Partial Differentiation, Differentiation and Continuity on $n$-Dimensional Real Normed Linear Spaces

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Summary. In this article, we aim to prove the characterization of differentiation by means of partial differentiation for vector-valued functions on $n$-dimensional real normed linear spaces (refer to [15] and [16]).

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The notation and terminology used in this paper have been introduced in the following papers: [2], [7], [1], [3], [4], [5], [17], [11], [13], [6], [9], [14], [10], [8], [12], and [18].

One can prove the following propositions:

1. Let $n$, $i$ be elements of $\mathbb{N}$, $q$ be an element of $\mathbb{R}^n$, and $p$ be a point of $\mathcal{E}_n^p$. If $i \in \text{Seg } n$ and $q = p$, then $|p_i| \leq |q|$.

2. For every real number $x$ and for every element $v_1$ of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ such that $v_1 = \langle x \rangle$ holds $\|v_1\| = |x|$.

3. Let $n$ be a non-empty element of $\mathbb{N}$, $x$ be a point of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and $i$ be an element of $\mathbb{N}$. If $1 \leq i \leq n$, then $\|\text{Proj}(i,n)(x)\| \leq \|x\|$.
(4) For every non empty element $n$ of $\mathbb{N}$ and for every element $x$ of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ and for every element $i$ of $\mathbb{N}$ holds $\|(\text{Proj}(i, n))(x)\| = |(\text{proj}(i, n))(x)|$.

(5) Let $n$ be a non empty element of $\mathbb{N}$, $x$ be an element of $\mathcal{R}^n$, and $i$ be an element of $\mathbb{N}$. If $1 \leq i \leq n$, then $|(\text{proj}(i, n))(x)| \leq |x|$.

(6) Let $m$, $n$ be non empty elements of $\mathbb{N}$, $s$ be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and $i$ be an element of $\mathbb{N}$. Suppose $1 \leq i \leq n$. Then $\text{Proj}(i, n)$ is a bounded bounded linear operator from $\langle \mathcal{E}^n, \| \cdot \| \rangle$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$ and $(\text{BdLinOpsNorm}(\langle \mathcal{E}^n, \| \cdot \| \rangle, \langle \mathcal{E}^1, \| \cdot \| \rangle))(\text{Proj}(i, n)) \leq 1$.

(7) Let $m$, $n$ be non empty elements of $\mathbb{N}$, $s$ be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and $i$ be an element of $\mathbb{N}$. Suppose $1 \leq i \leq n$. Then

(i) $\text{Proj}(i, n) \cdot s$ is a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$, and

(ii) $(\text{BdLinOpsNorm}(\langle \mathcal{E}^m, \| \cdot \| \rangle, \langle \mathcal{E}^1, \| \cdot \| \rangle))(\text{Proj}(i, n) \cdot s) \leq (\text{BdLinOpsNorm}(\langle \mathcal{E}^n, \| \cdot \| \rangle, \langle \mathcal{E}^1, \| \cdot \| \rangle))(\text{Proj}(i, n)) \cdot (\text{BdLinOpsNorm}(\langle \mathcal{E}^m, \| \cdot \| \rangle, \langle \mathcal{E}^n, \| \cdot \| \rangle))(s)$.

(8) For every non empty element $n$ of $\mathbb{N}$ and for every element $i$ of $\mathbb{N}$ holds $\text{Proj}(i, n)$ is homogeneous.

(9) Let $n$ be a non empty element of $\mathbb{N}$, $x$ be an element of $\mathcal{R}^n$, $r$ be a real number, and $i$ be an element of $\mathbb{N}$. Then $(\text{proj}(i, n))(r \cdot x) = r \cdot (\text{proj}(i, n))(x)$.

(10) Let $n$ be a non empty element of $\mathbb{N}$, $x$, $y$ be elements of $\mathcal{R}^n$, and $i$ be an element of $\mathbb{N}$. Then $(\text{proj}(i, n))(x + y) = (\text{proj}(i, n))(x) + (\text{proj}(i, n))(y)$.

(11) Let $n$ be a non empty element of $\mathbb{N}$, $x$, $y$ be points of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and $i$ be an element of $\mathbb{N}$. Then $\text{Proj}(i, n))(x - y) = (\text{Proj}(i, n))(x) - (\text{Proj}(i, n))(y)$.

(12) Let $n$ be a non empty element of $\mathbb{N}$, $x$, $y$ be elements of $\mathcal{R}^n$, and $i$ be an element of $\mathbb{N}$. Then $(\text{proj}(i, n))(x - y) = (\text{proj}(i, n))(x) - (\text{proj}(i, n))(y)$.

(13) Let $m$, $n$ be non empty elements of $\mathbb{N}$, $s$ be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^n, \| \cdot \| \rangle$, $i$ be an element of $\mathbb{N}$, and $s_1$ be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$. If $s_1 = \text{Proj}(i, n) \cdot s$ and $1 \leq i \leq n$, then $\|s_1\| \leq \|s\|$.

(14) Let $m$, $n$ be non empty elements of $\mathbb{N}$, $s$, $t$ be points of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^n, \| \cdot \| \rangle$, $s_1$, $t_1$ be points of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$, and $i$ be an element of $\mathbb{N}$. If $s_1 = \text{Proj}(i, n) \cdot s$ and $t_1 = \text{Proj}(i, n) \cdot t$ and $1 \leq i \leq n$, then $\|s_1 - t_1\| \leq \|s - t\|$.

(15) Let $K$ be a real number, $n$ be an element of $\mathbb{N}$, and $s$ be an element of $\mathcal{R}^n$. Suppose that for every element $i$ of $\mathbb{N}$ such that $1 \leq i \leq n$ holds $|s(i)| \leq K$. Then $|s| \leq n \cdot K$. 

(16) Let $K$ be a real number, $n$ be a non empty element of $\mathbb{N}$, and $s$ be an element of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose that for every element $i$ of $\mathbb{N}$ such that $1 \leq i \leq n$ holds $\| \text{Proj}(i, n)(s) \| \leq K$. Then $\|s\| \leq n \cdot K$.

(17) Let $K$ be a real number, $n$ be a non empty element of $\mathbb{N}$, and $s$ be an element of $\mathcal{R}^n$. Suppose that for every element $i$ of $\mathbb{N}$ such that $1 \leq i \leq n$ holds $|\text{proj}(i, n)(s)| \leq K$. Then $|s| \leq n \cdot K$.

(18) Let $m, n$ be non empty elements of $\mathbb{N}$, $s$ be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and $K$ be a real number. Suppose that for every element $i$ of $\mathbb{N}$ and for every point $s_i$ of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$ such that $s_i = \text{Proj}(i, n) \cdot s$ and $1 \leq i \leq n$ holds $\|s_i\| \leq K$. Then $\|s\| \leq n \cdot K$.

(19) Let $m, n$ be non empty elements of $\mathbb{N}$, $s, t$ be points of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and $K$ be a real number. Suppose that for every element $i$ of $\mathbb{N}$ and for all points $s_i, t_i$ of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$ such that $s_i = \text{Proj}(i, n) \cdot s$ and $t_i = \text{Proj}(i, n) \cdot t$ and $1 \leq i \leq n$ holds $\|s_i - t_i\| \leq K$. Then $\|s - t\| \leq n \cdot K$.

(20) Let $m, n$ be non empty elements of $\mathbb{N}$, $f$ be a partial function from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$, $X$ be a subset of $\langle \mathcal{E}^m, \| \cdot \| \rangle$, and $i$ be an element of $\mathbb{N}$. Suppose $1 \leq i \leq m$ and $X$ is open. Then the following statements are equivalent

(i) $f$ is partially differentiable on $X$ w.r.t. $i$ and $f^i|X$ is continuous on $X$,

(ii) for every element $j$ of $\mathbb{N}$ such that $1 \leq j \leq n$ holds $\text{Proj}(j, n) \cdot f$ is partially differentiable on $X$ w.r.t. $i$ and $\text{Proj}(j, n) \cdot f^i|X$ is continuous on $X$.

(21) Let $m, n$ be non empty elements of $\mathbb{N}$, $f$ be a partial function from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and $X$ be a subset of $\langle \mathcal{E}^m, \| \cdot \| \rangle$. Suppose $X$ is open. Then $f$ is differentiable on $X$ and $f^i|X$ is continuous on $X$ if and only if for every element $j$ of $\mathbb{N}$ such that $1 \leq j \leq n$ holds $\text{Proj}(j, n) \cdot f$ is differentiable on $X$ and $(\text{Proj}(j, n) \cdot f)^i|X$ is continuous on $X$.

(22) Let $m, n$ be non empty elements of $\mathbb{N}$, $f$ be a partial function from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and $X$ be a subset of $\langle \mathcal{E}^m, \| \cdot \| \rangle$. Suppose $X$ is open. Then for every element $i$ of $\mathbb{N}$ such that $1 \leq i \leq m$ holds $f$ is partially differentiable on $X$ w.r.t. $i$ and $f^i|X$ is continuous on $X$ if and only if $f$ is differentiable on $X$ and $f^i|X$ is continuous on $X$.

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Differentiable Functions into Real Normed Spaces

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Summary. In this article, we formalize the differentiability of functions from the set of real numbers into a normed vector space [15].

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The notation and terminology used here have been introduced in the following papers: [13], [2], [3], [7], [10], [12], [1], [4], [11], [14], [6], [18], [19], [16], [9], [8], [17], [20], and [5].

For simplicity, we adopt the following rules: $F$ is a non trivial real normed space, $G$ is a real normed space, $X$ is a set, $x, x_0, r, p$ are real numbers, $n, k$ are elements of $\mathbb{N}$, $Y$ is a subset of $\mathbb{R}$, $Z$ is an open subset of $\mathbb{R}$, $s_1$ is a sequence of real numbers, $s_2$ is a sequence of $G$, $f, f_1, f_2$ are partial functions from $\mathbb{R}$ to the carrier of $F$, $h$ is a convergent to 0 sequence of real numbers, and $c$ is a constant sequence of real numbers.

Next we state two propositions:

1. If for every $n$ holds $\|s_2(n)\| \leq s_1(n)$ and $s_1$ is convergent and $\lim s_1 = 0$, then $s_2$ is convergent and $\lim s_2 = 0_G$.

2. $(s_1 \uparrow k) (s_2 \uparrow k) = (s_1 s_2) \uparrow k$.

Let us consider $F$ and let $I_1$ be a partial function from $\mathbb{R}$ to the carrier of $F$. We say that $I_1$ is rest-like if and only if:

(Def. 1) $I_1$ is total and for every $h$ holds $h^{-1} (I_1, h)$ is convergent and $\lim(h^{-1} (I_1, h)) = 0_F$. 
Let us consider $F$. Note that there exists a partial function from $\mathbb{R}$ to the carrier of $F$ which is rest-like.

