Delay-Dependent Filtering for Discrete-Time Systems with Finite Frequency Small Gain Specifications

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Abstract—This paper is concerned with the delay-dependent filtering problem for linear discrete-time multi-delay systems with small gain conditions in finite frequency ranges. A new multiplier method is developed to convert the resulting non-convex filtering synthesis conditions to the ones based on linear matrix inequalities (LMIs). Thus, sufficient conditions for the existence of feasible filters are given in terms of solutions to a set of LMIs. Finally, the procedures and advantages of the proposed approach in comparison with existing ones in the entire frequency range are illustrated via a numerical example.

I. INTRODUCTION

The topic of $H_\infty$ filtering for time-delay systems is a research subject of great practical and theoretical significance, which has received considerable attention in the past decades. It is known that the delay-dependent conditions are generally less conservative than the delay-independent ones, especially when delays are small. Therefore, recently, great interest has been devoted to delay-dependent $H_\infty$ filtering problems in various settings, such as continuous-time systems [1]-[7], and discrete-time systems [8]-[11]. To mention a few for various settings, such as continuous-time systems [1]-[7], and discrete-time systems [8]-[11].

It should be noticed that the above-mentioned results were given mainly based on time-domain methods from the viewpoint of the entire frequency domain. However, various engineering design problems can be formalized in terms of specifications expressed by frequency domain inequalities (FDIs). In particular, each design specification is often given not for the entire frequency range but rather for a certain frequency range of relevance. Recently, many results for systems with finite frequency specifications have been reported in the literature; see [12]-[18] and the references therein.

The objective of this paper is to design delay-dependent filters for discrete-time multi-delay such that the filtering error systems are asymptotic stable and satisfy small gain conditions in finite frequency ranges. To our knowledge, this problem has not been addressed in the literature. Here, we use the idea in [19] and Finsler’s Lemma to introduce additional slack multipliers into our analysis conditions, such that a kind of decoupling for the system matrices is exhibited. Then, a new multiplier method is developed to render the multi-objective problem convex. The main contribution of this paper is that sufficient conditions for the existence of feasible filters are given in terms of solutions to a set of LMIs.

The paper is organized as follows. Section II gives the problem statement and some preliminaries, and Section III presents the filtering analysis and design. In Section IV, a numerical example is proposed to illustrate the design procedure and their effectiveness. Some concluding remarks are shown in Section V.

The following notation is used throughout this paper. For a matrix $A$, its complex conjugate transpose is denoted by $A^*$. The Hermitian part of a square matrix $A$ is denoted by $\text{He}(A) := A + A^*$. The symbol $\mathbf{H}_n$ stands for the set of $n \times n$ Hermitian matrices. $I$ denotes the identity matrix with an appropriate dimension. For matrices $\Phi$ and $P$, $\Phi \otimes P$ means the Kronecker product. $\sigma_{\text{max}}(G)$ represents the maximum singular value of $G$. For matrices $G \in \mathbb{C}^{n \times m}$ and $\Pi \in \mathbb{H}_{n+m}$, a function $\sigma : \mathbb{C}^{n \times m} \times \mathbb{H}_{n+m} \rightarrow \mathbb{H}_m$ is defined by

$$\sigma(G, \Pi) := \begin{bmatrix} G & \Pi \\ I_m & I_m \end{bmatrix} .$$

II. PROBLEM STATEMENT AND PRELIMINARIES

Consider the following asymptotically stable discrete-time multi-delay system described by

$$x(k+1) = Ax(k) + \sum_{i=1}^{r} A_i x(k-d_i) + B\sigma(k)$$

$$z(k) = C_1 x(k) + \sum_{i=1}^{r} C_{1i} x(k-d_i) + D_1 \sigma(k)$$

$$y(k) = C_2 x(k) + \sum_{i=1}^{r} C_{2i} x(k-d_i) + D_2 \sigma(k)$$

(1)

