MIMO Radar Transmit Beampattern Design with Ripple and Transition Band Control

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Abstract—The waveform diversity of multiple-input-multiple-output (MIMO) radar systems offers many advantages in comparison to the phased-array counterpart. One of the advantages is that it allows one to design MIMO radar with flexible transmit beampatterns, which has several useful applications. Many researchers have proposed solutions to this problem in the recent decade. However, these designs pay little attention to certain aspects of the performance such as the ripples within the energy focusing section, the attenuation of the sidelobes, the width of the transition band, the angle step-size, and the required number of transmit antennas. In this paper, we first propose several methods which can indirectly or directly control the ripple levels within the energy focusing section and the transition bandwidth. These methods are based on existing problem formulations. More importantly, we reformulate the design as a feasibility problem (FP). Such a formulation enables a more flexible and efficient design which achieves the most preferable beampatterns with the least system cost. Using this formulation, an empirical MIMO radar beampattern formula is obtained. This MIMO radar beampattern formula is similar to Kaiser’s formula in conventional FIR filter design. The performances of the proposed methods and formulations are evaluated via numerical examples.

Index Terms—Multiple-input-multiple-output (MIMO) radar, transmit beamspace processing (TBP), transmit beampattern, covariance matrix, ripple control, MIMO radar beampattern formula.

I. INTRODUCTION

MUltiple-input multiple-output (MIMO) radar systems have gained much research attention within the recent decade [1]–[31]. A MIMO radar system can be characterized by a system with multiple transmit and receive antennas, colocated or widely separated, which simultaneously transmits different waveforms and processes the signals reflected to detect targets, or estimate parameters of the targets. Compared to its phased-array counterpart, MIMO radar enjoys the advantage of waveform diversity but has drawbacks in terms of signal-to-noise ratio (SNR) loss [4]. To preserve the waveform diversity of MIMO radar while taking the advantage of the coherent processing gain of phased-array radar, new configurations of radar architecture such as the phased-MIMO radar and hybrid MIMO phased-array radar have been proposed [6]–[8]. The signaling strategies discussed in these works use multiple beams (from subarrays), each of which is generated using a different waveform.

Besides these array partitioning based techniques, the MIMO radar transmit beampattern synthesis (or transmit energy focusing) based on signal correlations has also been receiving much attention recently [9]–[27]. This approach makes full use of the waveform diversity introduced by MIMO radar to enhance the direction of arrival (DOA) estimation. The method focuses the energy of transmitted waveforms within one or several predefined angle sections where multiple targets are likely to exist. With an appropriate design, the performance can be improved compared to conventional phased-array radar or MIMO radar without beampattern synthesis. In addition, the designed system does not need to sweep the field at transmitter as in conventional phased-array radar.

MIMO radar transmit beampattern is characterized by the covariance matrix of the transmitted waveforms, thus the objective is to design several transmitted waveforms to achieve the desired covariance matrix. In this situation, the desired covariance matrix usually lies between a scaled identity matrix (orthogonal waveforms) and a scaled all-one matrix (coherent waveforms), thus the common mode of operation that MIMO radar transmits orthogonal waveforms is altered.

There are usually two steps to achieve the design. Firstly, the optimal covariance matrix is obtained according to the desired beampattern. Secondly, the waveforms are optimized according to the obtained covariance matrix. The beampattern synthesis problem for MIMO radar was first formulated in [11] as a minimization of a cost function, with the solution based on a gradient search. In [9] and [10], a closed-form least-squares solution and a low-complexity method based on the modification of the cost function were proposed, respectively. In [12] (the full version of [11]), the authors proposed new optimization problems in which the positive semidefinite matrix constraint was parameterized by a set of coefficients, to achieve iterative solutions. In [13] and [14], the authors provided a more comprehensive analysis on transmit beampattern synthesis, in which several designing scenarios, such as maximum power design, beampattern matching design, and minimum sidelobe design, were investigated. In [21] and [22], the covariance matrix was parameterized using the
coordinates of a hypersphere, resulting in an iterative solution as well as a closed-form solution. More recently, the design has been extended to wideband applications [26]. It was noted in [12], [14] and [22] that the second design step, i.e., the design of the waveforms according to the optimized covariance matrix, is not easy to achieve. In [17]–[19], and their full version [20], a more efficient technique called transmit beamspace processing (TBP) was proposed. The TBP technique introduces a weighting matrix to several orthogonal waveforms used in the standard MIMO radar. In this way, the covariance matrix of the transmitted waveforms becomes the quadratic form of the weighting matrix, and once the covariance matrix is obtained, the weighting matrix can be easily derived using eigenvalue decomposition. In [25], the problem formulation in [12] and the TBP technique in [20] were combined to formulate a semidefinite programming (SDP) problem, resulting in lower sidelobes in the designed transmit beampattern.

Although many solutions and discussions have been made upon this problem, very little attention has been paid to several aspects of the performance such as the ripple within the energy focusing section, the sidelobe attenuation, the transition bandwidth, and the system efficiency in terms of the choice of design step-size of the angle and the number of transmit antennas. In this paper, a comprehensive discussion of transmit beampattern design for MIMO radar is provided. The major contributions of this paper are noted below:

- A weighted beampattern matching design method based on an \( \ell_1 \)-norm cost function is proposed, which can indirectly control the ripples within the energy focusing section and the transition bandwidth.
- Two closed-form solutions to the \( \ell_2 \)-norm cost function are proposed. They are based on matrix regularization (MR) and singular value decomposition (SVD) respectively. The former is able to indirectly control the ripples and the transition bandwidth.
- We reformulate the transmit beampattern design as a feasibility problem (FP). This formulation is superior to minimization based optimization problems because it has a full control of the ripple levels, the sidelobe levels, the desired power levels, the transition bandwidth, and the number of transmit antennas.
- A MIMO radar beampattern formula for antennas selection is obtained empirically based on the proposed FP. This formula provides an approximation of the number of transmit antennas required for different design specifications.

This paper is organized as follows. Section II introduces MIMO radar signal model, and the matrix form of the transmit beampattern. In Section III, the indirect and direct ripple control in transmit beampattern design is presented. In section IV, we first show the relationship between MIMO radar transmit beampattern design and conventional finite-impulse-response (FIR) filter design, and then reformulate the problem as an FP. The MIMO radar beampattern formula is also formulated. Section V provides the performance analysis using numerical examples. The beampattern formula is then empirically obtained and verified. The conclusions are drawn in Section VI. Note that all the optimization problems formulated without closed-form solutions are solved using public domain optimization tools [32].

Notations: variables and constants are denoted by lowercase and uppercase italic letters respectively. Vectors and matrices are denoted by lowercase and uppercase boldface letters respectively. The identity matrix with appropriate dimension is denoted as \( I \). We use \( \{\cdot\}^T \) as the transpose operator, \( \{\cdot\}^* \) as the conjugate operator, and \( \{\cdot\}^H \) as the transpose conjugate operator. The absolute value and \( \ell_p \)-norm are denoted by \( |\cdot| \) and \( \|\cdot\|_p \) respectively. The trace of a matrix is denoted by \( \text{tr}\{\cdot\} \). The Kronecker product operator is expressed as \( \otimes \), and diagonal matrices are denoted by \( \text{diag}\{\cdot\} \). The operation of stacking columns of a matrix on top of each other is represented as \( \text{vec}\{\cdot\} \).