Let us consider $F$. A rest of $F$ is a rest-like partial function from $\mathbb{R}$ to the carrier of $F$.

Let us consider $F$ and let $I_1$ be a function from $\mathbb{R}$ into the carrier of $F$. We say that $I_1$ is linear if and only if:

(Def. 2) There exists a point $r$ of $F$ such that for every real number $p$ holds

$$I_1(p) = p \cdot r.$$

Let us consider $F$. Note that there exists a function from $\mathbb{R}$ into the carrier of $F$ which is linear.

Let us consider $F$. A linear of $F$ is a linear function from $\mathbb{R}$ into the carrier of $F$.

We adopt the following rules: $R$, $R_1$, $R_2$ denote rests of $F$ and $L$, $L_1$, $L_2$ denote linears of $F$.

Next we state several propositions:

(3) $L_1 + L_2$ is a linear of $F$ and $L_1 - L_2$ is a linear of $F$.

(4) $rL$ is a linear of $F$.

(5) Let $h_1$, $h_2$ be partial functions from $\mathbb{R}$ to the carrier of $F$ and $s_2$ be a sequence of real numbers. If $\text{rng } s_2 \subseteq \text{dom } h_1 \cap \text{dom } h_2$, then $(h_1 + h_2) \cdot s_2 = (h_1 \cdot s_2) + (h_2 \cdot s_2)$ and $(h_1 - h_2) \cdot s_2 = (h_1 \cdot s_2) - (h_2 \cdot s_2)$.

(6) Let $h_1$, $h_2$ be partial functions from $\mathbb{R}$ to the carrier of $F$ and $s_2$ be a sequence of real numbers. If $h_1$ is total and $h_2$ is total, then $(h_1 + h_2) \cdot s_2 = (h_1 \cdot s_2) + (h_2 \cdot s_2)$ and $(h_1 - h_2) \cdot s_2 = (h_1 \cdot s_2) - (h_2 \cdot s_2)$.

(7) $R_1 + R_2$ is a rest of $F$ and $R_1 - R_2$ is a rest of $F$.

(8) $rR$ is a rest of $F$.

Let us consider $F$, $f$ and let $x_0$ be a real number. We say that $f$ is differentiable in $x_0$ if and only if:

(Def. 3) There exists a neighbourhood $N$ of $x_0$ such that $N \subseteq \text{dom } f$ and there exist $L$, $R$ such that for every $x$ such that $x \in N$ holds $f_x - f_{x_0} = L(x - x_0) + R_{x-x_0}$.

Let us consider $F$, $f$ and let $x_0$ be a real number. Let us assume that $f$ is differentiable in $x_0$. The functor $f'(x_0)$ yielding a point of $F$ is defined by the condition (Def. 4).

(Def. 4) There exists a neighbourhood $N$ of $x_0$ such that $N \subseteq \text{dom } f$ and there exist $L$, $R$ such that $f'(x_0) = L(1)$ and for every $x$ such that $x \in N$ holds

$$f_x - f_{x_0} = L(x - x_0) + R_{x-x_0}.$$

Let us consider $F$, $f$, $X$. We say that $f$ is differentiable on $X$ if and only if:

(Def. 5) $X \subseteq \text{dom } f$ and for every $x$ such that $x \in X$ holds $f \restriction X$ is differentiable in $x$. 

The following propositions are true:

(9) If \( f \) is differentiable on \( X \), then \( X \) is a subset of \( \mathbb{R} \).

(10) \( f \) is differentiable on \( Z \) iff \( Z \subseteq \text{dom} \, f \) and for every \( x \) such that \( x \in Z \) holds \( f \) is differentiable in \( x \).

(11) If \( f \) is differentiable on \( Y \), then \( Y \) is open.

Let us consider \( F, f, X \). Let us assume that \( f \) is differentiable on \( X \). The functor \( f' | X \) yields a partial function from \( \mathbb{R} \) to the carrier of \( F \) and is defined by:

(Def. 6) \( \text{dom}(f' | X) = X \) and for every \( x \) such that \( x \in X \) holds \( f' | X(x) = f'(x) \).

The following propositions are true:

(12) Suppose \( Z \subseteq \text{dom} \, f \) and there exists a point \( r \) of \( F \) such that \( \text{rng} \, f = \{ r \} \). Then \( f \) is differentiable on \( Z \) and for every \( x \) such that \( x \in Z \) holds \( (f' | Z)_x = 0_F \).

(13) Let \( x_0 \) be a real number and \( N \) be a neighbourhood of \( x_0 \). Suppose \( f \) is differentiable in \( x_0 \) and \( N \subseteq \text{dom} \, f \). Let given \( h, c \). Suppose \( \text{rng} \, c = \{ x_0 \} \) and \( \text{rng} \, (h + c) \subseteq N \). Then \( h^{-1} ((f_*(h + c)) - (f_*c)) \) is convergent and \( f'(x_0) = \lim(h^{-1} ((f_*(h + c)) - (f_*c))) \).

(14) If \( f_1 \) is differentiable in \( x_0 \) and \( f_2 \) is differentiable in \( x_0 \), then \( f_1 + f_2 \) is differentiable in \( x_0 \) and \( (f_1 + f_2)'(x_0) = f_1'(x_0) + f_2'(x_0) \).

(15) If \( f_1 \) is differentiable in \( x_0 \) and \( f_2 \) is differentiable in \( x_0 \), then \( f_1 - f_2 \) is differentiable in \( x_0 \) and \( (f_1 - f_2)'(x_0) = f_1'(x_0) - f_2'(x_0) \).

(16) For every real number \( r \) such that \( f \) is differentiable in \( x_0 \) holds \( r \, f \) is differentiable in \( x_0 \) and \( (r \, f)'(x_0) = r \cdot f'(x_0) \).

(17) Suppose \( Z \subseteq \text{dom} \,(f_1 + f_2) \) and \( f_1 \) is differentiable on \( Z \) and \( f_2 \) is differentiable on \( Z \). Then \( f_1 + f_2 \) is differentiable on \( Z \) and for every \( x \) such that \( x \in Z \) holds \( (f_1 + f_2)'_Z(x) = f_1'(x) + f_2'(x) \).

(18) Suppose \( Z \subseteq \text{dom} \,(f_1 - f_2) \) and \( f_1 \) is differentiable on \( Z \) and \( f_2 \) is differentiable on \( Z \). Then \( f_1 - f_2 \) is differentiable on \( Z \) and for every \( x \) such that \( x \in Z \) holds \( (f_1 - f_2)'_Z(x) = f_1'(x) - f_2'(x) \).

(19) Suppose \( Z \subseteq \text{dom} \,(r \, f) \) and \( f \) is differentiable on \( Z \). Then \( r \, f \) is differentiable on \( Z \) and for every \( x \) such that \( x \in Z \) holds \( (r \, f)'_Z(x) = r \cdot f'(x) \).

(20) If \( Z \subseteq \text{dom} \, f \) and \( f | Z \) is constant, then \( f \) is differentiable on \( Z \) and for every \( x \) such that \( x \in Z \) holds \( f'_Z(x) = 0_F \).

(21) Let \( r, p \) be points of \( F \) and given \( Z, f \). Suppose \( Z \subseteq \text{dom} \, f \) and for every \( x \) such that \( x \in Z \) holds \( f_x = x \cdot r + p \). Then \( f \) is differentiable on \( Z \) and for every \( x \) such that \( x \in Z \) holds \( f'_Z(x) = r \).

(22) For every real number \( x_0 \) such that \( f \) is differentiable in \( x_0 \) holds \( f \) is continuous in \( x_0 \).

(23) If \( f \) is differentiable on \( X \), then \( f | X \) is continuous.
(24) If $f$ is differentiable on $X$ and $Z \subseteq X$, then $f$ is differentiable on $Z$.

(25) There exists a rest $R$ of $F$ such that $R_0 = 0$ and $R$ is continuous in 0.

Let us consider $F$ and let $f$ be a partial function from $\mathbb{R}$ to the carrier of $F$. We say that $f$ is differentiable if and only if:

(Def. 7) $f$ is differentiable on $\text{dom } f$.

Let us consider $F$. Note that there exists a function from $\mathbb{R}$ into the carrier of $F$ which is differentiable.

One can prove the following proposition

(26) Let $f$ be a differentiable partial function from $\mathbb{R}$ to the carrier of $F$. If $Z \subseteq \text{dom } f$, then $f$ is differentiable on $Z$.

References


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Conway’s Games and Some of their Basic Properties

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Summary. We formulate a few basic concepts of J. H. Conway’s theory of games based on his book [6]. This is a first step towards formalizing Conway’s theory of numbers into Mizar, which is an approach to proving the existence of a FIELD (i.e., a proper class that satisfies the axioms of a real-closed field) that includes the reals and ordinals, thus providing a uniform, independent and simple approach to these two constructions that does not go via the rational numbers and hence does for example not need the notion of a quotient field.

In this first article on Conway’s games, we provide a definition of games, their birthdays (or ranks), their trees (a notion which is not in Conway’s book, but is useful as a tool), their negates and their signs, together with some elementary properties of these notions. If one is interested only in Conway’s numbers, it would have been easier to define them directly, but going via the notion of a game is a more general approach in the sense that a number is a special instance of a game and that there is a rich theory of games that are not numbers.

The main obstacle in formulating these topics in Mizar is that all definitions are highly recursive, which is not entirely simple to translate into the Mizar language. For example, according to Conway’s definition, a game is an object consisting of left and right options which are themselves games, and this is by definition the only way to construct a game. This cannot directly be translated into Mizar, but a theorem is included in the article which proves that our definition is equivalent to Conway’s.

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The terminology and notation used here have been introduced in the following articles: [1], [4], [7], [5], [2], [3], [9], and [8].
1. Construction of Days

We adopt the following rules: $x$, $z$ are sets, $a_1$, $b_1$ are ordinal numbers, and $n$ is a natural number.

We consider lefts-rights as systems

\[ \langle \text{left options, right options} \rangle, \]

where the left options and the right options constitute sets.

The functor $0$ is defined as follows:

(Def. 1) \[ 0 = (\emptyset, \emptyset). \]

Let us observe that there exists a left-right which is strict.

Let us consider $a_1$. The functor ConwayDay $a_1$ yields a set and is defined by the condition (Def. 2).