where $x(k) \in \mathbb{R}^n$ is the state vector, $\sigma(k) \in \mathbb{R}^{ma}$ is the noise input, $z(k) \in \mathbb{R}^n$ is the signal to be estimated and $y(k) \in \mathbb{R}^n$ is the measured output, respectively. The positive integers $d_i (i = 1, 2, \ldots, r)$ are the constant delays of the system. $A, A_i,$ and $D_i$ are real matrices of appropriate dimensions.
Theorem 2: Consider the linear time-delay system (1). Let 
ψ := [0 ψ_{12} ψ_{22}] be given by (8). Then, an admissible filter of 
the form (2) exists with the small gain condition (6) for 
any delays \( d_i \) satisfying \( 0 < d_i \leq \bar{d} \) if there exist symmetric matrices \( P_i > 0 \), \( X_{ij} > 0 \) \( (i = 1, \ldots, r) \), \( P = P^* > 0 \), \( Q = Q^* > 0 \), \( Z_{ji} > Z_{jr} > 0 \) \( (j = 1, 2) \) and the matrices \( \bar{X}_1, \bar{X}_2 \), \( \bar{Y}_1, \bar{Y}_2 \), \( \bar{G}_k(k = 1, \ldots, 5) \), \( \bar{A}_F, \bar{B}_F, \bar{C}_F, \bar{D}_F \) such that

\[
\begin{bmatrix}
\Lambda_1 & \cdots & \Lambda_r \\
\end{bmatrix} \Phi \odot P_i + \Psi_{0} \odot \sum_{i=1}^{r} d_i Z_{i1} + \Phi \odot P + \Psi \odot Q + \Psi_{0} \odot \sum_{i=1}^{r} d_i Z_{i2} + \Phi \odot P + \Psi \odot Q + \Psi_{0} \odot \sum_{i=1}^{r} d_i Z_{i2}
\end{bmatrix} < 0,
\]

(11)

Proof. See Appendix I.

Based on Theorem 1, we present the following sufficient conditions for the existence of a desired filter of form (2).

Theorem 2: Consider the linear time-delay system (1). Let 
ψ := [0 ψ_{12} ψ_{22}] be given by (8). Then, an admissible filter of 
the form (2) exists with the small gain condition (6) for 
any delays \( d_i \) satisfying \( 0 < d_i \leq \bar{d} \) if there exist symmetric matrices \( P_i > 0 \), \( X_{ij} > 0 \) \( (i = 1, \ldots, r) \), \( P = P^* > 0 \), \( Q = Q^* > 0 \), \( Z_{ji} > Z_{jr} > 0 \) \( (j = 1, 2) \) and the matrices \( \bar{X}_1, \bar{X}_2 \), \( \bar{Y}_1, \bar{Y}_2 \), \( \bar{G}_k(k = 1, \ldots, 5) \), \( \bar{A}_F, \bar{B}_F, \bar{C}_F, \bar{D}_F \) such that

\[
\begin{bmatrix}
\Lambda_1 & \cdots & \Lambda_r \\
\end{bmatrix} \Phi \odot P_i + \Psi_{0} \odot \sum_{i=1}^{r} d_i Z_{i1} + \Phi \odot P + \Psi \odot Q + \Psi_{0} \odot \sum_{i=1}^{r} d_i Z_{i2} + \Phi \odot P + \Psi \odot Q + \Psi_{0} \odot \sum_{i=1}^{r} d_i Z_{i2}
\end{bmatrix} < 0,
\]

(12)
\[ \begin{bmatrix} \Gamma_1 & \Gamma_2 & 0 & 0 \\ \Gamma_3 & 0 & 0 & 0 \\ \Gamma_4 & \Gamma_5 & \Gamma_6 & \Gamma_7 \end{bmatrix} + \text{He} \begin{bmatrix} \Xi_1 & \Xi_2 & \Xi_3 & \Xi_4 \\ \Xi_5 & \Xi_6 & \Xi_7 & \Xi_8 \\ \Xi_9 & \Xi_{10} & \Xi_{11} & UB \end{bmatrix} < 0, \]  