II. MIMO RADAR SIGNAL MODEL AND TRANSMIT BEAMSPACE PROCESSING

Consider a MIMO radar system with \( N_T \) transmit antennas and \( N_T \) orthogonal waveforms of sample length \( N \). The \( i \)-th waveform sequence is denoted as \( s_i \), where \( i \in \{0,1,\cdots,N_T-1\} \). Let the transmit signal matrix be \( \mathbf{S} \in \mathbb{C}^{N_T \times N} \), then for the omnidirectional transmission,

\[
\mathbf{S} = [s_0, s_1, \cdots, s_{N_T-1}]^T,
\]

where \( \|s_i\|^2 = 1 \) is the elemental power constraint. Suppose the transmit array is a uniform linear array (ULA) with inter element spacing of half the carrier wavelength, and the counterclockwise angle 90° corresponds to the broadside, the steering vector is then expressed as

\[
e^{-j \pi \cos \theta} \cdots e^{-j (N_T-1) \pi \cos \theta}^T,
\]

where \( \theta \) denotes the azimuth angle. The transmit beampattern is defined as the power distribution along different steering directions, i.e.,

\[
P(\theta) = e^T(\theta)\mathbf{R}e^*(\theta),
\]

where

\[
\mathbf{R} = \lim_{N \to \infty} \mathbf{SS}^H
\]

is the covariance matrix of the transmitted waveforms. In this paper, the covariance matrix is approximated using a finite number of samples, and it is assumed that \( \mathbf{R} \approx \mathbf{SS}^H \approx I \).

The original design of MIMO radar transmit beampattern consists of two steps which first optimizes the covariance matrix \( \mathbf{R} \) followed by the corresponding waveform design. In this case, the design of \( N_T \) partially correlated waveforms is needed to achieve the optimized \( \mathbf{R} \), which can be found in [12], [14] and [22]. Instead of transmitting \( N_T \) partially correlated waveforms, the TBP technique [20] introduces a weighting matrix \( \mathbf{W} \in \mathbb{C}^{N_T^2 \times K} \) \( K \leq N_T \) at the transmitter. In this way, only \( K \) orthogonal waveforms are needed for \( N_T \) antennas. Denote the matrix stacking the \( K \) orthogonal waveforms as

\[
\mathbf{S}_{\text{TBP}} = [s_0, s_1, \cdots, s_{K-1}]^T \in \mathbb{C}^{K \times N},
\]
then the transmit signal matrix (1) modifies to \( S = WS_{\text{TBP}} \), and the transmit beampattern becomes

\[
P(\theta) = e^T(\theta)Re^*(\theta) = e^T(\theta)WS_{\text{TBP}}S^H_{\text{TBP}}W^H e^*(\theta) \approx e^T(\theta)WW^H e^*(\theta),
\]

(6)

where \( S_{\text{TBP}}S^H_{\text{TBP}} \approx I \), and the covariance matrix of the transmitted waveforms becomes \( R = WW^H \). It can be seen from (6) that by using TBP, it becomes unnecessary to design partially correlated waveforms. The weighting matrix can be easily obtained via simple matrix manipulations such as eigenvalue decomposition. Another advantage of the TBP technique is that the number of orthogonal waveforms required can be less than that of the transmit antennas, more explicitly [25].

\[
K_{\text{min}} = \text{rank}\{R\}. \tag{7}
\]

Hence the TBP technique simplifies the transmit beampattern design by avoiding complicated design of partially correlated waveforms as well as reducing the number of required waveforms. Note that the TBP technique is irrelevant to the first design step, i.e., optimizing the covariance matrix \( R \). Here we introduce the TBP technique to emphasize its efficiency in the second design step. In addition, the use of TBP technique reveals the possibility that the transmit beampattern design for MIMO radar can be mapped to conventional FIR filter design, which will be illustrated in Section IV, and in this way the FP formulation is proposed. In the following content, we focus on the design of \( R \). We now express the transmit beampattern in matrix form. According to the properties of trace and Kronecker product, (3) can be rewritten in an inner product form:

\[
P(\theta) = e^T(\theta)Re^*(\theta) = \text{tr}\{e^*(\theta)e^T(\theta)R\} = [\text{vec}\{e^*(\theta)e^T(\theta)\}]^H \text{vec}\{R\} = [e^H(\theta) \otimes e^T(\theta)] \text{vec}\{R\}. \tag{8}
\]

Let \( \Delta \theta \) be an appropriately chosen step-size for the angle resolution in the design, then more explicitly we have

\[
\begin{bmatrix}
P(0^\circ) \\
P(\Delta \theta) \\
P(2\Delta \theta) \\
\vdots \\
P(180^\circ)
\end{bmatrix} =
\begin{bmatrix}
e^T(0^\circ) \otimes e^H(0^\circ) \\
e^T(\Delta \theta) \otimes e^H(\Delta \theta) \\
e^T(2\Delta \theta) \otimes e^H(2\Delta \theta) \\
\vdots \\
e^T(180^\circ) \otimes e^H(180^\circ)
\end{bmatrix}
\text{vec}\{R\}, \tag{9}
\]

which represents a set of linear equations. Denote (9) as

\[
p = A \text{vec}\{R\}, \tag{10}
\]

where \( A \in \mathbb{C}^{[(180/\Delta \theta)+1] \times N_k^2} \), then the linear equations corresponding to the angle section(s) where the transmit energy is supposed to be focused can be extracted to form \( p_{\text{in}} = A_{\text{in}} \text{vec}\{R\} \), and those left is \( p_{\text{out}} = A_{\text{out}} \text{vec}\{R\} \). There are various other formulations to the design problem as noted in [13], [20], [21], and the references therein. However, all these formulations can be equivalently represented using (9), because (9) reflects the ultimate objective of the design, namely, the control of the spatial energy level w.r.t. each angle.

### III. The Design of the Covariance Matrix

A standard phased-array radar transmits coherent waveforms. With the elemental power constraint, the covariance matrix is an all-one matrix, thus the transmit beampattern is equivalent to the delay-and-sum response. For a standard MIMO radar transmitting orthogonal waveforms, the covariance matrix is an identity matrix, resulting in a uniform spatial response. The waveform diversity of MIMO radar enables flexible control over the transmit beampattern by generating a covariance matrix between the two extremes.

#### A. Existing Design Formulations

Generally, the design of transmit beampattern for MIMO radar can be considered as a constrained optimization problem. In the literature, two common design categories can be identified, which are the beampattern matching design and the maximum power design. The commonly used cost functions and constraints are noted in the following.

