\]
\[
- \sum_{l=0}^{r} a_{mlr} \int_{0}^{1} (-1)^{m+r+1-l} \frac{1}{(m+r+1-l)!} D_{x-i}^{m+r+2} f(i + t(x-i)) \, dt.
\]
Since \(\sum_{i \in \mathbb{Z}^s} L(\cdot - i) = 1\), we multiply by \(L(x-i)\) both sides of the previous equality and sum over \(i \in \mathbb{Z}^s\) to obtain
\[
\sum_{i \in \mathbb{Z}^s} \left( \sum_{l=0}^{r} \frac{a_{mlr}}{l!} D_{x-i}^l f(i) \right) L(x-i) = f(x) + N(x) + T(x) - R(x),
\]
where
\[
N(x) := \sum_{i \in \mathbb{Z}^s} \sum_{k=1}^{m} \left( \sum_{l=0}^{k'} (-1)^{k-l} \frac{a_{mlr}}{l! (k-l)!} \right) D_{x-i}^k f(x) L(x-i).
\]
Since
\[
D_{x-i}^k f(x) = \sum_{|\alpha|=k} \nu_\alpha \partial^\alpha f(x)(x-i)^\alpha
\]
for some \(\nu_\alpha \in \mathbb{R}\), we have
\[
N(x) = \sum_{k=1}^{m} \left( \sum_{l=0}^{k'} (-1)^{k-l} \frac{a_{mlr}}{l! (k-l)!} \right) \sum_{|\alpha|=k} D_{x-i}^k f(x) \nu_\alpha \partial^\alpha f(x) \sum_{i \in \mathbb{Z}^s} (x-i)^\alpha L(x-i),
\]
and then \(N(x) = 0\) because
\[
\sum_{i \in \mathbb{Z}^s} (x-i)^\alpha L(x-i) = Q_d [ (x - \cdot)^\alpha ] (x) = 0,
\]
due to the exactness of \(Q_d\) on \(\mathbb{P}_m\). The proof is complete. \(\square\)

Taking into account that
\[
D_{x-i}^{m+r+1} f(x) = \sum_{|\alpha|=m+r+1} \frac{(m+r+1)!}{\alpha!} \partial^\alpha f(x)(x-i)^\alpha,
\]
we obtain that
\[
T(x) = K_{m,r} \sum_{|\alpha|=m+r+1} \partial^\alpha f(x) Q_d[m_\alpha(x - \cdot)](x),
\]
with
\[
K_{m,r} := (m+r+1)! \sum_{l=0}^{r} (-1)^l a_{mlr} \frac{1}{l!(m+r+1-l)!}.
\]
Since \(\tau\) is a uniform partition of \(\mathbb{R}^s\), there exists \(\xi \in [0, 1]^s\) such that \(x = \xi + k\) for some \(k \in \mathbb{Z}^s\) and by a simple calculation we get
\[
T(x) = K_{m,r} \sum_{|\alpha|=m+r+1} \partial^\alpha f(x) Q_d[m_\alpha(\xi - \cdot)](\xi).
\]
From Theorem 3, the quasi-interpolation error for the normalized monomial \( m_\alpha \), \(|\alpha| = m + r + 1\), is given by

\[
Q_{D,r}m_\alpha(\xi) - m_\alpha(\xi) = \sum_{l=0}^{r} \frac{(-1)^{m+r+1-l}a_{mrl}}{l!(m + r + 1 - l)!} \sum_{i \in \mathbb{Z}} D_{\xi-i}^{m+r+1}m_\alpha(\xi)L(\xi - i).
\]

Noting that, for \(|\alpha| = m + r + 1\)

\[
D_{\xi-i}^{m+r+1}m_\alpha(\xi) = (m + r + 1)(\xi - i)^\alpha,
\]

we get

\[
Q_{D,r}m_\alpha(\xi) - m_\alpha(\xi) = (m + r + 1)! \sum_{l=0}^{r} \frac{(-1)^{r-l}a_{mrl}}{l!(m + r + 1 - l)!} \sum_{i \in \mathbb{Z}} (\xi - i)^\alphaL(\xi - i) = K_{m,r}Q_d(m_\alpha(\xi - \cdot))(\xi).
\]

Consequently, \( T \) is given by

\[
T(x) = \sum_{|\alpha|=m+r+1} \partial^\alpha f(x)(Q_{D,r}m_\alpha(\xi) - m_\alpha(\xi)),
\]

and then, by Theorem 3, we obtain the following error estimate.