(13)

\[ \Delta_1 := -P - \sum_{i=1}^{r} d_i^{-1} Z_{1i} + \sum_{i=1}^{r} X_{1i}, \]

\[ \Delta_2 := \begin{bmatrix} A^* Y_1^* + C_2^* B_F^* - Y_1^* \quad A^* V_1^* + C_2^* B_F^* - V_1^* \\ \hat{A}_F - G_2^* \quad \hat{A}_F - G_2^* \end{bmatrix}, \]

\[ \Delta_3 := \begin{bmatrix} A^* Y_2^* + C_2^* B_F^* - Y_2^* \quad A^* V_2^* + C_2^* B_F^* - V_2^* \\ \hat{A}_F - G_2^* \quad \hat{A}_F - G_2^* \end{bmatrix}, \]

\[ \Delta_4 := \begin{bmatrix} -X_{11} - d_1^{-1} Z_{11} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \]

\[ \Delta_5 := \begin{bmatrix} A_1^* Y_1^* + C_2^* B_F^* & A_1^* Y_1^* + C_2^* B_F^* & \vdots & \vdots \\ A_1^* Y_1^* + C_2^* B_F^* & A_1^* Y_1^* + C_2^* B_F^* & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ A_1^* Y_1^* + C_2^* B_F^* & A_1^* Y_1^* + C_2^* B_F^* & \vdots & \vdots \end{bmatrix}, \]

\[ \Delta_6 := \begin{bmatrix} A_1^* Y_1^* + C_2^* B_F^* & A_1^* Y_1^* + C_2^* B_F^* & \vdots & \vdots \\ A_1^* Y_1^* + C_2^* B_F^* & A_1^* Y_1^* + C_2^* B_F^* & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ A_1^* Y_1^* + C_2^* B_F^* & A_1^* Y_1^* + C_2^* B_F^* & \vdots & \vdots \end{bmatrix}, \]

\[ \Delta_7 := \begin{bmatrix} -\text{He}(Y_1) & -G_2 - V_1^* & \vdots & \vdots \\ \vdots & \text{He}(G_2) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ -\text{He}(Y_1) & -G_2 - V_1^* & \vdots & \vdots \end{bmatrix}, \]

\[ \Delta_8 := \begin{bmatrix} -H_e(Y_1) & -G_2 - V_1^* & \vdots & \vdots \\ \vdots & \text{He}(G_2) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ -H_e(Y_1) & -G_2 - V_1^* & \vdots & \vdots \end{bmatrix}, \]

\[ \Delta_9 := \begin{bmatrix} P - P_r + \sum_{i=1}^{r} d_i Z_{1i} \\ 0 \end{bmatrix}, \]

\[ \Gamma_1 := [P - P_r + \sum_{i=1}^{r} d_i Z_{1i}, 0], \]

\[ \Gamma_2 := \Psi_{12} Q - \sum_{i=1}^{r} d_i Z_{2i}, \quad \Gamma_3 := [0 \ 1], \]

\[ \Psi_{12} Q - \sum_{i=1}^{r} d_i Z_{2i} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \]

\[ \Gamma_4 := \begin{bmatrix} 0 & 0 & \vdots & \vdots \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \vdots & \ddots \end{bmatrix}, \]

\[ \Gamma_5 := \begin{bmatrix} 0 \\ -d_{r}^{-1} Z_{1r} \\ \vdots \\ -d_{r}^{-1} Z_{1r} \\ -X_{21} - d_{r}^{-1} Z_{21} \end{bmatrix}, \]

\[ \Gamma_6 := \begin{bmatrix} 0 \\ -d_{r}^{-1} Z_{21} \\ \vdots \\ -d_{r}^{-1} Z_{21} \\ -X_{21} - d_{r}^{-1} Z_{21} \end{bmatrix}, \]

Moreover, an admissible filter is given by

\[ A_F = G_2^{-1} \hat{A}_F, B_F = G_2^{-1} \hat{B}_F, C_F = G_5^{-1} \hat{C}_F, D_F = G_5^{-1} \hat{D}_F. \]

(15)