(Def. 2) There exists a transfinite sequence $f$ such that $a_1 \in \text{dom } f$ and $f(a_1) = \text{ConwayDay } a_1$ and for every $b_1$ such that $b_1 \in \text{dom } f$ holds $f(b_1) = \{ (x, y) : x \text{ ranges over subsets of } \bigcup \text{rng}(f \mid b_1), y \text{ ranges over subsets of } \bigcup \text{rng}(f \mid b_1) \}.$

We now state three propositions:

1. $z \in \text{ConwayDay } a_1$ if and only if there exists a strict left-right $w$ such that $z = w$ and for every $x$ such that $x \in (\text{the left options of } w) \cup (\text{the right options of } w)$ there exists $b_1$ such that $b_1 \in a_1$ and $x \in \text{ConwayDay } b_1$.

2. \( \text{ConwayDay } 0 = \{0\}. \)

3. If $a_1 \subseteq b_1$, then $\text{ConwayDay } a_1 \subseteq \text{ConwayDay } b_1$.

Let us consider $a_1$. One can verify that $\text{ConwayDay } a_1$ is non empty.

2. Games

Let us consider $x$. We say that $x$ is Conway game-like if and only if:

(Def. 3) There exists $a_1$ such that $x \in \text{ConwayDay } a_1$.

Let us consider $a_1$. Observe that every element of $\text{ConwayDay } a_1$ is Conway game-like.

Let us note that $0$ is Conway game-like.

Let us note that there exists a left-right which is Conway game-like and strict and there exists a set which is Conway game-like.

A Conway game is a Conway game-like set.

$0$ is an element of $\text{ConwayDay } 0$.

The element $1$ of $\text{ConwayDay } 1$ is defined as follows:

(Def. 4) \[ 1 = (\{0\}, \emptyset). \]

The element $\ast$ of $\text{ConwayDay } 1$ is defined as follows:

(Def. 5) \[ \ast = (\{0\}, \{0\}). \]
In the sequel $g, g_0, g_1, g_2, g_3, g_4, g_5, g_6$ are Conway games.

Next we state the proposition

(4) $g$ is a strict strict left-right.

One can verify that every left-right which is Conway game-like is also strict.

Let us consider $g$. The left options of $g$ is defined as follows:

(Def. 6) There exists a left-right $w$ such that $g = w$ and the left options of $g = \text{the left options of } w$.

The right options of $g$ is defined by:

(Def. 7) There exists a left-right $w$ such that $g = w$ and the right options of $g = \text{the right options of } w$.

Let us consider $g$. The options of $g$ is defined by:

(Def. 8) The options of $g = (\text{the left options of } g) \cup (\text{the right options of } g)$.

Next we state the proposition

(5) $g_1 = g_2$ if and only if the following conditions are satisfied:

(i) the left options of $g_1 = \text{the left options of } g_2$, and

(ii) the right options of $g_1 = \text{the right options of } g_2$.

One can check the following observations:

* the left options of $0$ is empty,
* the right options of $0$ is empty, and
* the right options of $1$ is empty.

We now state four propositions:

(6) $g = 0$ iff the options of $g = \emptyset$.

(7) $x \in \text{the left options of } 1$ iff $x = 0$.

(8)(i) $x \in \text{the options of } *$ iff $x = 0$,

(ii) $x \in \text{the left options of } *$ iff $x = 0$, and

(iii) $x \in \text{the right options of } *$ iff $x = 0$.

(9) $g \in \text{ConwayDay } a_1$ iff for every $x$ such that $x \in \text{the options of } g$ there exists $b_1$ such that $b_1 \in a_1$ and $x \in \text{ConwayDay } b_1$.

Let $g$ be a set. Let us assume that $g$ is a Conway game. The functor ConwayRank$g$ yields an ordinal number and is defined as follows:

(Def. 9) $g \in \text{ConwayDay } \text{ConwayRank } g$ and for every $b_1$ such that $b_1 \in \text{ConwayRank } g$ holds $g \notin \text{ConwayDay } b_1$.

One can prove the following propositions:

(10) If $g \in \text{ConwayDay } a_1$ and $x \in \text{the options of } g$, then $x \in \text{ConwayDay } a_1$.

(11) If $g \in \text{ConwayDay } a_1$ and if $x \in \text{the left options of } g$ or $x \in \text{the right options of } g$, then $x \in \text{ConwayDay } a_1$.

(12) $g \in \text{ConwayDay } a_1$ iff ConwayRank$g \subseteq a_1$. 

(13) ConwayRank $g \in a_1$ iff there exists $b_1$ such that $b_1 \in a_1$ and $g \in ConwayDay b_1$.

(14) If $g_3$ is in the options of $g$, then ConwayRank $g_3 \in ConwayRank g$.

(15) If $g_3$ is in the left options of $g$ or $g_3$ is in the right options of $g$, then ConwayRank $g_3 \in ConwayRank g$.

(16) $g \notin$ the options of $g$.

(17) If $x \in$ the options of $g$, then $x$ is a Conway game-like Conway game-like left-right.

(18) Suppose $x \in$ the left options of $g$ or $x \in$ the right options of $g$. Then $x$ is a Conway game-like Conway game-like left-right.

(19) Let $w$ be a strict left-right. Then $w$ is a Conway game if and only if for every $z$ such that $z \in$ (the left options of $w$) $\cup$ (the right options of $w$) holds $z$ is a Conway game.

3. Schemes of Induction

In this article we present several logical schemes. The scheme ConwayGameMinTot concerns a unary predicate $P$, and states that:

There exists $g$ such that $P[g]$ and for every $g_1$ such that ConwayRank $g_1 \in ConwayRank g$ holds not $P[g_1]$

provided the parameters meet the following condition:

- There exists $g$ such that $P[g]$.

The scheme ConwayGameMin concerns a unary predicate $P$, and states that:

There exists $g$ such that $P[g]$ and for every $g_3$ such that $g_3 \in$ the options of $g$ holds not $P[g_3]$

provided the parameters meet the following requirement:

- There exists $g$ such that $P[g]$.

The scheme ConwayGameInd concerns a unary predicate $P$, and states that:

For every $g$ holds $P[g]$

provided the following condition is satisfied:

- For every $g$ such that for every $g_3$ such that $g_3 \in$ the options of $g$ holds $P[g_3]$ holds $P[g]$.

4. Tree of a Game

Let $f$ be a function. We say that $f$ is Conway game-valued if and only if:

(Def. 10) For every $x$ such that $x \in$ dom $f$ holds $f(x)$ is a Conway game.

Let us consider $g$. Observe that $\langle g \rangle$ is Conway game-valued.

Let us observe that there exists a finite sequence which is Conway game-valued and non empty.
Let $f$ be a non empty finite sequence. Observe that every element of $\text{dom } f$ is natural and non empty.

Let $f$ be a Conway game-valued non empty function and let $x$ be an element of $\text{dom } f$. One can verify that $f(x)$ is Conway game-like.

Let $f$ be a Conway game-valued non empty finite sequence. We say that $f$ is Conway game chain-like if and only if:

(Def. 11) For every element $n$ of $\text{dom } f$ such that $n > 1$ holds $f(n - 1) \in \text{the options of } f(n)$.

One can prove the following proposition

(20) For every finite sequence $f$ and for every $n$ such that $n \in \text{dom } f$ and $n > 1$ holds $n - 1 \in \text{dom } f$.

Let us consider $g$. Observe that $\langle g \rangle$ is Conway game chain-like.

One can verify that there exists a Conway game-valued non empty finite sequence which is Conway game chain-like.

A Conway game chain is a Conway game chain-like Conway game-valued non empty finite sequence.

We now state three propositions:

(21) For every Conway game chain $f$ and for all elements $n$, $m$ of $\text{dom } f$ such that $n < m$ holds $\text{ConwayRank } f(n) \in \text{ConwayRank } f(m)$.

(22) For every Conway game chain $f$ and for all elements $n$, $m$ of $\text{dom } f$ such that $n \leq m$ holds $\text{ConwayRank } f(n) \subseteq \text{ConwayRank } f(m)$.

(23) For every Conway game chain $f$ such that $f(\text{len } f) \in \text{ConwayDay } a_1$ holds $f(1) \in \text{ConwayDay } a_1$.

Let us consider $g$. The tree of $g$ yielding a set is defined as follows:

(Def. 12) $z \in \text{the tree of } g$ iff there exists a Conway game chain $f$ such that $f(1) = z$ and $f(\text{len } f) = g$.

Let us consider $g$. One can verify that the tree of $g$ is non empty.

Let us consider $g$. The proper tree of $g$ yields a subset of the tree of $g$ and is defined as follows:

(Def. 13) The proper tree of $g = (\text{the tree of } g) \setminus \{g\}$.

We now state the proposition

(24) $g \in \text{the tree of } g$.

Let us consider $a_1$ and let $g$ be an element of $\text{ConwayDay } a_1$. Then the tree of $g$ is a subset of $\text{ConwayDay } a_1$.

Let us consider $g$. One can verify that every element of the tree of $g$ is Conway game-like.

One can prove the following propositions:

(25) For every Conway game chain $f$ and for every non empty natural number $n$ holds $f|n$ is a Conway game chain.
(26) Let $f_1, f_2$ be Conway game chains. Given $g$ such that $g = f_2(1)$ and $f_1(\text{len } f_1) \in$ the options of $g$. Then $f_1 \sim f_2$ is a Conway game chain.

(27) $x \in$ the tree of $g$ iff $x = g$ or there exists $g_3$ such that $g_3 \in$ the options of $g$ and $x \in$ the tree of $g_3$.

(28) If $g_3 \in$ the tree of $g$, then $g_3 = g$ or ConwayRank $g_3 \in$ ConwayRank $g$.

(29) If $g_3 \in$ the tree of $g$, then ConwayRank $g_3 \subseteq$ ConwayRank $g$.

(30) For every set $s$ such that $g \in s$ and for every $g_1$ such that $g_1 \in s$ holds the options of $g_1 \subseteq s$ holds the tree of $g \subseteq s$.

(31) If $g_1 \in$ the tree of $g_2$, then the tree of $g_1 \subseteq$ the tree of $g_2$.

(32) If $g_1 \in$ the tree of $g_2$, then the proper tree of $g_1 \subseteq$ the proper tree of $g_2$.

(33) The options of $g \subseteq$ the proper tree of $g$.

(34) The options of $g \subseteq$ the tree of $g$.

(35) If $g_1 \in$ the proper tree of $g_2$, then the tree of $g_1 \subseteq$ the proper tree of $g_2$.

(36) If $g_3 \in$ the options of $g$, then the tree of $g_3 \subseteq$ the proper tree of $g$.

(37) The tree of $0 = \{0\}$.

(38) $0 \in$ the tree of $g$.