**Proposition 4.** Let \( f \in C^{m+r+2}(\mathbb{R}^s) \). For any \( x \in \mathbb{R}^s \), we have

\[
\|Q_{D,r}f(x) - f(x)\| \leq \max_{|\alpha|=m+r+1} \|Q_{D,r}m_\alpha - m_\alpha\|_{\infty,[0,1]^s} |f|_{m+r+1}
\]

\[
+ C(m, r) \|D^{m+r+2}f\| \sum_{i \in \mathbb{Z}} \|x - i\|^{m+r+2} \|L(x - i)\|,
\]

where

\[
C(m, r) = \sum_{l=0}^{r} \frac{a_{mrl}}{l!(m + r + 2 - l)!}.
\]

Note that the constant \( \max_{|\alpha|=m+r+1} \|Q_{D,r}m_\alpha - m_\alpha\|_{\infty,[0,1]^s} \) depends on the sequence \( c \) and it is determined by how well the differential quasi-interpolant, which is exact on \( P_{m+r} \), approximates the normalized monomials \( m_\alpha \), \(|\alpha| = m + r + 1\).

Taking into account the estimate in Proposition 4, we can consider the construction of a good starting discrete quasi-interpolation operator \( Q_d \) by minimizing the objective function

\[
\max_{|\alpha|=m+r+1} \|Q_{D,r}m_\alpha - m_\alpha\|_{\infty,[0,1]^s},
\]

subject to the constraints yielding the exactness of \( Q_d \) on \( P_m \).

From Theorem 1, the quasi-interpolation error for the normalized monomial \( m_\alpha \), \(|\alpha| = m + r + 1\), satisfies the inequality

\[
|m_\alpha(x) - Q_{D,r}m_\alpha(x)| \leq \frac{m!r!}{(m + r)! (m + r + 1)!} \sum_{i \in \mathbb{Z}} D_{x-i}^{m+r+1}m_\alpha(i)L(x - i).
\]

Thus, we can consider the following problem:

**Problem 5.** Minimize

\[
F(x, c) := \max_{|\alpha| = m+r+1} \left| \sum_{i \in \mathbb{Z}} D_{x-i}^{m+r+1}m_\alpha(i)L(x - i) \right|, (x, c) \in [0, 1]^s \times \mathbb{R}^J,
\]

subject to the linear constraints yielding the exactness of \( Q_d \) on \( P_m \).
4. A $C^2$ quintic quasi-interpolant

Let $M$ be the centered box spline $M_{2,1}$ associated with the direction set $X_{2,1} = \{d_1, d_2, d_2, d_3, d_4\}$, where $d_1 = (1, 0)$, $d_2 = (0, 1)$, $d_3 = d_1 + d_2$ and $d_4 = -d_1 + d_2$. It is a $C^2(\mathbb{R}^2)$ quartic piecewise polynomial function defined on the triangulation spanned by the directions $d_1$, $1 \leq \ell \leq 4$, and supported on the octagon with vertices $(2, 1), (1, 2), (-1, 2), (-2, 1), (-1, -2), (1, -2)$ and $(2, -1)$ (cf. [10], Chapter 2 and [16], Chapter 12).

The space $\mathbb{P}_3$ of bivariate polynomials of total degree at most 3 is contained in $S(M_{2,1})$. Therefore, the proposed construction runs with $m = 3$, and moreover we will take $r = 1$. Although we look for a quasi-interpolant $Q_d$ given by (2) and (3) that minimizes $F(x, c)$ under the constraints yielding the exactness of $Q_d$ on $\mathbb{P}_3$, we will minimize an appropriate upper bound $U(c)$ in order to obtain a simpler minimization problem. The next result gives conditions on the sequence $c$ to achieve the required exactness (cf. [6]).