**Proof.** By Schur complement lemma, (9) is equivalent to

\[ \begin{bmatrix} -P - \sum_{i=1}^{r} d_i^{-1} Z_{1i} + \sum_{i=1}^{r} X_{1i} & -d_1^{-1} Z_{11} & \ldots & -d_1^{-1} Z_{1r} & -d_1^{-1} Z_{1r} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ -d_r^{-1} Z_{1r} & -d_r^{-1} Z_{1r} & \ldots & -d_r^{-1} Z_{1r} & -d_r^{-1} Z_{1r} \\ -d_r^{-1} Z_{21} & \ldots & \ldots & \ldots & \ldots \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ -d_r^{-1} Z_{21} & \ldots & \ldots & \ldots & \ldots \\ -X_{21} - d_r^{-1} Z_{21} & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots \end{bmatrix} < 0. \]

(16)

From Lemma 4 in Appendix III, it is equivalent to

\[ \begin{bmatrix} -P - \sum_{i=1}^{r} d_i^{-1} Z_{1i} + \sum_{i=1}^{r} X_{1i} & -d_1^{-1} Z_{11} & \ldots & -d_1^{-1} Z_{1r} & -d_1^{-1} Z_{1r} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ -d_r^{-1} Z_{1r} & -d_r^{-1} Z_{1r} & \ldots & -d_r^{-1} Z_{1r} & -d_r^{-1} Z_{1r} \\ -d_r^{-1} Z_{21} & \ldots & \ldots & \ldots & \ldots \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ -d_r^{-1} Z_{21} & \ldots & \ldots & \ldots & \ldots \\ -X_{21} - d_r^{-1} Z_{21} & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots \end{bmatrix} < 0. \]

(17)
Moreover, (10) is equivalent to

\[
\begin{bmatrix}
J^* & \Gamma_1 & \Gamma_2 & 0 & 0 \\
0 & \Gamma_3 & 0 & 0 & 0 \\
0 & 0 & \Gamma_4 & \Gamma_5 & \Gamma_6 & \Gamma_7 \\
\end{bmatrix} < 0.
\]

(18)

where \( J := \begin{bmatrix} A & A_1 & \ldots & A_r & B \\ C & C_1 & \ldots & C_r & D \end{bmatrix} \) and \( \Gamma_j \) \((j = 1, \ldots, 7)\) are defined by (14). By virtue of Lemma 5 in Appendix III, (18) is equivalent to

\[
\begin{bmatrix}
\Gamma_1 & \Gamma_2 & 0 & 0 \\
0 & \Gamma_3 & 0 & 0 & 0 \\
0 & 0 & \Gamma_4 & \Gamma_5 & \Gamma_6 & \Gamma_7 \\
\end{bmatrix} + \text{He}(W_3 [-I & J]) < 0.
\]

(19)

Set \( W_3 := \begin{bmatrix} G_1 & G_2 & 0 & 0 \\
G_3 & G_2 & 0 & 0 \\
G_4 & G_5 & 0 & 0 \\
U & 0 & 0 & 0 \\
\end{bmatrix} \) with \( U \in \mathbb{C}^{(2r+2) \times n_x} \). In order to solve the multi-objective problem subject to (17) and (19), we define \( W_1 = \begin{bmatrix} Y_1 & W_{12} \\
Y_2 & W_{22} \end{bmatrix} \) and \( W_2 = \begin{bmatrix} Y_2 & W_{12} \\
Y_2 & W_{22} \end{bmatrix} \).

From Lemma 6 in Appendix III, \( W_1 \) and \( W_2 \) can be recharacterized as \( W_1 = \begin{bmatrix} Y_1 & G_2 \\
Y_2 & G_2 \end{bmatrix} \) and \( W_2 = \begin{bmatrix} Y_2 & G_2 \\
V_2 & G_2 \end{bmatrix} \), respectively. Defining the new variables

\[
\hat{A}_F := G_2 A_F, \quad \hat{B}_F := G_2 B_F, \quad \hat{C}_F := G_5 C_F, \quad \hat{D}_F := G_5 D_F,
\]

we can obtain (12) and (13) from (17) and (19), respectively. Thus, the asymptotic stability is achieved.

Remark 3: Theorem 2 presents sufficient conditions for the existence of admissible filters such that the filtering error systems are asymptotically stable and satisfy the small gain conditions in finite frequency ranges. In order to solve the nonconvex multi-objective filtering synthesis problem, we set that \( W_1 \) and \( W_2 \) have the same elements in the second column and specify the structure of \( W_3 \) such that a common variable can be used, which will lead to some conservatism.