- In beampattern matching design, a desired transmit beampattern, \( P_d(\theta) \), is specified. The desired power level within the focusing angle section is denoted as \( P_0 \), and that outside this region is 0. Usually, \( P_2(\theta) \) is one or several rectangular functions. Similar to \( p_{\text{in}} \) and \( p_{\text{out}} \), we have \( p_{\text{in}} \) and \( p_{\text{out}} \), respectively. The commonly used cost functions in [9]-[13], [21], and [25], which have been argued to be better than others, are

\[
J_1(R) = ||p_d - A \text{vec}\{R\}||_2, \tag{11}
\]

\[
J_2(R) = ||p_d - A \text{vec}\{R\}||_1, \tag{12}
\]

where \( J_2(R) \) is the \( \ell_1 \)-norm version of \( J_1(R) \). Note that in some of the literature, there is a cross term added to \( J_1(R) \), e.g., (19) in [13] and (4) in [21]. This term is irrelevant to the covariance matrix design. Instead, it helps in the second design step to reduce the cross-correlations of the partially correlated waveforms. As we adopt the TBP technique for the design, which uses orthogonal waveforms, the use of the cross term is unnecessary.

- The maximum power design appears in [13] and [20]. The former maximizes the minimum power within the energy focusing section, whereas the latter maximizes the ratio of the focused power and the total power. Note that there is no desired focusing power level involved in the maximum power design technique. Hence the cost functions can be written as:

\[
J_3(R) = -\min \{A_{\text{in}} \text{vec}\{R\}\}, \tag{13}
\]

\[
J_4(R) = ||A_{\text{in}} \text{vec}\{R\}||_1/||A \text{vec}\{R\}||_1, \tag{14}
\]

On one hand \( R \) needs to be a covariance matrix, thus \( R \) is positive semidefinite. On the other hand, to improve the transmission efficiency, the elemental power constraint needs to be satisfied. These two constraints are expressed as

\[
C_1 : \ R \geq 0, \tag{15}
\]

\[
C_2 : \ \text{diag}\{R\} = 1. \tag{16}
\]
It should be noted that in the beamforming design, $C_2$ may lead to failure in achieving $P_0$ within the focusing section of the resulting transmit beampattern. The power levels of the transmit beampattern are implicitly governed by the power assigned to the waveforms (e.g., $C_2$ or other constraints on the power of the waveforms). For a given $N_T$ with $C_2$ satisfying that each waveform has unit power, it is intuitive to expect $P_0$ has an upper limit. If $P_0$ is chosen above this limit, then the desired power focusing level would be beyond that can be achieved by the given waveforms. In this paper, such conflict is illustrated in Section V, but a quantitative analysis, which is worth of further research attention, is not provided. Hence, if $C_2$ incurs problems in the design, it could be relaxed to

$$C_3: \quad \text{tr}\{R\} = N_T$$

(17)

to weaken the conditions on $R$. If the solution still fails to achieve $P_0$, then $C_3$ can be further relaxed in the design, e.g., a larger value could be assigned to $\text{tr}\{R\}$.

### B. Indirect Control of Ripples

1) The Weighted Beampattern Matching Design: It has been illustrated in [12] and [21] that the use of $J_2(R)$ outperforms $J_1(R)$ in terms of the ripple levels within the energy focusing section. Hence we propose a weighted cost function based only on $J_2(R)$, which is expressed as

$$J_2(R, \mu) = \mu \left\| p_{\text{d,in}} - A_{\text{in}} \text{vec}\{R\} \right\|_1 + \left(2 - \mu\right) \left\| p_{\text{d,out}} - A_{\text{out}} \text{vec}\{R\} \right\|_1,$$

(18)

where $0 \leq \mu \leq 2$. The value 2 is chosen to normalize the weight to each term in (18). It can be verified that $J_2(R, \mu) = J_2(R)$, thus $J_2(R)$ is a special case of $J_2(R, \mu)$. The value 2 can be replaced by any arbitrary positive value without altering the cost function. Note that $J_2(R, \mu)$ enables the trade-off between the designing error within and outside the energy focusing section(s). Hence the design becomes more flexible as it can emphasize more on either the energy focusing or the sidelobe reduction. If $\mu \rightarrow 2$, then the ripples within the focusing section would be small. If $\mu \rightarrow 0$, then the sidelobe levels would become small. Simulations in Section V will illustrate these points.

2) Closed-Form Solutions to $J_1(R)$: It has been noted in [9], [10], and [21] that $J_1(R)$ can be considered as a least-squares problem with the following linear equation representation

$$A \text{vec}\{R\} = p_d.$$

(19)

The rows in $A$ have $N_T^2$ elements with $(2N_T - 1)$ distinct values, which are

$$\{e^{j\pi \cos(1-N_T)\theta}, e^{j\pi \cos(2-N_T)\theta}, \ldots, e^{j\pi \cos(N_T-1)\theta}\}.$$ 

Hence if

$$\frac{180}{\Delta \theta} + 1 \geq 2N_T - 1,$$

(20)

which is usually satisfied, then $\text{rank}\{A\} = 2N_T - 1$. In general $A$ is rank deficient. The closed-form solution in [9] suffers from inversion of a rank deficient matrix. In [21] the authors have considered the positive semidefinite constraint and the elemental power constraint into the formulation, which complicates the formulation and hence the solutions.

We propose two closed-form solutions based on matrix regularization (MR) and singular value decomposition (SVD), respectively, without any modification of (19). The positive semidefinite constraint $C_1$ and elemental power constraint $C_2$ are relaxed. The two solutions are given by

$$\text{vec}\{R_{\text{MR}}\} = (A^H A + \lambda I)^{-1} A^H p_d,$$

(21)

$$\text{vec}\{R_{\text{SVD}}\} = V_1 \Sigma_1^{-1} U_1^H p_d + V_2 x,$$

(22)

where $\lambda$ is a regularization parameter, and $x$ is an arbitrary vector. $U_1$, $\Sigma_1$, $V_1$, and $V_2$ are from the SVD of $A$:

$$A = U \Sigma V^H$$

(23)

$$= \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^H,$$

(24)

where $\Sigma_1 \in \mathbb{R}^{(2N_T-1) \times (2N_T-1)}$, and $x$ is an arbitrary vector. We first investigate the similarity and difference between the two solutions, then we discuss how the solutions can be modified to satisfy $C_1$.

The SVD based solution (22) is the general solution using pseudo inverse for least-squares problems. In a general problem, one would use a minimum norm constraint on the variable, resulting in $x = 0$. The MR based solution (21) adds a perturbation parameter $\lambda$ to the pseudo inverse minimum norm solution, and in this way the ripples can be controlled. To illustrate this point, substitute (22) and (24) into the left-hand side of (19), the designed transmit beampattern using (22) is then given by

$$A \text{vec}\{R_{\text{SVD}}\} = U_1 \Sigma_1 V_1^H V_1 \Sigma_1^{-1} U_1^H p_d + U_1 \Sigma_1 V_1^H V_2 x = U_1 H_1 p_d,$$

(25)

where $V_1^H V_2 = 0$ because $V$ is a unitary matrix. Hence, $x$ does not alter the designed beampattern. For the MR based solution (21), it can be verified using (24) that if $\lambda \rightarrow 0$, then designed beampattern approaches (25), i.e., $U_1 H_1 p_d$, and if $\lambda \rightarrow \infty$, then the designed beampattern becomes $A A^H H_1 p_d$. This is how $\lambda$ indirectly controls the ripples.