**Lemma 6.** Let $Q_d$ be a discrete quasi-interpolant given by (2) and (3). It is exact on $\mathbb{P}_3$ if

$$\sum_{j \in J} c_j = 1, \sum_{j \in J} j_1 c_j = \sum_{j \in J} j_2 c_j = 0, \sum_{j \in J} j_1^2 c_j = \sum_{j \in J} j_2^2 c_j = -\frac{1}{3}, \sum_{j \in J} j_1 j_2 c_j = 0,$$

$$\sum_{j \in J} j_\alpha c_j = 0 \text{ for } |\alpha| = 3.$$

To bound every function $Q_{D,1}m_\alpha - m_\alpha$ we need some identities derived from the representation of the monomials in $S(M_{2,1})$ as linear combinations of the translates of $M_{2,1}$.

**Lemma 7.** Let $x = (x_1, x_2) \in \mathbb{R}^2$ and $i = (i_1, i_2) \in \mathbb{Z}^2$, then the following identities hold:

$$\sum_{i \in \mathbb{Z}^2} M_{2,1}(x - i) = 1,$$

$$\sum_{i \in \mathbb{Z}^2} (x_1 - i_1)M_{2,1}(x - i) = \sum_{i \in \mathbb{Z}^2} (x_2 - i_2)M_{2,1}(x - i) = 0,$$

$$\sum_{i \in \mathbb{Z}^2} (x_1 - i_1)^2 M_{2,1}(x - i) = \sum_{i \in \mathbb{Z}^2} (x_2 - i_2)^2 M_{2,1}(x - i) = \frac{1}{3},$$

$$\sum_{i \in \mathbb{Z}^2} (x_1 - i_1)(x_2 - i_2)M_{2,1}(x - i) = 0,$$

$$\sum_{i \in \mathbb{Z}^2} (x - i)^\alpha M_{2,1}(x - i) = 0, |\alpha| = 3.$$

**Proof.** It is well-known (cf. [10], p. 119) that, for every $|\alpha| \leq 3$,

$$m_\alpha(x) = \sum_{i \in \mathbb{Z}^2} g_\alpha(i) M_{2,1}(x - i),$$

where the polynomials $g_\alpha$ are recursively computed as follows:

$$g_0 = m_0, \quad g_\alpha = m_\alpha - \sum_{j \in \mathbb{Z}^2} M_{2,1}(j) \sum_{\beta \leq \alpha} m_{\alpha - \beta}(-j) g_\beta, |\alpha| > 0.$$
Since $M_{2,1}(0, 0) = 5/12$, $M_{2,1}(± 1, 0) = M_{2,1}(0, ± 1) = 1/8$, and $M_{2,1}(± 1, 1) = M_{2,1}(1, ± 1) = 1/48$, then we have

$$1 = \sum_{i \in \mathbb{Z}^2} M_{2,1}(\cdot - i),$$

$$x_1 = \sum_{i \in \mathbb{Z}^2} i_1 M_{2,1}(\cdot - i), x_2 = \sum_{i \in \mathbb{Z}^2} i_2 M_{2,1}(\cdot - i),$$

$$x_1^2 = \sum_{i \in \mathbb{Z}^2} \left( i_1^2 - \frac{1}{3} \right) M_{2,1}(\cdot - i), x_1 x_2 = \sum_{i \in \mathbb{Z}^2} i_1 i_2 M_{2,1}(\cdot - i), x_2^2 = \sum_{i \in \mathbb{Z}^2} \left( i_2^2 - \frac{1}{3} i_1 \right) M_{2,1}(\cdot - i),$$

From these identities we obtain the ones for $\sum_{i \in \mathbb{Z}^2} (x - i)^\alpha M_{2,1}(x - i)$, $|\alpha| \leq 3$. □

We are now in position to derive an upper bound $U(c)$ for $F(x, c)$.