IV. EXAMPLE

Consider a linear discrete-time single-delay system in the form of (1) with

\[
A = \begin{bmatrix} 1 & 0 \\
0 & 0.8 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.1 & -0.1 \\
-0.2 & -0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\
1 \end{bmatrix},
\]

\[
C_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad C_{11} = 0, \quad D_1 = 0,
\]

\[
C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad C_{21} = \begin{bmatrix} 0.5 & 0 \end{bmatrix}, \quad D_2 = 1.
\]

Our objective is to design a filter (2) such that the filtering error system (3) is asymptotically stable and satisfies

\[
\sigma_{\text{max}}(G(e^{j\omega})) < \gamma, \quad |\omega| \geq \frac{\pi}{5}.
\]

Here, \( \Psi \) can be given by \( \Psi := \begin{bmatrix} 0 & \Psi_{12} \\
\Psi_{12}^* & \Psi_{22} \end{bmatrix} \). The results are compared in Table 1. It is clearly demonstrated that, for the case with a \( H_\infty \) filter, the method obtained by Theorem 2 (Q=0) is less conservative than those in [9] and [10]. Furthermore, Theorem 2 which achieves the filter design with a small gain condition in the high frequency range produces much less conservative results than others.

V. CONCLUSION

In this paper, the filter design problem for linear discrete-time state-delayed systems with the small gain conditions in finite frequency ranges has been addressed. Sufficient conditions for the existence of feasible filters are given in terms of solutions to a set of LMIs. Finally, a numerical example is given to demonstrate the utility of the proposed design approach.

APPENDIX I

PROOF OF THEOREM 1

Under the condition of the theorem, we first show the asymptotic stability of system (3). To this end, we consider system (3) with \( \Theta(k) = 0 \). Now, choose a Lyapunov functional candidate as

\[
V(\tilde{x}(k)) = \tilde{x}^*(k) P_3 \tilde{x}(k) + \sum_{i=1}^{r} \sum_{j=k-d_i}^{k-1} \bar{\xi}^*(j) X_{1i} \bar{\xi}(j) + \sum_{i=1}^{r} \sum_{j=k-d_i}^{k-1} \bar{\xi}^*(j) Z_{1i} \bar{\xi}(j)
\]

(21)

where \( \bar{\xi}(k) := \tilde{x}(k+1) - \tilde{x}(k) \). Defining \( \Delta V := V(\tilde{x}(k+1)) - V(\tilde{x}(k)) \), we have

\[
\Delta V = \tilde{x}^*(k+1) P_3 \tilde{x}(k+1) - \tilde{x}^*(k) P_3 \tilde{x}(k) + \sum_{i=1}^{r} \tilde{x}^*(k) X_{1i} \tilde{x}(k) - \sum_{i=1}^{r} \tilde{x}^*(k-d_i) X_{1i} \tilde{x}(k-d_i) + \sum_{i=1}^{r} d_i \bar{\xi}^*(k) Z_{1i} \bar{\xi}(k)
\]

(22)

By virtue of Jensen integral inequality, it follows that

\[
- \sum_{j=k-d_i}^{k-1} \bar{\xi}^*(j) Z_{1i} \bar{\xi}(j) \leq - \frac{1}{d_i} \left( \sum_{j=k-d_i}^{k-1} \bar{\xi}(j) \right)^* Z_{1i} \left( \sum_{j=k-d_i}^{k-1} \bar{\xi}(j) \right) = - \frac{1}{d_i} \left[ k(k-x(k-d_i)) \right]^* Z_{1i} \left[ k(k-x(k-d_i)) \right].
\]