The scheme $\text{ConwayGameMin2}$ concerns a unary predicate $\mathcal{P}$, and states that:

There exists $g$ such that $\mathcal{P}[g]$ and for every $g_3$ such that $g_3 \in$ the proper tree of $g$ holds not $\mathcal{P}[g_3]$

provided the following condition is satisfied:

• There exists $g$ such that $\mathcal{P}[g]$.

5. Scheme about Definability of Functions by Recursion

Now we present two schemes. The scheme $\text{Func1RecUniq}$ deals with a binary functor $\mathcal{F}$ yielding a set, and states that:

Let given $g$ and $f_1, f_2$ be functions. Suppose that

(i) $\text{dom } f_1 = \text{the tree of } g$,

(ii) $\text{dom } f_2 = \text{the tree of } g$,

(iii) for every $g_1$ such that $g_1 \in \text{dom } f_1$ holds $f_1(g_1) = \mathcal{F}(g_1, f_1|\text{the proper tree of } g_1)$, and

(iv) for every $g_1$ such that $g_1 \in \text{dom } f_2$ holds $f_2(g_1) = \mathcal{F}(g_1, f_2|\text{the proper tree of } g_1)$.

Then $f_1 = f_2$

for all values of the parameter.

The scheme $\text{Func1RecEx}$ deals with a binary functor $\mathcal{F}$ yielding a set, and states that:
There exists a function $f$ such that $\text{dom } f = \text{the tree of } g$ and for every $g_1$ such that $g_1 \in \text{dom } f$ holds $f(g_1) = F(g_1, f|\text{the proper tree of } g_1)$ for all values of the parameter.

6. The Negative and Signs

Let us consider $g$. The functor $-g$ is defined by the condition (Def. 14).

(Def. 14) There exists a function $f$ such that

(i) $\text{dom } f = \text{the tree of } g$,
(ii) $-g = f(g)$, and
(iii) for every $g_1$ such that $g_1 \in \text{dom } f$ holds $f(g_1) = \{ \{ f(g_4); g_4 \text{ ranges over elements of the right options of } g_1 \text{; the right options of } g_1 \neq \emptyset \}, \{ f(g_7); g_7 \text{ ranges over elements of the left options of } g_1 \text{; the left options of } g_1 \neq \emptyset \} \}$.

Let us consider $g$. One can check that $-g$ is Conway game-like. Next we state three propositions:

(39)(i) For every $x$ holds $x \in \text{the left options of } -g$ iff there exists $g_4$ such that $g_4 \in \text{the right options of } g$ and $x = -g_4$, and
(ii) for every $x$ holds $x \in \text{the right options of } -g$ iff there exists $g_7$ such that $g_7 \in \text{the left options of } g$ and $x = -g_7$.

(40) $-\neg g = g$.

(41)(i) $g_3 \in \text{the left options of } -g$ iff $-g_3 \in \text{the right options of } g$,
(ii) $g_3 \in \text{the left options of } g$ iff $-g_3 \in \text{the right options of } -g$,
(iii) $g_3 \in \text{the right options of } -g$ iff $-g_3 \in \text{the left options of } g$, and
(iv) $g_3 \in \text{the right options of } g$ iff $-g_3 \in \text{the left options of } -g$.

Let us consider $g$. We say that $g$ is non-negative if and only if the condition (Def. 15) is satisfied.

(Def. 15) There exists $s$ such that

(i) $g \in s$, and
(ii) for every $g_1$ such that $g_1 \in s$ and for every $g_4$ such that $g_4 \in \text{the right options of } g_1$ there exists $g_8$ such that $g_8 \in \text{the left options of } g_4$ and $g_8 \in s$.

Let us consider $g$. We say that $g$ is non-positive if and only if:

(Def. 16) $-g$ is non-negative.

Let us consider $g$. We say that $g$ is zero if and only if:

(Def. 17) $g$ is non-negative and non-positive.

We say that $g$ is fuzzy if and only if:

(Def. 18) $g$ is not non-negative and $g$ is not non-positive.

Let us consider $g$. We say that $g$ is positive if and only if:

(Def. 19) $g$ is non-negative and $g$ is not zero.
We say that \( g \) is negative if and only if:

(Def. 20) \( g \) is non-positive and \( g \) is not zero.

One can verify the following observations:

* every Conway game which is zero is also non-negative and non-positive,
* every Conway game which is non-positive and non-negative is also zero,
* every Conway game which is negative is also non-positive and non zero,
* every Conway game which is non-positive and non zero is also negative,
* every Conway game which is positive is also non-negative and non zero,
* every Conway game which is non-negative and non zero is also positive,
* every Conway game which is fuzzy is also non non-negative and non non-positive, and
* every Conway game which is non non-negative and non non-positive is also fuzzy.

We now state several propositions:

(42) \( g \) is zero, or positive, or negative, or fuzzy.

(43) \( g \) is non-negative if and only if for every \( g_4 \) such that \( g_4 \in \) the right options of \( g \) there exists \( g_8 \) such that \( g_8 \in \) the left options of \( g_4 \) and \( g_8 \) is non-negative.

(44) \( g \) is non-positive if and only if for every \( g_7 \) such that \( g_7 \in \) the left options of \( g \) there exists \( g_6 \) such that \( g_6 \in \) the right options of \( g_7 \) and \( g_6 \) is non-positive.

(45)(i) \( g \) is non-negative iff for every \( g_4 \) such that \( g_4 \in \) the right options of \( g \) holds \( g_4 \) is fuzzy or positive, and

(ii) \( g \) is non-positive iff for every \( g_7 \) such that \( g_7 \in \) the left options of \( g \) holds \( g_7 \) is fuzzy or negative.

(46) \( g \) is fuzzy if and only if the following conditions are satisfied:

(i) there exists \( g_7 \) such that \( g_7 \in \) the left options of \( g \) and \( g_7 \) is non-negative, and

(ii) there exists \( g_4 \) such that \( g_4 \in \) the right options of \( g \) and \( g_4 \) is non-positive.

(47) \( g \) is zero if and only if the following conditions are satisfied:

(i) for every \( g_7 \) such that \( g_7 \in \) the left options of \( g \) holds \( g_7 \) is fuzzy or negative, and

(ii) for every \( g_4 \) such that \( g_4 \in \) the right options of \( g \) holds \( g_4 \) is fuzzy or positive.

(48) \( g \) is positive if and only if the following conditions are satisfied:

(i) for every \( g_4 \) such that \( g_4 \in \) the right options of \( g \) holds \( g_4 \) is fuzzy or positive, and

(ii) there exists \( g_7 \) such that \( g_7 \in \) the left options of \( g \) and \( g_7 \) is non-negative.
(49) $g$ is negative if and only if the following conditions are satisfied:

(i) for every $g_7$ such that $g_7 \in$ the left options of $g$ holds $g_7$ is fuzzy or negative, and

(ii) there exists $g_4$ such that $g_4 \in$ the right options of $g$ and $g_4$ is non-positive.

Let us observe that $0$ is zero.

Let us observe that $1$ is positive and $*$ is fuzzy.

One can check the following observations:

* there exists a Conway game which is zero,

* there exists a Conway game which is positive, and

* there exists a Conway game which is fuzzy.

Let $g$ be a non-positive Conway game. Note that $-g$ is non-negative.

Let $g$ be a non-negative Conway game. One can check that $-g$ is non-positive.

Let $g$ be a positive Conway game. Observe that $-g$ is negative.

One can check that there exists a Conway game which is negative.

Let $g$ be a negative Conway game. One can check that $-g$ is positive.

Let $g$ be a fuzzy Conway game. Note that $-g$ is fuzzy.

References


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Veblen Hierarchy

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Summary. The Veblen hierarchy is an extension of the construction of epsilon numbers (fixpoints of the exponential map: \( \omega^\omega = \varepsilon \)). It is a collection \( \varphi_\alpha \) of the Veblen Functions where \( \varphi_0(\beta) = \omega^\beta \) and \( \varphi_1(\beta) = \varepsilon^\beta \). The sequence of fixed points of \( \varphi_1 \) function form \( \varphi_2 \), etc. For a limit non empty ordinal \( \lambda \) the function \( \varphi_\lambda \) is the sequence of common fixed points of all functions \( \varphi_\alpha \) where \( \alpha < \lambda \).

The Mizar formalization of the concept cannot be done directly as the Veblen functions are classes (not (small) sets). It is done with use of universal sets (Tarski classes). Namely, we define the Veblen functions in a given universal set and \( \varphi_\alpha(\beta) \) as a value of Veblen function from the smallest universal set including \( \alpha \) and \( \beta \).

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The papers [16], [18], [2], [5], [14], [13], [9], [10], [15], [3], [4], [1], [8], [11], [19], [17], [6], [7], and [12] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we adopt the following convention: \( \alpha, \beta, \gamma, \delta \) denote ordinal numbers, \( \lambda \) denotes a non empty limit ordinal ordinal number, \( A \) denotes a non empty ordinal number, \( e \) denotes an element of \( A \), \( X, Y, x, y \) denote sets, and \( n \) denotes a natural number.

Next we state several propositions:

(1) Let \( \varphi \) be a function. Suppose \( \varphi \) is an isomorphism between \( \subseteq_X \) and \( \subseteq_Y \). Let given \( x, y \). If \( x, y \in X \), then \( x \subseteq y \) if \( \varphi(x) \subseteq \varphi(y) \).

(2) Let \( X, Y \) be ordinal-membered sets and \( \varphi \) be a function. Suppose \( \varphi \) is an isomorphism between \( \subseteq_X \) and \( \subseteq_Y \). Let given \( x, y \). If \( x, y \in X \), then \( x \in y \) if \( \varphi(x) \in \varphi(y) \).
(3) If \( (x, y) \in \subseteq_X \), then \( x \subseteq y \).

(4) For all transfinite sequences \( f_1, f_2 \) holds \( f_1 \subseteq f_1 \triangleleft f_2 \).

(5) For all transfinite sequences \( f_1, f_2 \) holds \( \text{rng}(f_1 \triangleleft f_2) = \text{rng} f_1 \cup \text{rng} f_2 \).

(6) \( \alpha \subseteq \beta \) iff \( \varepsilon_\alpha \subseteq \varepsilon_\beta \).

(7) \( \alpha \in \beta \) iff \( \varepsilon_\alpha \in \varepsilon_\beta \).

Let \( X \) be an ordinal-membered set. Note that \( \cup X \) is ordinal.

Let \( \varphi \) be an ordinal yielding function. Observe that \( \text{rng} \varphi \) is ordinal-membered.

Let us consider \( \alpha \). Note that \( \text{id}_\alpha \) is transfinite sequence-like and ordinal yielding.