Although the SVD based solution is not able to control the ripples, we present this solution here to show an advantage of this solution, especially under the unavoidable constraint $C_1$. Note that the solutions (21) and (22) are obtained without considering the constraint $C_1$ (being positive semidefinite), which must be satisfied. It is shown in the appendix that $R_{\text{MR}}$ and $R_{\text{SVD}}$ are Hermitian. However, they may have negative eigenvalues. For the MR based solution (21), the only way to solve this problem is to discard negative eigenvalues of $R_{\text{MR}}$, which may cause severe degradation of performance especially when the negative eigenvalues are large. However for the SVD based solution (22), it may be possible to take the advantage of the variable $x$ to force the eigenvalues of $R_{\text{SVD}}$ being nonnegative. Let $R_H$ be an arbitrary Hermitian matrix, then the problem of determining $x$ after obtaining (22) is stated...
as
\[
\begin{align*}
\text{find} & \quad \mathbf{x}, \mathbf{R}_H \\
\text{s.t.} & \quad \mathbf{R}_H = \mathbf{R}_H^H, \\
& \quad \text{vec} \{\mathbf{R}_H\} = \mathbf{V}_2 \mathbf{x}, \\
& \quad \mathbf{R}_{SVD} \succeq 0.
\end{align*}
\] (26)

As this paper mainly focuses on the ripple and transition band control in the transmit beampattern design, the solution to the above problem is not discussed here. The problem (26) is noted here for the future research as well as for the completeness of the solution (22).

In this subsection, two methods to indirectly control the ripples have been discussed. The control is indirect because there are no explicit constraints or objective functions to determine the ripple levels. Although the ripples can be affected by tuning the newly introduced parameters \(\mu\) or \(\lambda\), quantitative assessment of ripple levels is not known before the solution is obtained. Although the SVD based closed-form solution is unable to indirectly control the ripples, its relation to the MR solution and the existence of the arbitrary parameter \(\mathbf{x}\) is interesting and worth of further investigation.

### C. Direct Control of Ripples

Additional constraints can be imposed to explicitly control the ripple levels within the energy focusing section. The formulations using such constraints are referred to as direct control. Let the maximum tolerable ripple level be \(\delta\), then the additional constraints can be formulated as follows. For beampattern matching design, the ripple levels depend on \(P_0\), thus we set
\[
\begin{align*}
C_4 : \quad & \max \{\mathbf{A}_{in} \text{vec}\{\mathbf{R}\}\} \leq P_0 + 0.5\delta, \\
& \min \{\mathbf{A}_{in} \text{vec}\{\mathbf{R}\}\} \geq P_0 - 0.5\delta.
\end{align*}
\] (27)

For maximum power design, the peak level is independent of \(P_0\), thus the ripple constraint can be set as
\[
C_5 : \quad \max \{\mathbf{A}_{in} \text{vec}\{\mathbf{R}\}\} - \min \{\mathbf{A}_{in} \text{vec}\{\mathbf{R}\}\} \leq \delta.
\] (28)

Note that \(C_4\) and \(C_5\) can be used with any cost function. For \(p_d\)-dependent cost functions, \(C_4\) and \(C_5\) are identical. For \(p_x\)-independent cost functions, the use of \(C_4\) will modify the design to a \(p_{d^2}\)-dependent case. The relationship between the ripple level and the transition bandwidth can be explained using correlation properties. It can be seen from (9) that the adjacent rows in \(\mathbf{A}\) are correlated because of the small variation \(\Delta \theta\). If \(\delta\) is small, then \(\text{vec}\{\mathbf{R}\}\) has nearly identical Hermitian angles [33] with the rows corresponding to the passband. This will make the Hermitian angles between \(\text{vec}\{\mathbf{R}\}\) and the adjacent rows near the passband quite close to each other, resulting in a wide transition band. On the contrary, if \(\delta\) is increased, the Hermitian angles between \(\text{vec}\{\mathbf{R}\}\) and stopband rows increase, so that the transition bandwidth would be shortened.

### D. Convexity

In the previous subsections, 5 cost functions and 5 constraints have been reviewed or proposed. Different combinations of the cost functions and constraints form different constrained optimization problems. To ensure that the problems formulated are efficiently solvable, the cost functions and constraints need to satisfy convexity. Here the convexity is briefly discussed and details can be found in [34]. Let \(\text{Re}\{\cdot\}\) and \(\text{Im}\{\cdot\}\) denote the real and imaginary parts of a complex variable, then it can be shown that
\[
\begin{align*}
J_1^2(\mathbf{R}) &= \left\| \text{Re} \{\mathbf{p}_d\} - \begin{bmatrix} \text{Re}\{\mathbf{A}\} \\ -\text{Im}\{\mathbf{A}\} \end{bmatrix}^T \begin{bmatrix} \text{Re}\{\text{vec}\{\mathbf{R}\}\} \\ \text{Im}\{\text{vec}\{\mathbf{R}\}\} \end{bmatrix} \right\|^2_2 \\
&+ \left\| \text{Im} \{\mathbf{p}_d\} - \begin{bmatrix} \text{Im}\{\mathbf{A}\} \\ \text{Re}\{\mathbf{A}\} \end{bmatrix}^T \begin{bmatrix} \text{Re}\{\text{vec}\{\mathbf{R}\}\} \\ \text{Im}\{\text{vec}\{\mathbf{R}\}\} \end{bmatrix} \right\|^2_2,
\end{align*}
\] (29)

which is a convex function. Similarly, \(J_2(\mathbf{R})\) and \(J_5(\mathbf{R}, \mu)\) are convex. For \(J_3(\mathbf{R})\), \(C_4\), and \(C_5\), the portion \(\mathbf{A}_{in} \text{vec}\{\mathbf{R}\}\) can be verified to be an affine function by substituting real and imaginary parts into \(\mathbf{A}_{in}\) and \(\text{vec}\{\mathbf{R}\}\). Hence \(J_3(\mathbf{R})\), \(C_4\), and \(C_5\) are convex. \(C_1\), \(C_2\), and \(C_3\) are well-formed constraints that can be used with any of the convex cost functions. Hence combinations of the above cost functions and constraints can be efficiently solved using the tools (e.g., the solver SeDuMi or SDPT3) provided in [32]. Note that \(J_4(\mathbf{R})\) is not convex, but an efficient solution via eigenvalue decomposition for \(J_4(\mathbf{R})\) with \(C_1\) and \(C_2\) are provided in [20]. In this paper, we mainly use the tools in [32] to solve the optimization problems. We therefore emphasize more on appropriate convex problem formulations and hence the computational complexity analysis for the solutions is not provided.