**Lemma 8.** It holds

$$F(x, c) \leq \frac{1}{3} \max \{u_\alpha, |\alpha| = 5\} =: U(c),$$

where

$$u_{5,0}(c) = \sum_{j \in J} |c_j| \left( 8 + 20 |j_1| + 10|j_1|^3 + 3|j_1|^5 \right),$$

$$u_{4,1}(c) = 5 \sum_{j \in J} |c_j| \left( 8 + 16 |j_1| + 4|j_1|^2 + 6j_1^2 |j_2| + 3j_1^3 |j_2|^2 \right),$$

$$u_{3,2}(c) = 10 \sum_{j \in J} |c_j| \left( 8 + 12 |j_1| + 8|j_2| + |j_1|^3 + 3|j_1|^3 j_2 \right),$$

$$u_{2,3}(c) = 10 \sum_{j \in J} |c_j| \left( 8 + 8 |j_1| + 12|j_2| + |j_1|^3 + 3j_1^2 |j_2|^2 \right),$$

$$u_{1,4}(c) = 5 \sum_{j \in J} |c_j| \left( 8 + 4 |j_1| + 16|j_2| + 6|j_1| j_2^2 + 3j_1 j_2^3 \right),$$

$$u_{0,5}(c) = \sum_{j \in J} |c_j| \left( 8 + 20 |j_2| + 10|j_2|^3 + 3|j_2|^5 \right).$$

**Proof.** We have

$$\sum_{i \in \mathbb{Z}^2} D^5_{x-i} m_\alpha(i)L(x - i) = \frac{5!}{\alpha!} \sum_{i \in \mathbb{Z}^2} (x - i)^\alpha \sum_{j \in J} c_j M(x - i - j) = \frac{5!}{\alpha!} \sum_{j \in J} c_j \sum_{i \in \mathbb{Z}^2} (x - i + j)^\alpha M_{2,1}(x - i).$$

Therefore,

$$\left| \sum_{i \in \mathbb{Z}^2} D^5_{x-i} m_\alpha(i)L(x - i) \right| \leq \frac{5!}{\alpha!} \sum_{j \in J} |c_j| \left| \sum_{i \in \mathbb{Z}^2} (x - i + j)^\alpha M_{2,1}(x - i) \right|.$$

Since $x \in [0, 1]^2$ every sum runs over $I := \{-1, 0, 1\} \times \{-1, 0, 1\}$, and then $|x_k - i_k| \leq 2$, $k = 1, 2$. For $\alpha = (5, 0)$, using Lemma 7 we get

$$\sum_{i \in \mathbb{Z}^2} (x - i + j)^5 M(x - i) = \sum_{i \in I} \left( (x_1 - i_1)^5 + 5j_1(x_1 - i_1)^4 \right) M_{2,1}(x - i) + \frac{10}{3} j_1^3 + j_1^5.$$
Since
\[
\left| \sum_{i \in I} (x_1 - i_1)^5 M_{2,1}(x - i) \right| \leq \left| \sum_{i \in I} |x_1 - i_1|^5 M_{2,1}(x - i) \right| \leq 8 \left| \sum_{i \in I} (x_1 - i_1)^2 M_{2,1}(x - i) \right| = \frac{8}{3},
\]
and, similarly
\[
\sum_{i \in I} (x_1 - i_1)^4 M_{2,1}(x - i) \leq \frac{4}{3},
\]
we obtain
\[
\left| \sum_{i \in \mathbb{Z}^2} (x - i + j)^q M(x - i) \right| \leq \frac{1}{3} \left( 8 + 20 |j_1| + 10 |j_1|^3 + 3 |j_1|^5 \right).
\]
The other terms are bounded in a similar manner, and the claim follows. \(\square\)