Let us consider \( \alpha \). Observe that \( \text{id}_\alpha \) is increasing.

Let us consider \( \alpha \). Note that \( \text{id}_\alpha \) is continuous.

Let us observe that there exists a sequence of ordinal numbers which is non empty, increasing, and continuous.

Let \( \varphi \) be a transfinite sequence. We say that \( \varphi \) is normal if and only if:

(Def. 1) \( \varphi \) is an increasing continuous sequence of ordinal numbers.

Let \( \varphi \) be a sequence of ordinal numbers. Let us observe that \( \varphi \) is normal if and only if:

(Def. 2) \( \varphi \) is increasing and continuous.

One can verify the following observations:

* every transfinite sequence which is normal is also ordinal yielding,
* every sequence of ordinal numbers which is normal is also increasing and continuous, and
* every sequence of ordinal numbers which is increasing and continuous is also normal.

Let us observe that there exists a transfinite sequence which is non empty and normal.

Next we state the proposition

(8) For every sequence \( \varphi \) of ordinal numbers such that \( \varphi \) is non-decreasing holds \( \varphi|_\alpha \) is non-decreasing.

Let us consider \( X \). The functor \( \text{ord-type} X \) yields an ordinal number and is defined by:

(Def. 3) \( \text{ord-type} X = \overline{\subseteq}_{\text{On} X} \).

Let \( X \) be an ordinal-membered set. Then \( \text{ord-type} X \) can be characterized by the condition:

(Def. 4) \( \text{ord-type} X = \overline{\subseteq} X \).

Let \( X \) be an ordinal-membered set. One can verify that \( \subseteq X \) is well-ordering.

Let \( E \) be an empty set. Observe that \( \text{On} E \) is empty.

Let \( E \) be an empty set. One can verify that \( \overline{E} \) is empty.

Next we state four propositions:
(9) ord-type $\emptyset = 0$.
(10) ord-type $\{\alpha\} = 1$.
(11) If $\alpha \neq \beta$, then ord-type $\{\alpha, \beta\} = 2$.
(12) ord-type $\alpha = \alpha$.

Let us consider $X$. The functor numbering $X$ yields a sequence of ordinal numbers and is defined as follows:

(Def. 5) numbering $X = \text{the canonical isomorphism between } \subseteq_{\text{ord-type } X} \text{ and } \subseteq_{\text{On } X}$.

Next we state four propositions:

(13) dom numbering $X = \text{ord-type } X$ and rng numbering $X = \text{On } X$.
(14) For every ordinal-membered set $X$ holds rng numbering $X = X$.
(15) Card dom numbering $X = \text{Card On } X$.
(16) numbering $X$ is an isomorphism between $\subseteq_{\text{ord-type } X}$ and $\subseteq_{\text{On } X}$.

In the sequel $\varphi$ denotes a sequence of ordinal numbers.

One can prove the following propositions:

(17) If $\varphi = \text{numbering } X$ and $x, y \in \text{dom } \varphi$, then $x \subseteq y$ iff $\varphi(x) \subseteq \varphi(y)$.
(18) If $\varphi = \text{numbering } X$ and $x, y \in \text{dom } \varphi$, then $x \in y$ iff $\varphi(x) \in \varphi(y)$.

Let us consider $X$. Note that numbering $X$ is increasing.

Let $X, Y$ be ordinal-membered sets. One can check that $X \cup Y$ is ordinal-membered.

Let $X$ be an ordinal-membered set and let $Y$ be a set. Observe that $X \setminus Y$ is ordinal-membered.

The following three propositions are true:

(19) Let $X, Y$ be ordinal-membered sets. Suppose that for all $x, y$ such that $x \in X$ and $y \in Y$ holds $x \in y$. Then (numbering $X$) $\cap$ numbering $Y$ is an isomorphism between $\subseteq_{\text{ord-type } X + \text{ord-type } Y}$ and $\subseteq_{X \cup Y}$.
(20) For all ordinal-membered sets $X, Y$ such that for all $x, y$ such that $x \in X$ and $y \in Y$ holds numbering($X \cup Y$) = (numbering $X$) $\cap$ numbering $Y$.
(21) For all ordinal-membered sets $X, Y$ such that for all $x, y$ such that $x \in X$ and $y \in Y$ holds ord-type($X \cup Y$) = ord-type $X + \text{ord-type } Y$.

2. Fixpoints of a Normal Function

We now state the proposition

(22) For every function $\varphi$ such that $x$ is a fixpoint of $\varphi$ holds $x \in \text{rng } \varphi$.

Let $\varphi$ be a sequence of ordinal numbers. The functor criticals $\varphi$ yielding a sequence of ordinal numbers is defined as follows:
(Def. 6) criticals$\varphi = \text{numbering}\{\alpha \in \text{dom } \varphi : \alpha \text{ is a fixpoint of } \varphi\}$.

Next we state three propositions:

(23) On$\{\alpha \in \text{dom } \varphi : \alpha \text{ is a fixpoint of } \varphi\} = \{\alpha \in \text{dom } \varphi : \alpha \text{ is a fixpoint of } \varphi\}$.

(24) If $x \in \text{dom } \text{criticals } \varphi$, then $(\text{criticals } \varphi)(x)$ is a fixpoint of $\varphi$.

(25) $\text{rng } \text{criticals } \varphi = \{\alpha \in \text{dom } \varphi : \alpha \text{ is a fixpoint of } \varphi\}$ and $\text{rng } \text{criticals } \varphi \subseteq \text{rng } \varphi$.

Let us consider $\varphi$. Note that $\text{criticals } \varphi$ is increasing.

One can prove the following propositions:

(26) If $x \in \text{dom } \text{criticals } \varphi$, then $x \subseteq (\text{criticals } \varphi)(x)$.

(27) $\text{dom } \text{criticals } \varphi \subseteq \text{dom } \varphi$.

(28) If $\beta$ is a fixpoint of $\varphi$, then there exists $\alpha$ such that $\alpha \in \text{dom } \text{criticals } \varphi$ and $\beta = (\text{criticals } \varphi)(\alpha)$.

(29) If $\alpha \in \text{dom } \text{criticals } \varphi$ and $\beta$ is a fixpoint of $\varphi$ and $(\text{criticals } \varphi)(\alpha) \in \beta$, then $\text{succ } \alpha \in \text{dom } \text{criticals } \varphi$.

(30) If $\text{succ } \alpha \in \text{dom } \text{criticals } \varphi$ and $\beta$ is a fixpoint of $\varphi$ and $(\text{criticals } \varphi)(\alpha) \in \beta$, then $(\text{criticals } \varphi)(\text{succ } \alpha) \subseteq \beta$.

(31) Suppose $\varphi$ is normal and $\bigcup X \in \text{dom } \varphi$ and $X$ is non empty and for every $x$ such that $x \in X$ there exists $y$ such that $x \subseteq y$ and $y \in X$ and $y$ is a fixpoint of $\varphi$. Then $\bigcup X = \varphi(\bigcup X)$.

(32) If $\varphi$ is normal and $\bigcup X \in \text{dom } \varphi$ and $X$ is non empty and for every $x$ such that $x \in X$ holds $x$ is a fixpoint of $\varphi$, then $\bigcup X = \varphi(\bigcup X)$.

(33) If $\lambda \subseteq \text{dom } \text{criticals } \varphi$ and $\alpha$ is a fixpoint of $\varphi$ and for every $x$ such that $x \in \lambda$ holds $(\text{criticals } \varphi)(x) \in \alpha$, then $\lambda \in \text{dom } \text{criticals } \varphi$.

(34) If $\varphi$ is normal and $\lambda \in \text{dom } \text{criticals } \varphi$, then $(\text{criticals } \varphi)(\lambda) = \bigcup(\text{criticals } \varphi\mid \lambda)$.

Let $\varphi$ be a normal sequence of ordinal numbers. Observe that $\text{criticals } \varphi$ is continuous.

One can prove the following proposition

(35) For all sequences $f_1$, $f_2$ of ordinal numbers such that $f_1 \subseteq f_2$ holds criticals$f_1 \subseteq \text{criticals } f_2$.

3. Fixpoints in a Universal Set

In the sequel $U$, $W$ are universal classes.

Let us consider $U$. Note that there exists a transfinite sequence of ordinals of $U$ which is normal.

Let us consider $U$, $\alpha$. A ordinal-sequence from $\alpha$ to $U$ is a function from $\alpha$ into $\text{On } U$. 
Let us consider $U$, $\alpha$. Observe that every ordinal-sequence from $\alpha$ to $U$ is transfinite sequence-like and ordinal yielding.

Let us consider $U$, $\alpha$, let $\varphi$ be an ordinal-sequence from $\alpha$ to $U$, and let us consider $x$. Then $\varphi(x)$ is an ordinal of $U$.

Next we state two propositions:

(36) If $\alpha \in U$, then for every ordinal-sequence $\varphi$ from $\alpha$ to $U$ holds $\bigcup \varphi \in U$.

(37) If $\alpha \in U$, then for every ordinal-sequence $\varphi$ from $\alpha$ to $U$ holds $\sup \varphi \in U$.

In this article we present several logical schemes. The scheme CriticalNumber2 deals with a universal class $A$, an ordinal $B$ of $A$, a ordinal-sequence $C$ from $\omega$ to $A$, and a unary functor $F$ yielding an ordinal number, and states that:

$B \subseteq \bigcup C$ and $F(\bigcup C) = \bigcup C$ and for every $\beta$ such that $B \subseteq \beta$ and $\beta \in A$ and $F(\beta) = \beta$ holds $\bigcup C \subseteq \beta$

provided the parameters satisfy the following conditions:

- $\omega \in A$.
- For every $\alpha$ such that $\alpha \in A$ holds $F(\alpha) \in A$.
- For all $\alpha$, $\beta$ such that $\alpha \in \beta$ and $\beta \in A$ holds $F(\alpha) \in F(\beta)$.
- Let $\alpha$ be an ordinal of $A$. Suppose $\alpha$ is non empty and limit ordinal. Let $\varphi_1$ be a sequence of ordinal numbers. If $\text{dom} \varphi_1 = \alpha$ and for every $\beta$ such that $\beta \in \alpha$ holds $\varphi_1(\beta) = F(\beta)$, then $F(\alpha)$ is the limit of $\varphi_1$.
- $C(0) = B$, and
- For every $\alpha$ such that $\alpha \in \omega$ holds $C(\text{succ} \alpha) = F(C(\alpha))$.