### IV. Covariance Matrix Design as a Feasibility Problem

#### A. Motivation

The design of the transmit beampattern for phased-array radar with ULA transmitter can be related to the design of digital finite-impulse-response (FIR) filters, where the tapped delay line is replaced by spatial delay, and the filter length is corresponding to the number of transmit antennas [23], [24]. The MIMO radar transmit beampattern design based on TBP can then be considered as a generalization of conventional filter design, which is a multiple-input-single-output (MISO) filter. This is depicted in Fig. 1, where \(\tau = e^{j\pi \cos \delta}\), and the filter coefficient \(w_{m,k}\) is the \((m,k)\)th element of the weighting matrix \(\mathbf{W}\). According to (6), the transmit beampattern is in fact the energy density spectrum resulting from the filter coefficients. Another observation is that the positive semidefinite constraint and the power constraint are required in MIMO radar design, but they are unnecessary in conventional FIR filter design.

It is well known that in a typical FIR filter design problem, the passband and stopband ripples and the transition bandwidth are key parameters to characterize the performance of the filter.
Reducing the ripples and the transition bandwidth results in increasing the filter length. For a set of given specifications, an empirical formula (Kaiser’s formula) provides an approximation of the filter length [35]. As noted earlier in the context of MIMO radar transmit beampattern design, the ripple control has seldom been considered. This motivates us to study the ripple control performance and its relationship with the transition bandwidth in MIMO radar transmit beampattern design. The ripples exist due to the truncation of the ideal impulse response, which is referred to as the Gibbs phenomenon [35]. For a radar system, the filter length, i.e., the number of transmit antennas is limited, thus it is intuitive to expect that if the number of transmit antennas is fixed, then the ripples will result in larger transition bandwidth. If the number of transmit antennas can be adjusted, then there exists a trade-off among the ripple levels, transition bandwidth, and the number of transmit antennas. In this paper, such a trade-off will be quantitatively investigated using an expression similar to Kaiser’s formula.

B. Design Reformulated as a Feasibility Problem

In the previous sections, \( \mathbf{A} \) is partitioned into \( \mathbf{A}_{\text{in}} \) and \( \mathbf{A}_{\text{out}} \), indicating a design with zero transition bandwidth. In this section, a transition band \( \Delta B \) is introduced. For simplicity, the desired beampatterns are chosen symmetrical w.r.t. 90°. Let the passband edges be \( \theta_p \) (90° < \( \theta_p \) < 180°) and 180° - \( \theta_p \), thus the stopband edges are \( \theta_S \equiv \theta_p + \Delta B \) and 180° - \( \theta_S \). The rows in \( \mathbf{A} \) which correspond to the passband and stopband are now denoted as \( \mathbf{A}_p(\theta_p) \) and \( \mathbf{A}_s(\theta_S) \) respectively. Because \( \theta_S \equiv \theta_p + \Delta B \), \( \mathbf{A}_s(\theta_S) \) is a function of both \( \theta_p \) and \( \Delta B \), and therefore denoted as \( \mathbf{A}_s(\theta_p, \Delta B) \). Let the stopband attenuation be \( \varepsilon \), then the FP is formulated as

\[
\begin{align*}
\min \{ \mathbf{A}_p(\theta_p) \mathbf{vec}(\mathbf{R}) \} & \geq P_0 - 0.5\delta, \\
\max \{ \mathbf{A}_s(\theta_p, \Delta B) \mathbf{vec}(\mathbf{R}) \} & \leq \varepsilon.
\end{align*}
\]

Let \( N_{\text{bot}} \) denote the initial estimate of the minimum number of transmit antennas, and \( N_{\text{top}} \) denote the maximum number of transmit antennas allowed in the radar system, then the algorithm to find the number of transmit antennas is summarized as follows:

**Step 0:** Set designing parameters: \( \theta_p, \Delta B, \delta, P_0 \), and \( \varepsilon \).

**Step 1:** Start at initial value \( N_T \leftarrow N_{\text{bot}} \).

**Step 2:** Obtain the solution to (30) using the tools in [32].

**Step 3:** If the problem is not feasible, then \( N_T \leftarrow N_T + 1 \), and go to **Step 4**. Else if the problem is feasible, then the problem is solved, and the \( \mathbf{R} \) obtained is the optimal solution.

**Step 4:** If \( N_T \leq N_{\text{top}} \), go to **Step 2**. Else if \( N_T > N_{\text{top}} \), then the problem cannot be solved, \( N_{\text{top}} \) should be increased.

Note that the FP formulation depends on \( P_0 \). If \( P_0 \) is chosen inappropriately, then the elemental power constraint or trace constraint will conflict with \( P_0 \), and the design will fail. An alternative formulation independent of \( P_0 \) can be easily obtained by subtracting the 3rd and 4th constraints of (30) if necessary, i.e.,

\[
\max \{ \mathbf{A}_p(\theta_p) \mathbf{vec}(\mathbf{R}) \} - \min \{ \mathbf{A}_p(\theta_p) \mathbf{vec}(\mathbf{R}) \} \leq \delta.
\]

C. The MIMO Radar Beampattern Formula

In conventional FIR filter designs, the empirical formula obtained by Kaiser [35] provides an efficient way to determine the approximate filter length required to achieve the ripples and transition bandwidth requirements. As shown in Fig. 1, the design of transmit beampattern for MIMO radar is a generalized MISO FIR filter design problem with the tap delay nonlinearly mapped into propagation delay. In this case, the estimation of the number of transmit antennas (filter length) is a more complicated problem which cannot be easily solved. In this section, we establish a relationship between the estimated number of transmit antennas and other system parameters, to which we refer as the MIMO radar transmit beampattern formula. The parameters involved in the design are \( N_T \), \( \theta_p \), \( \Delta B \), \( \delta \), \( \varepsilon \), \( P_0 \), and \( \Delta \theta \).

1) **The Nonlinear Mapping:** According to Fig. 2, the relationship between the conventional FIR filter and MIMO radar transmit beampattern is that \( e^{-j\pi 2f} \) is mapped to \( e^{j\pi \cos \theta} \).

Because such mapping is nonlinear, the locations of \( \theta_p \) and \( \theta_S \) affect the design, while in Kaiser’s formula, the design is independent of the locations of passband and stopband edges. The parameters mapped to an equivalent FIR filter are as follows: passband edge: 0.5 \(|\cos \theta_p|\), stopband edge: 0.5 \(|\cos \theta_S|\), transition bandwidth: 0.5 \(|\cos \theta_p| \pm |\cos \theta_S|\). The estimated number of transmit antennas is denoted as \( N_T(\theta_p, \Delta B, \delta, \varepsilon, P_0) \).