(23)
Therefore, it follows from (9) that for all $0 < d_i \leq \bar{d}_i$
\[ \Delta V \leq \xi^* \phi \xi < 0 \] (24)
where $\xi := [x^*(t) \ x^*(t-d_1) \ \ldots \ x^*(t-d_r)]^*$ and $\phi$ is
defined as the left side of (9). This implies that the system
(3) with $\sigma(k) = 0$ is asymptotically stable for any delays $d_i$
satisfying $0 < d_i \leq \bar{d}_i$. Next, we shall establish the
performance. The specification (6) can be denoted by the FDI
\[ \begin{bmatrix} \xi^* \\ \frac{\mathcal{C}}{0} \ D^* \end{bmatrix} \Pi \begin{bmatrix} \bar{C} \\ 0 \end{bmatrix} \begin{bmatrix} \xi^* \\ \frac{\mathcal{D}}{0} \ l \end{bmatrix} < 0 \quad \omega \in \Omega \] (25)
where
\[ \xi := (e^{j \omega I} - \sum_{i=1}^r e^{-d_i j \omega A_i})^{-1} B, \quad \mathcal{C} := C + \sum_{i=1}^r e^{-d_i j \omega} C_i. \] (26)
It is equivalent to
\[ E^* \Theta E < 0 \quad \omega \in \Omega \] (27)
where
\[ E := \begin{bmatrix} \xi \\ e^{-d_1 j \omega} \xi \\ \vdots \\ e^{-d_r j \omega} \xi \\ l \end{bmatrix}, \quad \Theta := \begin{bmatrix} C^* \ 0 \\ C_1^* \ 0 \\ \vdots \ \\ C_r^* \ 0 \\ D^* \ l \end{bmatrix} \begin{bmatrix} C^* \ 0 \end{bmatrix} \begin{bmatrix} C^* \ 0 \end{bmatrix} + \begin{bmatrix} X_2^* \ 0 \\ 0 \ 0 \end{bmatrix} \] (28)
with $X_2$ defined by (11). Defining
\[ G_1 := \{ \xi \in C^{(2r+2)n+n^2} : \xi = E \eta, \eta \in C^{n^2}, \eta \neq 0, \omega \in \Omega \}, \] (29)
(27) can be denoted by
\[ \xi^* \Theta \xi < 0, \quad \forall \xi \in G_1. \] (30)
Furthermore, for $\forall \xi \in G_1$, we have $\Gamma_\lambda F \xi = 0(\lambda \in \Lambda)$ where
\[ \Gamma_\lambda := \begin{bmatrix} I & -\lambda I \\ 0 & -I \end{bmatrix} \quad (\lambda \in \Lambda) \quad (\lambda = \infty), \] (31)
\[ F := \begin{bmatrix} A & A_1 & \ldots & A_r & B \end{bmatrix}. \] (32)
Defining
\[ G_2 := \{ \xi \in C^{(2r+2)n+n^2} : \xi \neq 0, \Gamma_\lambda F \xi = 0, \lambda \in \Lambda \}, \] (33)
we can see that $G_1 \subset G_2$. From Lemma 1 in Appendix II,
the set $G_2$ can be characterized by (37) with $M$ in (38). Then define
\[ G := \{ \xi \in C^{(2r+2)n+n^2} : \xi \neq 0, \xi^* M \xi \geq 0 \forall M \in M, \xi^* N \xi \geq 0 \} \] (34)
where $M$ is given by (38) and
\[ N := \begin{bmatrix} Z_2 & 0 \\ 0 & 0 \end{bmatrix} + F^* (\Psi_0 \otimes \sum_{i=1}^r \bar{d}_i Z_i) F \] (35)
with $Z_2$ defined by (11). It is easy to see that $G \subset G_2$. Then,
let us show the relationship between $G$ and $G_1$. For $\forall \xi \neq 0 \in G_1$ and any delays $d_i$ satisfying $0 < d_i \leq \bar{d}_i$, we have
\[ \xi^* N \xi = \eta^* E^* N E \eta = \sum_{i=1}^r 2(-\bar{d}^{-1}_i + \bar{d}^{-1}_i \cos \omega + \bar{d}_i - \bar{d}_i \cos \omega) \eta^* \xi^* Z_i \xi \eta \]
\[ = \sum_{i=1}^r 4(-\bar{d}_i \sin^2 \frac{\omega}{2} + \bar{d}_i \sin^2 \frac{\omega}{2}) \eta^* \xi^* Z_i \xi \eta \]
\[ \geq \sum_{i=1}^r 4(-\bar{d}_i \sin^2 \frac{\omega}{2} \eta^* \xi^* Z_i \xi \eta \geq 0 \] (36)
in view of the fact that $d_i \geq 1$ are positive integers. Due to $G_1 \subset G_2$, it follows that $\xi \in G$. Hence, we show that $G_1 \subset G$ and $G_2$. Thus (30) holds if $\xi^* \Theta \xi < 0$ for $\forall \xi \in G$. By Lemma 2 in Appendix II, $M$ is admissible and rank-one separable. From Lemma 3 in Appendix II, $\xi^* \Theta \xi < 0\forall \xi \in G$ is equivalent to $\Theta + M + N \eta < 0$. Replacing $\tau Z_i$ with $Z_i$, it is equivalent to $\Theta + M + N \eta < 0$. Thus, the proof is completed.