The scheme CriticalNumber3 deals with a universal class $A$, an ordinal $B$ of $A$, and a unary functor $F$ yielding an ordinal number, and states that:

There exists an ordinal $\alpha$ of $A$ such that $B \in \alpha$ and $F(\alpha) = \alpha$

provided the parameters satisfy the following conditions:

- $\omega \in A$.
- For every $\alpha$ such that $\alpha \in A$ holds $F(\alpha) \in A$.
- For all $\alpha$, $\beta$ such that $\alpha \in \beta$ and $\beta \in A$ holds $F(\alpha) \in F(\beta)$, and
- Let $\alpha$ be an ordinal of $A$. Suppose $\alpha$ is non empty and limit ordinal. Let $\varphi_1$ be a sequence of ordinal numbers. If $\text{dom} \varphi_1 = \alpha$ and for every $\beta$ such that $\beta \in \alpha$ holds $\varphi_1(\beta) = F(\beta)$, then $F(\alpha)$ is the limit of $\varphi_1$.

In the sequel $F$, $\varphi_1$ denote normal transfinite sequences of ordinals of $W$.

One can prove the following propositions:

(38) If $\omega$, $\beta \in W$, then there exists $\alpha$ such that $\beta \in \alpha$ and $\alpha$ is a fixpoint of $\varphi_1$.

(39) If $\omega \in W$, then criticals $F$ is a transfinite sequence of ordinals of $W$.

(40) If $\varphi$ is normal, then for every $\alpha$ such that $\alpha \in \text{dom criticals} \varphi$ holds $\varphi(\alpha) \subseteq (\text{criticals} \varphi)(\alpha)$.
4. Sequences of Sequences of Ordinals

Let us consider $U$ and let $\alpha, \beta$ be ordinals of $U$. Then $\alpha^\beta$ is an ordinal of $U$.

Let us consider $U, \alpha$. Let us assume that $\alpha \in U$. The functor $U \exp \alpha$ yielding a transfinite sequence of ordinals of $U$ is defined as follows:

(Def. 7) For every ordinal $\beta$ of $U$ holds $(U \exp \alpha)(\beta) = \alpha^\beta$.

Let us note that $\omega$ is non trivial.

Let us consider $U$. One can verify that there exists an ordinal of $U$ which is non trivial and finite.

Let us note that there exists an ordinal number which is non trivial and finite.

Let us consider $U$ and let $\alpha$ be a non trivial ordinal of $U$. Note that $U \exp \alpha$ is normal.

Let $\psi$ be a function. We say that $\psi$ is ordinal-sequence-valued if and only if:

(Def. 8) If $x \in \text{rng } \psi$, then $x$ is a sequence of ordinal numbers.

Let $\varphi$ be a sequence of ordinal numbers. Observe that $\langle \varphi \rangle$ is ordinal-sequence-valued.

Let $\varphi$ be a function. We say that $\varphi$ has the same dom if and only if:

(Def. 9) $\text{rng } \varphi$ has common domain.

Let $\varphi$ be a function. Observe that $\{\varphi\}$ has common domain.

Let $\varphi$ be a function. Note that $\langle \varphi \rangle$ has the same dom.

Let us mention that there exists a transfinite sequence which is non empty and ordinal-sequence-valued and has the same dom.

Let $\psi$ be an ordinal-sequence-valued function and let us consider $x$. Note that $\psi(x)$ is relation-like and function-like.

Let $\psi$ be an ordinal-sequence-valued function and let us consider $x$. Observe that $\psi(x)$ is transfinite sequence-like and ordinal yielding.

Let $\psi$ be a transfinite sequence and let us consider $\alpha$. Note that $\psi \restriction \alpha$ is transfinite sequence-like.

Let $\psi$ be an ordinal-sequence-valued transfinite sequence. The functor criticals $\psi$ yields a sequence of ordinal numbers and is defined by:

(Def. 10) $\text{criticals } \psi = \text{numbering}\{\alpha \in \text{dom } \psi(0) : \alpha \in \text{dom } \psi(0) \land \bigwedge_\varphi (\varphi \in \text{rng } \psi \Rightarrow \alpha \text{ is a fixpoint of } \varphi)\}$.

In the sequel $\psi$ is an ordinal-sequence-valued transfinite sequence.

The following propositions are true:

(41) Let given $\psi$. Then $\{\alpha \in \text{dom } \psi(0) : \alpha \in \text{dom } \psi(0) \land \bigwedge_\varphi (\varphi \in \text{rng } \psi \Rightarrow \alpha \text{ is a fixpoint of } \varphi)\}$ is ordinal-membered.
(42) If $\alpha \in \text{dom } \psi$ and $\beta \in \text{dom criticals } \psi$, then $(\text{criticals } \psi)(\beta)$ is a fixpoint of $\psi(\alpha)$.

(43) If $x \in \text{dom criticals } \psi$, then $x \subseteq (\text{criticals } \psi)(x)$.

(44) If $\varphi \in \text{rng } \psi$, then $\text{dom criticals } \psi \subseteq \text{dom } \varphi$.

(45) If $\text{dom } \psi \neq \emptyset$ and for every $\gamma$ such that $\gamma \in \text{dom } \psi$ holds $\beta$ is a fixpoint of $\psi(\gamma)$, then there exists $\alpha$ such that $\alpha \in \text{dom criticals } \psi$ and $\beta = (\text{criticals } \psi)(\alpha)$.

(46) Suppose $\text{dom } \psi \neq \emptyset$ and $\lambda \subseteq \text{dom criticals } \psi$ and for every $\varphi$ such that $\varphi \in \text{rng } \psi$ holds $\alpha$ is a fixpoint of $\varphi$ and for every $x$ such that $x \in \lambda$ holds $(\text{criticals } \psi)(x) \in \alpha$. Then $\lambda \in \text{dom criticals } \psi$.

(47) For every $\psi$ such that $\text{dom } \psi \neq \emptyset$ and for every $\alpha$ such that $\alpha \in \text{dom } \psi$ holds $\psi(\alpha)$ is normal holds if $\lambda \in \text{dom criticals } \psi$, then $(\text{criticals } \psi)(\lambda) = \bigcup(\text{criticals } \psi|\lambda)$.

(48) For every $\psi$ such that $\text{dom } \psi \neq \emptyset$ and for every $\alpha$ such that $\alpha \in \text{dom } \psi$ holds $\psi(\alpha)$ is normal holds criticals $\psi$ is continuous.

(49) Let given $\psi$. Suppose $\text{dom } \psi \neq \emptyset$ and for every $\alpha$ such that $\alpha \in \text{dom } \psi$ holds $\psi(\alpha)$ is normal. Let given $\alpha$, $\varphi$. If $\alpha \in \text{dom criticals } \psi$ and $\varphi \in \text{rng } \psi$, then $\varphi(\alpha) \subseteq (\text{criticals } \psi)(\alpha)$.

(50) Let $g_1, g_2$ be ordinal-sequence-valued transfinite sequences. If $\text{dom } g_1 = \text{dom } g_2$ and $\text{dom } g_1 \neq \emptyset$ and for every $\alpha$ such that $\alpha \in \text{dom } g_1$ holds $g_1(\alpha) \subseteq g_2(\alpha)$, then criticals $g_1 \subseteq$ criticals $g_2$.

Let $\psi$ be an ordinal-sequence-valued transfinite sequence. The functor $\text{lims } \psi$ yielding a sequence of ordinal numbers is defined as follows:

(Def. 11) $\text{dom lims } \psi = \text{dom } \psi(0)$ and for every $\alpha$ such that $\alpha \in \text{dom lims } \psi$ holds $(\text{lims } \psi)(\alpha) = \bigcup\{\psi(\beta)(\alpha); \beta \text{ ranges over elements of dom } \psi : \beta \in \text{dom } \psi\}$.

Next we state the proposition

(51) Let $\psi$ be an ordinal-sequence-valued transfinite sequence. Suppose $\text{dom } \psi \neq \emptyset$ and $\text{dom } \psi \in U$ and for every $\alpha$ such that $\alpha \in \text{dom } \psi$ holds $\psi(\alpha)$ is a transfinite sequence of ordinals of $U$. Then $\text{lims } \psi$ is a transfinite sequence of ordinals of $U$.

5. Veblen Hierarchy

Let us consider $x$. The functor $OS(x)$ yielding a sequence of ordinal numbers is defined as follows:

(Def. 12) $OS(x) = \begin{cases} x, & \text{if } x \text{ is a sequence of ordinal numbers}, \\ \emptyset, & \text{otherwise}. \end{cases}$

The functor $OSV(x)$ yields an ordinal-sequence-valued transfinite sequence and is defined by:
Let us consider $U$. The functor $U$-Veblen yielding an ordinal-sequence-valued transfinite sequence is defined by the conditions (Def. 14).

(Def. 14)(i) $\text{dom}(U$-Veblen) = $On$

(ii) $U$-Veblen(0) = $U$ exp $\omega$,

(iii) for every $\beta$ such that $\text{succ} \beta \in On$ holds $U$-Veblen(succ $\beta$) = \text{criticals} $U$-Veblen($\beta$), and

(iv) for every $\lambda$ such that $\lambda \in On$ holds $U$-Veblen($\lambda$) = \text{criticals}($U$-Veblen|$\lambda$).

Let us mention that there exists a universal class which is uncountable.

The following propositions are true:

(52) For every universal class $U$ holds $U$ is uncountable iff $\omega \in U$.

(53) If $\alpha \in \beta$ and $\beta, \omega \in U$ and $\gamma \in \text{dom} U$-Veblen($\beta$), then $U$-Veblen($\beta$)($\gamma$) is a fixpoint of $U$-Veblen($\alpha$).

(54) If $\lambda \in U$ and for every $\gamma$ such that $\gamma \in \lambda$ holds $\alpha$ is a fixpoint of $U$-Veblen($\gamma$), then $\alpha$ is a fixpoint of lims($U$-Veblen|$\lambda$).

(55) If $\alpha \subseteq \beta$ and $\beta, \omega \in U$ and $\gamma \in \text{dom} U$-Veblen($\beta$) and for every $\gamma$ such that $\gamma \in \beta$ holds $U$-Veblen($\gamma$) is normal, then $U$-Veblen($\alpha$)($\gamma$) $\subseteq$ $U$-Veblen($\beta$)($\gamma$).

(56) Suppose $\lambda, \alpha \in U$ and $\beta \in \lambda$ and for every $\gamma$ such that $\gamma \in \lambda$ holds $U$-Veblen($\gamma$) is a normal transfinite sequence of ordinals of $U$. Then lims($U$-Veblen|$\lambda$)($\alpha$) is a fixpoint of $U$-Veblen($\beta$).

(57) If $\omega \in U$, then $U$-Veblen(0) is a normal transfinite sequence of ordinals of $U$.