Note that \( \theta_S = \theta_p + \Delta B \). In Kaiser’s formula the passband and stopband edges do not affect the estimation of the filter length. Instead, the transition bandwidth does, which appears in its denominator [35]. However, due to the nonlinear mapping from filter tap delay to antenna propagation delay, the locations of the passband and stopband edges become nontrivial. To eliminate the effect of the locations of passband and stopband edges in the beampattern formula, we set \( \theta_p = 90° \) in the
initial design. In fact, any other value of $\theta_p \in (90^\circ, 180^\circ)$ can be chosen for the initial design. Using the nonlinear mapping, it can be shown that $\hat{N}_T(120^\circ, \Delta B, \delta, \varepsilon, P_0)$ is equal to the estimation of the number of transmit antennas for any arbitrary location $\theta_p, \Delta B$, with an approximately chosen $\Delta B$, i.e.,

$$\hat{N}_T(\theta_p, \Delta \tilde{B}, \delta, \varepsilon, P_0) = \hat{N}_T(120^\circ, \Delta B, \delta, \varepsilon, P_0).$$  \hspace{1cm} (32)

Equating the equivalent transition bandwidth by letting $|\cos(\tilde{\theta}_p + \Delta \tilde{B})| - |\cos \tilde{\theta}_p| = |\cos(120 + \Delta B)| - |\cos 120|$, we have

$$\Delta B = \arccos \left\{ \cos(\tilde{\theta}_p + \Delta \tilde{B}) - \cos \tilde{\theta}_p + \cos 120^\circ \right\} - 120^\circ. \hspace{1cm} (33)$$

In other words, if the design is for passband edge at $\tilde{\theta}_p$ and transition bandwidth of $\Delta \tilde{B}$, then the estimated number of antennas is equivalent to the estimation using $\theta_p = 120^\circ$ and $\Delta B$ evaluated via (33).

2) Kaiser’s Formula Like Expression: The properties observed in Kaiser’s formula [35] are also applicable for the beampattern formula, i.e.,

$$\hat{N}_T \propto \log_{10}(\delta \varepsilon), \hspace{1cm} (34)$$

and

$$\hat{N}_T \propto \frac{1}{\Delta B}. \hspace{1cm} (35)$$

The step-size $\Delta \theta$ for the angles in the design corresponds to the spectral resolution of an FIR filter frequency response. If it is chosen too large, then the beampattern may not be able to preserve the energy between two consecutive design angles. If it is chosen small, then the complexity of the design increases as the dimension of $A$ increases. For simplicity, we set $\Delta \theta = 1^\circ$ in the design. The estimated number of transmitted antennas can then be expressed as

$$\hat{N}_T(120^\circ, \Delta B, \delta, \varepsilon, P_0) \approx a f(P_0) \log_{10}(\delta \varepsilon) + b \frac{c}{\Delta B}, \hspace{1cm} (36)$$

which indicates that the MIMO radar transmit beampattern formula has the form of the Kaiser’s formula, with the inclusion of an additional variable $P_0$. It will be shown in Section V D that the function $f$ and the coefficients $a$, $b$, and $c$ are empirically obtained via simulations.

V. PERFORMANCE ANALYSIS WITH NUMERICAL EXAMPLES

In this section, the advantages of the proposed cost function $J_5(R, \mu)$ is first illustrated. Then the performances of the closed-form solutions presented in Section III are compared and analyzed. After that, the performances considering the direct ripple control are investigated by comparing different combinations of cost functions and constraints. All the optimization problems without closed-form solutions are solved using public domain optimization tools [32]. The parameters for simulations are set as follows: $N_T = 10$, $\Delta \theta = 1^\circ$, and the angle section of energy focusing is $\Theta_1 = [60^\circ, 120^\circ]$. Note that in the literature, the design of multiple beams has been considered. However, this can also be achieved via the individual design of each beam and then appropriately adding them together. Hence it is sufficient to discuss only the single beam design. $C_1$ is considered as a global constraint hence omitted. The simulations to obtain $a$, $b$, $c$ and $f$ for the beampattern formula are provided at the end of this section.

A. Performances of Indirect Ripple Control

The transmit beampatterns resulting from various methods without the ripple control constraints are shown in Fig. 3. Fig. 3(a) compares the proposed method using $J_5(R, \mu)$, with two popular designs using $J_1(R)$ and $J_2(R)$. It is indicated that different choices of $\mu$ put different weights on the error within and outside the energy focusing section. If $\mu < 1$, then minimizing the sidelobes is more important than minimizing the error within the focusing section, and vice versa if $\mu > 1$. The the beampattern designed using $J_5(R, 1.5)$ appears to be the best solution as it has minimum passband ripples and satisfactory sidelobes observed from Fig. 3(a) and (b). Note that although the solution using $J_5(R, 0.5)$ results in large ripples, it has minimum sidelobes as shown in Fig. 3(b).

The failure of the design is illustrated in Fig. 3(c), where $P_0(=30)$ is inappropriately chosen under the constraint $C_3$. As mentioned at the end of Section III A, the value of $P_0$ should not be chosen beyond the capability of the transmitter where the total transmit energy is constrained by $N_T = 10$. It can be seen that the solutions of $J_1(R)$ and $J_2(R)$ can only have power level around 20. A few points in the beampatterns designed using $J_5(R, \mu)$ are reaching 30. Generally, all the methods presented in Fig. 3(c) fail the design. This problem can be solved via the relaxation of $C_3$, which is shown in Fig. 3(d). It can also be solved by increasing the total transmit energy, which is straightforward and not presented via simulations. A further observation with $J_5(R, \mu)$ is shown in Fig. 3(e), where it is seen that the ripple is small when $\mu$ approaches 2, while the sidelobes are relatively high. The ripple is large when $\mu$ approaches 0, while the sidelobes become relatively low. However, such a trade-off is empirical and the ripple control is not considered in the design formulations.

The optimized transmit beampatterns based on maximum power design are shown in Fig. 3(f), where the solution to
B. Performances of Direct Ripple Control

\( J_3(\mathbf{R}) \) is obtained using [32], and the solution to \( J_3(\mathbf{R}) \) is based on the eigenvalue decomposition of \( \sum_\theta \mathbf{e}(\theta)\mathbf{e}(\theta)^H \), where \( \mathbf{e}(\theta) \) is the steering vector as defined in (2). Note that in maximum power design, the peak level of the power is independent of \( P_0 \). Comparing Fig. 3(f) with Fig. 3(a), it can be seen that the solution to \( J_1(\mathbf{R}) \) has a similar performance with the solution to \( J_3(\mathbf{R}, 0.5) \), although the former is not tunable. The solution to \( J_1(\mathbf{R}) \) has a wider mainbeam with lower energy focusing levels. In addition, the two methods have similar sidelobe levels.

The performances of the closed-form solutions to \( J_1(\mathbf{R}) \) are shown in Fig. 4. The iterative solution to \( J_2(\mathbf{R}) \) is also provided for comparison. It can be seen that both solutions have certain degradation. The SVD based solution has larger ripples and higher sidelobes, whereas the MR based solution has slightly less focused power compared to the \( J_2(\mathbf{R}) \) solution. In general, both solutions are quite satisfactory. Fig. 4(c) shows the beampatterns versus different values of \( \lambda \). According to (22) and (43), if \( \lambda = 0 \) and \( x = 0 \), then \( \mathbf{R}_{\text{MR}} = \mathbf{R}_{\text{SVD}} \). Hence the advantage of \( \mathbf{R}_{\text{MR}} \) is that it can control the ripples via \( \lambda \), albeit indirectly. The advantage of \( \mathbf{R}_{\text{SVD}} \) is that the variable \( x \) may be chosen as noted in (26) to force \( \mathbf{R}_{\text{SVD}} \) being positive semidefinite to avoid the degradation caused by discarding negative eigenvalues.