**APPENDIX II**

Some preliminaries are required for the proof of Theorem 1.

**Lemma 1:** Let $F \in C^{4m \times [(2r+2)n+n^2]}$ and $\Phi, \Psi \in H_2$ be given such that $\Lambda$ in (6) represents curves. Define $\Lambda$ and $\Gamma_\lambda$ by (6) and (31), respectively. Then, the set $G_2$ defined in (33) can be characterized by
\[ G_2 := \{ \xi \in C^{(2r+2)n+n^2} : \xi \neq 0, \xi^* M \xi \geq 0 \forall M \in M \}, \] (37)
with $M := \{ F^* (\Phi \otimes \Psi \otimes Q) F : P, Q \in H_{2n}, Q > 0 \}$. (38)

**Lemma 2:** Let matrices $\Phi, \Psi \in H_2$ and $F \in C^{4m \times [(2r+2)n+n^2]}$ be given such that $\Lambda$ in (6) represents curves. Then, the set $M$ defined by (38) is admissible and rank-one separable.

**Remark 2:** Lemmas 1 and 2 basically follow from [12].

**Lemma 3** (Further generalization of the strict S-procedure): Let admissible sets $M_i \subset H_q (i = 1, 2, \ldots, p)$ and $N_i \in H_q (i = 1, 2, \ldots, s)$ be given and define $G$ by
\[ G := \{ \xi \in C^q : \xi \neq 0, \xi^* M_i \xi \geq 0 \forall M_i \in M_i, \xi^* N_i \xi \geq 0 \}. \] (39)
Then, if and only if $M_i \in H_q$ are rank-one separable, the following statements are equivalent.

i) $\xi^* \Theta \xi < 0 \quad \forall \xi \in G$.

ii) There exist $M_i \in M_i$ and $\tau_i \geq 0$ such that $\Theta + \sum_{i=1}^p M_i + \sum_{i=1}^s \tau_i N_i < 0$ for an arbitrary $\Theta \in H_q$.

**APPENDIX III**

Lemmas 4-6 are presented for the proof of Theorem 2.

**Lemma 4** Consider the system described by (3). Then the following statements are equivalent:

i) There exist symmetric matrices $P_i > 0, X_{ii} > 0 (i = 1, \ldots, r)$ and $Z_i > 0$ such that (16) holds.

ii) There exist symmetric matrices $P_i > 0, X_{ii} > 0 (i = 1, \ldots, r)$ and $Z_i > 0$ such that (16) holds.
1,..., r) and matrices $W_j(j = 1, 2)$ such that (17) holds.

Proof. (i)⇒(ii): Choose $W_1 = W_1^* = \sum_{i=1}^r \tilde{\alpha}_i Z_{1i}$ and $W_2 = W_2^* = P_i$.