It only remains to minimize \(U(c)\) under the linear equalities in Lemma 6 for a given subset \(J\). We choose
\[J = \{(0,0), (1,0), (0,1), (-1,0), (0,-1), (1,1), (-1,1), (-1,-1), (1,-1)\},\]
and then we write \(c = (c_{0,0}, c_{1,0}, c_{0,1}, c_{-1,0}, c_{0,-1}, c_{1,1}, c_{-1,1}, c_{-1,-1}, c_{1,-1})\). It is an easy matter to prove that the minimum of \(U(c)\) in the subset of \(\mathbb{R}^9\) defined by the equalities in Lemma 6 is attained only at \(\tilde{c} = \left(\frac{4}{7}, 0, 0, 0, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}\right)\). Let \(\tilde{Q}_d\) and \(\tilde{Q}_{D,1}\) be its associated discrete and differential quasi-interpolants, respectively. We can write
\[
\tilde{Q}_d f = \sum_{i \in \mathbb{Z}^2} f(i) \tilde{L}(x - i),
\]
with
\[
\tilde{L}(x) = \frac{4}{3} M_{2,1}(x) - \frac{1}{12} \left( M_{2,1}(x \pm d_3) + M_{2,1}(x \pm d_4) \right),
\]
and
\[
\tilde{Q}_{D,1} f = \sum_{i \in \mathbb{Z}^2} \left( f(i) + \frac{1}{4} D_{x-i} f(i) \right) \tilde{L}(x - i).
\]
To test the differential quasi-interpolation operator \(\tilde{Q}_{D,1}\), we consider the Franke’s function (cf. [12])
\[
F(x, y) = 0.75 \exp \left( -\frac{1}{4} \left( (9x - 2)^2 + (9y - 2)^2 \right) \right) + 0.75 \exp \left( -\frac{1}{49} (9x + 1)^2 - \frac{1}{10} (9y + 1) \right) + 0.5 \exp \left( -\frac{1}{4} \left( (9x - 7)^2 + (9y - 3)^2 \right) \right) - 0.2 \exp \left( -(9x - 4)^2 - (9y - 7)^2 \right),
\]
and the first order differential quasi-interpolants \(P_{D,1}\) and \(\tilde{P}_{D,1}\) associated with the following discrete operators (cf. [17]):
\[
P_d f = \sum_{i \in \mathbb{Z}^2} \left( \frac{43}{24} f(i) - \frac{5}{24} (f(i \pm d_1) + f(i \pm d_2)) + \frac{1}{96} (f(i \pm d_3) + f(i \pm d_4)) \right) M_{2,1}(x - i),
\]
\[
\tilde{P}_d f = \sum_{i \in \mathbb{Z}^2} \left( \frac{19}{12} f(i) - \frac{1}{8} (f(i \pm d_1) + f(i \pm d_2)) - \frac{1}{48} (f(i \pm d_3) + f(i \pm d_4)) \right) M_{2,1}(x - i).
\]
Fig. 1 shows the graphs of the quasi-interpolation errors \(|\tilde{Q}_{D,1}^h F - F|\) for several values of \(h\). They illustrate the nearly-optimal approximation order of the new quasi-interpolation scheme.
We recall that the scaled operator $Q_{h,D,r}^b$ associated with the operator $Q_{D,r}$ defined in (5) is given by

$$Q_{h,D,r}^b f(x) = \sum_{i \in \mathbb{Z}} \left( \sum_{l=0}^r a_{ml} h^l D_{x-i} f(ih) \right) L \left( \frac{x}{h} - i \right).$$

The operators $P_{D,1}$ and $\tilde{P}_{D,1}$ provide slightly smaller errors for Franke’s function than $\tilde{Q}_d$. However, the computational cost of $\tilde{Q}_d$ is smaller than the ones of $P_d$ and $\tilde{P}_d$. Moreover, $\|P_d\|_\infty = 2.6666$ and $\|\tilde{P}_d\|_\infty = 2.1666$ (cf. [17]) and the new scheme $\tilde{Q}_d$ has a smaller infinity norm because $\|\tilde{Q}_d\|_\infty \leq \|\tilde{c}\|_1 = \frac{5}{3}$.

5. Concluding remarks

In this paper, we have derived an error estimate for the differential quasi-interpolation operator constructed form a discrete quasi-interpolation operator exact on a space of polynomials. The method used in the construction increases the degree of the reproduced polynomials and then the error estimate involves the quasi-interpolation error for the first non-reproduced monomials. The result on the error permits to define a minimization problem whose solution could give an optimal starting discrete quasi-interpolant.

The $C^2$ quartic box spline has been considered to test the method and quite interesting results has been obtained, although additional work is required to produce better results, particularly in the three dimensional case.

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References