(58) Suppose $\omega, \alpha \in U$ and $U$-Veblen($\alpha$) is a normal transfinite sequence of ordinals of $U$. Then $U$-Veblen(succ $\alpha$) is a normal transfinite sequence of ordinals of $U$.

(59) Suppose $\lambda \in U$ and for every $\alpha$ such that $\alpha \in \lambda$ holds $U$-Veblen($\alpha$) is a normal transfinite sequence of ordinals of $U$. Then $U$-Veblen($\lambda$) is a normal transfinite sequence of ordinals of $U$.

(60) If $\omega, \alpha \in U$, then $U$-Veblen($\alpha$) is a normal transfinite sequence of ordinals of $U$.

(61) If $\omega \in U$ and $U \subseteq W$ and $\alpha \in U$, then $U$-Veblen($\alpha$) $\subseteq$ $W$-Veblen($\alpha$).

(62) If $\omega, \alpha, \beta \in U$ and $\omega, \alpha, \beta \in W$, then $U$-Veblen($\beta$)($\alpha$) = $W$-Veblen($\beta$)($\alpha$).

(63) Suppose $\lambda \in U$ and for every $\alpha$ such that $\alpha \in \lambda$ holds $U$-Veblen($\alpha$) is a normal transfinite sequence of ordinals of $U$. Then lims($U$-Veblen|$\lambda$) is a non-decreasing continuous transfinite sequence of ordinals of $U$.

Let us consider $\alpha$. One can verify that $T(\alpha \cup \omega)$ is uncountable.
Let us consider $\alpha, \beta$. The functor $\varphi_{\alpha}(\beta)$ yielding an ordinal number is defined by:

(Def. 15) $\varphi_{\alpha}(\beta) = T(\alpha \cup \beta \cup \omega)$-Veblen$(\alpha)(\beta)$.

Let us consider $n, \beta$. Then $\varphi_n(\beta)$ is an ordinal number and it can be characterized by the condition:

(Def. 16) $\varphi_n(\beta) = T(\beta \cup \omega)$-Veblen$(n)(\beta)$.

Next we state several propositions:

(64) $\alpha \in T(\alpha \cup \beta \cup \gamma)$.

(65) If $\omega, \alpha, \beta \in U$, then $\varphi_{\beta}(\alpha) = U$-Veblen$(\beta)(\alpha)$.

(66) $\varphi_0(\alpha) = \omega^\alpha$.

(67) $\varphi_{\beta}(\varphi_{\text{succ}}(\alpha)) = \varphi_{\text{succ}}(\beta)(\alpha)$.

(68) If $\beta \in \gamma$, then $\varphi_{\beta}(\varphi_{\gamma}(\alpha)) = \varphi_{\gamma}(\alpha)$.

(69) $\alpha \subseteq \beta$ iff $\varphi_{\gamma}(\alpha) \subseteq \varphi_{\gamma}(\beta)$.

(70) $\alpha \in \beta$ iff $\varphi_{\gamma}(\alpha) \in \varphi_{\gamma}(\beta)$.

(71) $\varphi_{\alpha}(\beta) \in \varphi_{\gamma}(\delta)$ iff $\alpha = \gamma$ and $\beta \in \delta$ or $\alpha \in \gamma$ and $\beta \in \varphi_{\gamma}(\delta)$ or $\gamma \in \alpha$ and $\varphi_{\alpha}(\beta) \in \delta$.

6. Epsilon Numbers

In the sequel $U$ is an uncountable universal class.

We now state several propositions:

(72) $U$-Veblen$(1) = \text{criticals}(U \exp \omega)$.

(73) $\varphi_1(\alpha)$ is epsilon.

(74) For every epsilon ordinal number $e$ there exists $\alpha$ such that $e = \varphi_1(\alpha)$.

(75) $\varphi_1(0) = \varepsilon_0$.

(76) If $\varphi_1(\alpha) = \varepsilon_\alpha$, then $\varphi_1(\text{succ } \alpha) = \varepsilon_{\text{succ } \alpha}$.

(77) If for every $\alpha$ such that $\alpha \in \lambda$ holds $\varphi_1(\alpha) = \varepsilon_\alpha$, then $\varphi_1(\lambda) = \varepsilon_\lambda$.

(78) $\varphi_1(\alpha) = \varepsilon_\alpha$.

References


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Sorting by Exchanging

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Summary. We show that exchanging of pairs in an array which are in incorrect order leads to sorted array. It justifies correctness of Bubble Sort, Insertion Sort, and Quicksort.

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The notation and terminology used here have been introduced in the following papers: [20], [6], [10], [1], [7], [16], [11], [12], [9], [13], [8], [17], [18], [3], [4], [2], [14], [21], [22], [19], [5], and [15].

1. Preliminaries

We use the following convention: $a$, $b$, $c$, $d$ are ordinal numbers, $k$ is a natural number, and $x$, $y$, $z$, $t$, $X$, $Y$, $Z$ are sets.

Next we state several propositions:

(1) $x \in (a + b) \setminus a$ iff there exists $c$ such that $x = a + c$ and $c \in b$.

(2) Suppose $a \in b$ and $c \in d$. Then $c \neq a$ and $c \neq b$ and $d \neq a$ and $d \neq b$ or $c \in a$ and $d = a$ or $c \in a$ and $d = b$ or $c = a$ and $d \in b$ or $c = a$ and $d = b$ or $c = a$ and $b \in d$ or $a \in c$ and $d = b$ or $c = b$ and $b \in d$.

(3) If $x \notin y$, then $(y \cup \{x\}) \setminus y = \{x\}$.

(4) $\text{succ } x \setminus x = \{x\}$.

(5) Let $f$ be a function, $r$ be a binary relation, and given $x$. Then $x \in f \circ r$ if and only if there exist $y$, $z$ such that $\langle y, z \rangle \in r$ and $\langle y, z \rangle \in \text{dom } f$ and $f(y, z) = x$.

(6) If $a \setminus b \neq \emptyset$, then $\inf(a \setminus b) = b$ and $\sup(a \setminus b) = a$ and $\bigcup(a \setminus b) = \bigcup a$.

(7) If $a \setminus b$ is non empty and finite, then there exists a natural number $n$ such that $a = b + n$. 
2. Arrays

Let $f$ be a set. We say that $f$ is segmental if and only if:

(Def. 1) There exist $a$, $b$ such that $\pi_1(f) = a \setminus b$.

In the sequel, $f$, $g$ denote functions.

Next we state two propositions:

(8) If $\text{dom } f = \text{dom } g$ and $f$ is segmental, then $g$ is segmental.

(9) If $f$ is segmental, then for all $a$, $b$, $c$ such that $a \subseteq b \subseteq c$ and $a$, $c \in \text{dom } f$
holds $b \in \text{dom } f$.

Let us observe that every function which is transfinite sequence-like is also segmental and every function which is finite sequence-like is also segmental.

Let us consider $a$ and let $s$ be a set. We say that $s$ is $a$-based if and only if:

(Def. 2) If $b \in \pi_1(s)$, then $a \in \pi_1(s)$ and $a \subseteq b$.

We say that $s$ is $a$-limited if and only if:

(Def. 3) $a = \sup \pi_1(s)$.

The following propositions are true:

(10) $f$ is $a$-based and segmental iff there exists $b$ such that $\text{dom } f = b \setminus a$ and $a \subseteq b$.

(11) $f$ is $b$-limited, non empty, and segmental iff there exists $a$ such that $\text{dom } f = b \setminus a$ and $a \in b$.

Let us observe that every function which is transfinite sequence-like is also 0-based and every function which is finite sequence-like is also 1-based.

The following three propositions are true:

(12) $f$ is inf $\text{dom } f$-based.

(13) $f$ is sup $\text{dom } f$-limited.

(14) If $f$ is $b$-limited and $a \in \text{dom } f$, then $a \in b$.

Let us consider $f$. The functor $\text{base } f$ yielding an ordinal number is defined as follows:

(Def. 4)(i) $f$ is $f$-based if there exists $a$ such that $a \in \text{dom } f$,

(ii) $\text{base } f = 0$, otherwise.

The functor $\text{limit } f$ yielding an ordinal number is defined by:

(Def. 5)(i) $f$ is $f$-limited if there exists $a$ such that $a \in \text{dom } f$,

(ii) $\text{limit } f = 0$, otherwise.

Let us consider $f$. The functor $\text{length } f$ yields an ordinal number and is defined as follows:

(Def. 6) $\text{length } f = \text{limit } f - \text{base } f$.

We now state four propositions:

(15) $\text{base } \emptyset = 0$ and $\text{limit } \emptyset = 0$ and $\text{length } \emptyset = 0$. 
(16) \( \text{limit } f = \sup \text{dom } f. \)
(17) \( f \) is limit \( f \)-limited.
(18) Every empty set is \( a \)-based.

Let us consider \( a, X, Y \). One can check that there exists a transfinite sequence which is \( Y \)-defined, \( X \)-valued, \( a \)-based, segmental, finite, and empty.

An array is a segmental function.
Let \( A \) be an array. Observe that \( \text{dom } A \) is limit ordinal.
One can prove the following proposition

(19) For every array \( f \) holds \( f \) is 0-limited iff \( f \) is empty.

Let us observe that every array which is 0-based is also transfinite sequence-like.

Let us consider \( X \). An array of \( X \) is a \( X \)-valued array.
Let \( X \) be a 1-sorted structure. An array of \( X \) is an array of the carrier of \( X \).
Let us consider \( a, X \). An array of \( a, X \) is a \( a \)-defined array of \( X \).
In the sequel \( A, B, C \) denote arrays.
We now state several propositions:
(20) \( \text{base } f = \inf \text{dom } f. \)
(21) \( f \) is base \( f \)-based.
(22) \( \text{dom } A = \text{limit } A \setminus \text{base } A. \)
(23) If \( \text{dom } A = a \setminus b \) and \( A \) is non empty, then \( \text{base } A = b \) and \( \text{limit } A = a. \)
(24) For every transfinite sequence \( f \) holds \( \text{base } f = 0 \) and \( \text{limit } f = \text{dom } f \) and \( \text{length } f = \text{dom } f. \)

Let us consider \( a, b, X \). One can check that there exists an array of \( a, X \) which is \( b \)-based, natural-valued, integer-valued, real-valued, complex-valued, and finite.

Let us consider \( a, x \). Observe that \( \{\langle a, x \rangle\} \) is segmental.
Let us consider \( a \) and let \( x \) be a natural number. Note that \( \{\langle a, x \rangle\} \) is natural-valued.