(ii)⇒(i): Assume that (17) holds. Hence, $\text{He}(W_1) - \sum_{i=1}^r \tilde{\alpha}_i Z_{1i} > 0$ and $\text{He}(W_2) - P_i > 0$. It implies that $W_j(j = 1, 2)$ are both nonsingular matrices. Furthermore, we have $\sum_{i=1}^r \tilde{\alpha}_i Z_{1i} - W_1\sum_{i=1}^r \tilde{\alpha}_i Z_{1i}^{-1} - (\sum_{i=1}^r \tilde{\alpha}_i Z_{1i} - W_1)^+ \geq 0$ and $(P_i - W_2)P_i^{-1}(P_i - W_2)^+ \geq 0$. Therefore, $W_1(\sum_{i=1}^r \tilde{\alpha}_i Z_{1i}^{-1}) W_1^* \geq \text{He}(W_1) - \sum_{i=1}^r \tilde{\alpha}_i Z_{1i}^{-1}$ and $W_2^* W_1^* \geq \text{He}(W_2)$ and $P_i - \text{He}(W_2)$ with $-W_1(\sum_{i=1}^r \tilde{\alpha}_i Z_{1i}^{-1}) W_1^*$ and $-W_2^* W_1^*$ in (17), respectively. Multiplying it by $\text{diag}\{I, I, \ldots, I, \sum_{i=1}^r \tilde{\alpha}_i Z_{1i}^{-1}, P_i W_1^{-1}\}$ on the left and its complex conjugate transpose on the right, respectively, it is equivalent to (16). Thus, the proof is completed.

**Lemma 5** (Finsler’s Lemma): Let $x \in \mathbb{R}^n$, symmetric matrix $Z \in \mathbb{R}^{n \times n}$, and $\Gamma \in \mathbb{R}^{n \times n}$ such that rank($\Gamma$) = $r < n$. Then the following statements are equivalent:

i) $x^T \Gamma x < 0$, for any $x \neq 0$ and $\Gamma x = 0$.

ii) $\exists \Gamma \in \mathbb{R}^{n \times n}$: Let $\tilde{\alpha}_s$, $s = 1, 2$.

**Lemma 6:** Consider the system described by (3). Then the following statements are equivalent:

i) There exist symmetric matrices $P_i > 0$, $X_{1i} > 0$, and matrices $W_j = \begin{bmatrix} Y_j & W_{1j} \\ Y_j & W_{2j} \end{bmatrix}$ (j = 1, 2) and a filter described by (2) such that (17) holds.

ii) There exist symmetric matrices $P_i > 0$, $X_{1i} > 0$, and matrices $\tilde{W}_j = \begin{bmatrix} Y_j & G_2 \\ Y_j & G_2 \end{bmatrix}$ (j = 1, 2) and a filter described by (2) with $\tilde{A}_F = \hat{A}_F$ and $\tilde{B}_F = \hat{B}_F$ such that

\[
\begin{bmatrix}
-P_i - \sum_{i=1}^r \tilde{\alpha}_i Z_{1i} + \sum_{i=1}^r \tilde{\alpha}_i X_{1i} & \tilde{\alpha}_1^{-1} Z_{11} & \ldots & \tilde{\alpha}_1^{-1} Z_{11} \\
& \tilde{\alpha}_1^{-1} Z_{12} - \tilde{X}_{11} & \ldots & \\
& \vdots & \ddots & \\
& \vdots & \ddots & \tilde{\alpha}_1^{-1} Z_{1r} - \tilde{X}_{1r} \\
\end{bmatrix}

\begin{bmatrix}
\hat{A}_F & \hat{\alpha}_1 W_1^* & \hat{\alpha}_1 W_2^* \\
0 & \hat{\alpha}_1 W_1^* & \hat{\alpha}_1 W_2^* \\
\vdots & \vdots & \vdots \\
0 & \sum_{i=1}^r \tilde{\alpha}_i Z_{1i} - \text{He}(W_1) & 0 \\
\end{bmatrix}

< 0
\]

where $T := \begin{bmatrix} I & 0 \\ 0 & \{W_{12} W_{21}^{-1} \} \end{bmatrix}$.

Proof: Let $W_{12}$ and $W_{22}$ are nonsingular. Letting $V_j = W_{12} W_{22}^{-1} Y_j$ and $G_2 = W_{12}(W_{22}^{-1})^* W_{12}$, it follows that $\tilde{W}_j = TW_j T^*$.

\[
\tilde{W}_j = \begin{bmatrix} Y_j & G_2 \\ V_j & G_2 \end{bmatrix}.
\]

The proof is completed.

**REFERENCES**