B. Performances of Direct Ripple Control

It is shown using Fig. 3 and Fig. 4 that none of the existing methods is able to control the passband ripples, whereas the proposed method using cost function \( J_3(\mathbf{R}, \mu) \), and the closed-form solution \( \mathbf{R}_{\text{MR}} \) allow ripple control. However, the ripple control is indirect, i.e., we empirically change the values of \( \mu \) and \( \lambda \) without knowing the effect on the ripple levels of the resultant beampatterns. It should be noted that the solution to \( J_2(\mathbf{R}) \) appears to have very small ripples, but if \( \Theta_1 \) and \( P_0 \) are changed, the resultant ripples are not guaranteed to be small. Hence the use of constraints \( C_4 \) and \( C_5 \) is necessary, so that a certain level of ripple tolerance can be allowed in the design.

Fig. 5 provides a few examples of the transmit beampattern design using ripple control constraints \( C_4 \) and \( C_5 \). \( C_4 \) is used for beampattern matching design and \( C_5 \) is used for maximum power design. It can be seen from Fig. 5(a) that the ripple levels within the passband are perfectly constrained within \( \delta \). However, this is achieved by compromising the transition bandwidth. Since there can be some tolerance within the passband, we can reduce the passband width accordingly to adjust the beampattern. This is shown in Fig. 5(b). Note that although \( \Theta_1 \) is modified to range \([68^\circ, 112^\circ]\) for the solution, the desired passband is still \([60^\circ, 120^\circ]\). The optimized beampatterns from the maximum power design using \( J_3(\mathbf{R}) \) with the trace constraint \( C_3 \) are shown in Fig. 5(c). We observe that the trade-off between the passband ripples and transition bandwidth can be achieved when the total energy constraint is imposed.

In general, the proposed cost function, closed-form solutions, as well as the ripple level constraints enable flexible
design of the transmit beampattern. The ripple levels can be controlled indirectly (via tuning \( \mu \) or \( \lambda \)) or directly (via tuning \( \delta \)), and as a result one can freely choose the constraints and parameters for various design criteria such as minimum sidelobes, minimum passband ripples, narrowest transition bandwidth, etc. The design can be easily extended to multiple-passband beampatterns.

### C. Feasibility Problem Solution Examples

The simulation results using the proposed FP formulation are shown in Fig. 6. The design parameters are set as follows: \( \theta_p = 120^\circ \), \( \Delta B = 10^\circ \), \( \delta = \varepsilon = 0.1 \), and \( \Delta \theta = 1^\circ \). To compare the results with the methods reviewed and proposed in Section III, we first solve the FP formulation to obtain the minimum number of antennas \( N_T \). Using \( N_T \), we then perform the optimization formulated by \( J_2 \) under two situations: 1) The simulation result using \( J_2 \) and \( C_4 \) where \( P_0 = 25 \) without \( C_3 \) is shown in Fig. 6 (a). The resultant trace of the covariance matrix is 14.3068, and \( N_T = 21 \). 2) The simulation result using \( J_2 \), \( C_5 \), and \( C_3 \), without \( P_0 \), is shown in Fig. 6 (b), where the maximum power focusing level is 40.33, and \( N_T = 23 \). These figures also indicate the possibility of improving the design using FP formulation upon the cost function based optimizations. It can be seen that the FP solutions have extra sidelobe suppression than the direct control \( l1 \)-norm cost function based solutions. Another observation is that the sidelobes of the FP solution do not decay away from the mainlobe, but the sidelobes using cost function \( J_2(R) \) do. This is because the sidelobe attenuation constraint in the FP formulation (30) only aims to suppress the sidelobes below \( \varepsilon \), while the cost function formulation \( J_2(R) \) minimizes the errors, resulting in the sidelobes approaching the desired value, i.e., 0, along the angle axis away from the mainlobe. The sidelobes for FP solution are strictly bounded under \(-25 \) dB, which is more preferable for rejection of echoes from targets existing outside the energy focusing region at the receiver. Moreover, the FP solution method is more flexible, because \( \Delta B \) and \( \varepsilon \) can be tuned in the design, which is not possible in any other method discussed in Section III. Hence it is indicated that the FP method yields the optimal solution which satisfies all the design constraints in a single iteration of the algorithm.

### D. Beampattern Formula Verification

To obtain the beampattern formula, we use \( P_0 \)-dependent constraints and exclude the constraints on the signal power, i.e., \( C_2 \) and \( C_3 \), in the FP formulation (30). This is because the normalized ripple level depends on the desired peak level \( P_0 \) in the energy focusing section, which should be known a prior. The exclusion of \( C_2 \) and \( C_3 \) also avoids their conflict with \( P_0 \). Note that the values of \( P_0 \) and \( \varepsilon \) together determine

Fig. 4. Optimized beampatterns with indirect ripple control using closed-form solutions to \( J_2(R) \), \( P_0 = 20 \), \( x = 0 \). All the traces are normalized to \( N_T \). The solution of \( \{ J_2, C_3 \} \) is provided for comparison. (a) Comparison between iterative solution and closed-form solutions. (b) The dB view of (a). (c) Comparison between MR solutions with different choices of \( \lambda \) and the SVD solution.

Fig. 5. Optimized beampatterns with direct ripple control using closed-form solutions to \( J_2 \), \( \theta_1 \) under different values of \( P_0 \). \( \delta = 0.1 \). (b) The same as (a) except \( \theta_1 \) is modified to \([68^\circ, 112^\circ]\). (c) Maximum power design using \( J_3(R) \) under different values of \( \delta \), where \( \theta_1 \) is modified to \([65^\circ, 115^\circ]\).
the sidelobe attenuation in dB. Fig. 7 shows the simulation results and their approximations to obtain the MIMO radar beampattern formula. From Fig. 7 (a) we observe the logarithmic relationship between \( N_T \) and \( P_0 \), thus (36) can be modified as

\[
N_T(120^\circ, \Delta B, \delta, \varepsilon, P_0) \approx \frac{a(\log_{10}(\varepsilon)) (\log_{10} P_0) + b}{c\Delta B}.
\]

Except for the new parameter \( P_0 \), we observe from Fig. 7 (b) and (c) that the properties of Kaiser’s formula are preserved in MIMO radar scenario, where \( \Delta B \) is inversely proportional to \( N_T \), and \( \log_{10}(\delta \varepsilon) \) is linear related to \( N_T \). According to the figures, \( a, b, \) and \( c \) can be related using the following approximations, i.e.,

\[
\begin{align*}
-2a \log_{10} P_0 + b & \approx 8 \log_{10} P_0 + 11, \\
-2a \log_{10} 30 + b & \approx 220, \\
c \Delta B & = 8 \log_{10}(\delta \varepsilon) + b, \\
10c & \approx -4.2 \log_{10}(\delta \varepsilon) + 14.
\end{align*}
\]

(38)

The above approximated equations are solved empirically, and the coefficients are approximated as

\[
\begin{align*}
a & \approx -34.22c, \\
b & \approx 128.7c.
\end{align*}
\]

(39)

Substituting (39) into (37), the beampattern formula is expressed as

\[
N_T(120^\circ, \Delta B, \delta, \varepsilon, P_0) \approx -34.22 (\log_{10}(\delta \varepsilon)) (\log_{10} P_0) + 128.7. \\
\]

(40)

The experiment results verifying this formula are shown in Table I, where \( P_0 = 50 \). The first two rows show that changing \( \delta \) and \( \varepsilon \) will not affect the filter length as long as \( \delta \varepsilon \) is constant. The 3rd and 4th rows compare the results using the nonlinear mapping equation (33) when \( \theta_T = 100^\circ \neq 120^\circ \). The last two rows show the design with more strict conditions, where \( \Delta B = 8^\circ \) and \( \varepsilon = 0.001 \). The nonlinear mapping is again verified by these two examples. We observe slight mismatch between the estimated value and the true value, but this is acceptable as Kaiser’s formula also has such mismatch. Generally, it can be seen from the last two columns that the estimated number of transmit antennas is quite close to the true values.

**VI. CONCLUSIONS**

The transmit beampattern design for MIMO radar has been receiving much attention in the recent decade. However, the solutions proposed in the literature have paid little attention to several aspects of the performance such as the ripple control, transition bandwidth issue, the sidelobe attenuation, the angle step-size of the design, and the minimum number of transmit antennas, etc. This paper first propose several methods for indirect and direct control of the passband ripples. More importantly, it is then shown that the MIMO radar transmit beampattern design is similar to, but more complicated than traditional low-pass FIR filter design problem. Hence we reformulate this problem as an FP problem, which is able to design the beampattern with quantitative specifications of

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**TABLE I**

**MIMO RADAR BEAMPATTERN FORMULA VERIFICATIONS**

<table>
<thead>
<tr>
<th>( \theta_T )</th>
<th>( \Delta B )</th>
<th>( \delta )</th>
<th>( \varepsilon )</th>
<th>( N_T )</th>
<th>( N_T )</th>
</tr>
</thead>
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<td>10°</td>
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<td>0.01</td>
<td>30.3116</td>
</tr>
<tr>
<td>120°</td>
<td>10°</td>
<td>10°</td>
<td>0.1</td>
<td>0.1</td>
<td>30.3116</td>
</tr>
<tr>
<td>100°</td>
<td>10°</td>
<td>11.9415°</td>
<td>0.1</td>
<td>0.1</td>
<td>20.5148</td>
</tr>
<tr>
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<td>12°</td>
<td>0.1</td>
<td>0.1</td>
<td>20.4148</td>
</tr>
<tr>
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<td>8.0710°</td>
<td>0.1</td>
<td>0.001</td>
<td>44.7496</td>
</tr>
<tr>
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<td>8°</td>
<td>8°</td>
<td>0.1</td>
<td>0.001</td>
<td>45.1569</td>
</tr>
</tbody>
</table>
the passband ripple $\delta$, stopband attenuation $\varepsilon$, and transition bandwidth $\Delta B$. Moreover, with the FP formulation, we then empirically obtain the MIMO radar beampattern formula, which is similar to Kaiser’s formula for FIR filter length estimation. The design angle step-size is set to $\Delta \theta = 1^\circ$ in the simulations. In fact, if the design criteria are not so strict, then $\Delta \theta$ can be increased to reduce the dimension of $\mathbf{A}$. On the other hand, if the design requires strict ripple control, substantial sidelobe attenuation and very narrow transition bandwidth, then $\Delta \theta$ needs to be further reduced. Hence, the MIMO radar beampattern formula as a function of $\Delta \theta$ is worth of further investigation. In addition, there are several other issues that are worth of further investigation, i.e., the quantitative assessment of the conflict between the constraint $C_2$ and power level $P_0$, the performance analysis of the free variable $\mathbf{x}$ in (22), and the computational complexity analysis for all the optimization problems involved in this paper.

**APPENDIX**

**Proof of $\tilde{\mathbf{R}}_{MR}$ and $\tilde{\mathbf{R}}_{SVD}$ Being Hermitian**

Proof: According to (9), the rows in $\mathbf{A}$, i.e., $\mathbf{e}^T(\theta) \otimes \mathbf{e}^H(\theta)$, $\theta \in \{0^\circ, \Delta \theta, \cdots, 180^\circ\}$, can be represented as

$$
\mathbf{e}^T(\theta) \otimes \mathbf{e}^H(\theta) = \{ \text{vec}[\mathbf{e}^*(\theta)\mathbf{e}^T(\theta)] \}^T,
$$

(41)

where, $\mathbf{e}^*(\theta)\mathbf{e}^T(\theta)$ is a Hermitian matrix. Hence each column in

$$
\mathbf{A}^H = \mathbf{V} \Sigma^H \mathbf{U}^H
$$

(42)

is the vectorization of a Hermitian matrix. Substituting (23) into (21), we have

$$
\text{vec} \{ \tilde{\mathbf{R}}_{MR} \} = (\mathbf{A}^H \mathbf{A} + \lambda \mathbf{I})^{-1} \mathbf{A}^H \mathbf{p}_d
$$

$$
= (\mathbf{V} \Sigma^H \mathbf{U}^H \mathbf{U} \Sigma \mathbf{V}^H + \lambda \mathbf{I})^{-1} \mathbf{V} \Sigma^H \mathbf{U}^H \mathbf{p}_d
$$

$$
= \left[ \mathbf{V} \left( \Sigma^H \Sigma + \lambda \mathbf{I} \right) \mathbf{V}^H \right]^{-1} \mathbf{V} \Sigma^H \mathbf{U}^H \mathbf{p}_d
$$

$$
= \mathbf{V} \left( \Sigma^H \Sigma + \lambda \mathbf{I} \right)^{-1} \Sigma^H \mathbf{U}^H \mathbf{p}_d.
$$

(43)

Note that (42) can be considered as a set of linear operations $(\Sigma^H \mathbf{U}^H)$ on the columns of $\mathbf{V}$, yielding the columns in $\mathbf{A}^H$ to be the vectorization of Hermitian matrices. In (43), $\Sigma^H \mathbf{U}^H$ performs the same column operations as in (42) on $\mathbf{V} \left( \Sigma^H \Sigma + \lambda \mathbf{I} \right)^{-1}$ instead of $\mathbf{V}$. Note that $(\Sigma^H \Sigma + \lambda \mathbf{I})^{-1}$ is a diagonal matrix which only scales the columns of $\mathbf{V}$. Hence all the columns in $\mathbf{V} \left( \Sigma^H \Sigma + \lambda \mathbf{I} \right)^{-1} \Sigma^H \mathbf{U}^H$ are the vectorization of Hermitian matrices, and any linear combinations (any $\mathbf{p}_d$) of them will result in a vector that can be reordered as a Hermitian matrix, i.e., $\tilde{\mathbf{R}}_{MR}$ is Hermitian. $\tilde{\mathbf{R}}_{SVD}$ can be easily verified to be Hermitian in the same way when $\mathbf{x} = \mathbf{0}$. ■

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