Let us consider \( a \) and let \( x \) be a real number. One can check that \( \{\langle a, x \rangle\} \) is real-valued.

Let us consider \( a, X \) be a non empty set, and let \( x \) be an element of \( X \). Note that \( \{\langle a, x \rangle\} \) is \( X \)-valued.

Let us consider \( a, x \). One can check that \( \{\langle a, x \rangle\} \) is \( a \)-based and \( \text{succ } a \)-limited.

Let us consider \( b \). One can verify that there exists an array which is non empty, \( b \)-based, natural-valued, integer-valued, real-valued, complex-valued, and \( X \)-valued.

Let \( s \) be a transfinite sequence. We introduce \( s \text{ last} \) as a synonym of \( s \text{ last} \).
Let \( A \) be an array. The functor \( s \text{ last } A \) is defined by:
3. Descending Sequences

Let $f$ be a function. We say that $f$ is descending if and only if:

(Def. 7) Last $A = A(\bigcup \text{dom } A)$.

We now state four propositions:

(25) For every finite array $f$ such that for every $a$ such that $a, \text{succ } a \in \text{dom } f$ holds $f(\text{succ } a) \subseteq f(a)$ holds $f$ is descending.

(26) For every array $f$ such that length $f = \omega$ and for every $a$ such that $a, \text{succ } a \in \text{dom } f$ holds $f(\text{succ } a) \subseteq f(a)$ holds $f$ is descending.

(27) For every transfinite sequence $f$ such that $f$ is descending and $f(0)$ is finite holds $f$ is finite.

(28) Let $f$ be a transfinite sequence. Suppose $f$ is descending and $f(0)$ is finite and for every $a$ such that $f(a) \neq \emptyset$ holds $\text{succ } a \in \text{dom } f$. Then last $f = \emptyset$.

The scheme $A$ deals with a transfinite sequence $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

$\mathcal{A}$ is finite

provided the following conditions are met:

- $\mathcal{F}(\mathcal{A}(0))$ is finite, and
- For every $a$ such that $\text{succ } a \in \text{dom } \mathcal{A}$ and $\mathcal{F}(\mathcal{A}(a))$ is finite holds $\mathcal{F}(\mathcal{A}(\text{succ } a)) \subseteq \mathcal{F}(\mathcal{A}(a))$.

4. Swap

Let us consider $X$, let $f$ be a $X$-defined function, and let $a, b$ be sets. One can verify that $\text{Swap}(f, a, b)$ is $X$-defined.

Let $X$ be a set, let $f$ be a $X$-valued function, and let $x, y$ be sets. Observe that $\text{Swap}(f, x, y)$ is $X$-valued.

One can prove the following propositions:

(29) If $x, y \in \text{dom } f$, then $(\text{Swap}(f, x, y))(x) = f(y)$.

(30) For every array $f$ of $X$ such that $x, y \in \text{dom } f$ holds $(\text{Swap}(f, x, y))_x = f_y$.

(31) If $x, y \in \text{dom } f$, then $(\text{Swap}(f, x, y))(y) = f(x)$.

(32) For every array $f$ of $X$ such that $x, y \in \text{dom } f$ holds $(\text{Swap}(f, x, y))_y = f_x$.

(33) If $z \neq x$ and $z \neq y$, then $(\text{Swap}(f, x, y))(z) = f(z)$.

(34) For every array $f$ of $X$ such that $z \in \text{dom } f$ and $z \neq x$ and $z \neq y$ holds $(\text{Swap}(f, x, y))_z = f_z$. 

(35) If \( x, y \in \text{dom } f \), then \( \text{Swap}(f, x, y) = \text{Swap}(f, y, x) \).

Let \( X \) be a non empty set. Observe that there exists a \( X \)-valued non empty function which is onto.

Let \( X \) be a non empty set, let \( f \) be an onto \( X \)-valued non empty function, and let \( x, y \) be elements of \( \text{dom } f \). Observe that \( \text{Swap}(f, x, y) \) is onto.

Let us consider \( A \) and let us consider \( x, y \). Observe that \( \text{Swap}(A, x, y) \) is segmental.

One can prove the following proposition

(36) If \( x, y \in \text{dom } A \), then \( \text{Swap}(\text{Swap}(A, x, y), x, y) = A \).

Let \( A \) be a real-valued array and let us consider \( x, y \). Note that \( \text{Swap}(A, x, y) \) is real-valued.

5. Permutations

Let \( A \) be an array. An array is called a permutation of \( A \) if:

(Def. 9) There exists a permutation \( f \) of \( \text{dom } A \) such that it = \( A \cdot f \).

We now state several propositions:

(37) For every permutation \( B \) of \( A \) holds \( \text{dom } B = \text{dom } A \) and \( \text{rng } B = \text{rng } A \).

(38) \( A \) is a permutation of \( A \).

(39) If \( A \) is a permutation of \( B \), then \( B \) is a permutation of \( A \).

(40) If \( A \) is a permutation of \( B \) and \( B \) is a permutation of \( C \), then \( A \) is a permutation of \( C \).

(41) \( \text{Swap}(\text{id}_X, x, y) \) is one-to-one.

Let \( X \) be a non empty set and let \( x, y \) be elements of \( X \). Observe that \( \text{Swap}(\text{id}_X, x, y) \) is one-to-one, \( X \)-valued, and \( X \)-defined.

Let \( X \) be a non empty set and let \( x, y \) be elements of \( X \). Observe that \( \text{Swap}(\text{id}_X, x, y) \) is onto and total.

Let \( X, Y \) be non empty sets, let \( f \) be a function from \( X \) into \( Y \), and let \( x, y \) be elements of \( X \). Then \( \text{Swap}(f, x, y) \) is a function from \( X \) into \( Y \).

We now state three propositions:

(42) If \( x, y \in X \) and \( f = \text{Swap}(\text{id}_X, x, y) \) and \( X = \text{dom } A \), then \( \text{Swap}(A, x, y) = A \cdot f \).

(43) If \( x, y \in \text{dom } A \), then \( \text{Swap}(A, x, y) \) is a permutation of \( A \) and \( A \) is a permutation of \( \text{Swap}(A, x, y) \).

(44) Suppose \( x, y \in \text{dom } A \) and \( A \) is a permutation of \( B \). Then \( \text{Swap}(A, x, y) \) is a permutation of \( B \) and \( A \) is a permutation of \( \text{Swap}(B, x, y) \).
6. Exchanging

Let \( O \) be a relational structure and let \( A \) be an array of \( O \). We say that \( A \) is ascending if and only if:

(Def. 10) For all \( a, b \) such that \( a, b \in \text{dom } A \) and \( a \in b \) holds \( A_a \leq A_b \).

The functor inversions \( A \) is defined by:

(Def. 11) \( \text{inversions } A = \{ \langle a, b \rangle; a \text{ ranges over elements of } \text{dom } A, b \text{ ranges over elements of } \text{dom } A : a \in b \land A_a \not\leq A_b \} \).

Let \( O \) be a relational structure. One can verify that every empty array of \( O \) is ascending. Let \( A \) be an empty array of \( O \). Note that inversions \( A \) is empty.

In the sequel \( O \) denotes a connected non empty poset and \( R, Q \) denote arrays of \( O \).

The following proposition is true

(45) For every \( O \) and for all elements \( x, y \) of \( O \) holds \( x > y \) iff \( x \not\leq y \).

Let us consider \( O, R \). Then inversions \( R \) can be characterized by the condition:

(Def. 12) \( \text{inversions } R = \{ \langle a, b \rangle; a \text{ ranges over elements of } \text{dom } R, b \text{ ranges over elements of } \text{dom } R : a \in b \land R_a > R_b \} \).

Next we state two propositions:

(46) \( \langle x, y \rangle \in \text{inversions } R \) iff \( x, y \in \text{dom } R \) and \( x \in y \) and \( R_x > R_y \).

(47) \( \text{inversions } R \subseteq \text{dom } R \times \text{dom } R \).

Let us consider \( O \) and let \( R \) be a finite array of \( O \). One can verify that inversions \( R \) is finite.

The following three propositions are true:

(48) \( R \) is ascending iff \( \text{inversions } R = \emptyset \).

(49) If \( \langle x, y \rangle \in \text{inversions } R \), then \( \langle y, x \rangle \not\in \text{inversions } R \).

(50) If \( \langle x, y \rangle, \langle y, z \rangle \in \text{inversions } R \), then \( \langle x, z \rangle \in \text{inversions } R \).

Let us consider \( O, R \). One can verify that inversions \( R \) is relation-like.

Let us consider \( O, R \). Observe that inversions \( R \) is asymmetric and transitive.

Let us consider \( O \) and let \( a, b \) be elements of \( O \). Let us note that the predicate \( a < b \) is antisymmetric.

The following propositions are true:

(51) If \( \langle x, y \rangle \in \text{inversions } R \), then \( \langle x, y \rangle \not\in \text{inversions } \text{Swap}(R, x, y) \).

(52) If \( x, y \in \text{dom } R \) and \( z \neq x \) and \( z \neq y \) and \( t \neq x \) and \( t \neq y \), then \( \langle z, t \rangle \in \text{inversions } R \) iff \( \langle z, t \rangle \in \text{inversions } \text{Swap}(R, x, y) \).

(53) If \( \langle x, y \rangle \in \text{inversions } R \), then \( \langle z, y \rangle \in \text{inversions } R \) and \( z \in x \) iff \( \langle z, x \rangle \in \text{inversions } \text{Swap}(R, x, y) \).

(54) If \( \langle x, y \rangle \in \text{inversions } R \), then \( \langle z, x \rangle \in \text{inversions } R \) iff \( z \in x \) and \( \langle z, y \rangle \in \text{inversions } \text{Swap}(R, x, y) \).
(55) If \( (x, y) \in \text{inversions } R \) and \( z \in y \) and \( (x, z) \in \text{inversions } \text{Swap}(R, x, y) \), then \( (x, z) \in \text{inversions } R \).

(56) If \( (x, y) \in \text{inversions } R \) and \( x \in z \) and \( (z, y) \in \text{inversions } \text{Swap}(R, x, y) \), then \( (z, y) \in \text{inversions } R \).

(57) If \( (x, y) \in \text{inversions } R \) and \( y \in z \) and \( (x, z) \in \text{inversions } \text{Swap}(R, x, y) \), then \( (y, z) \in \text{inversions } R \).

(58) If \( (x, y) \in \text{inversions } R \), then \( y \in z \) and \( (x, z) \in \text{inversions } \text{Swap}(R, x, y) \).

Let us consider \( O, R, x, y \). The functor \( \subseteq R_{x,y} \) yields a function and is defined by:

\[
\text{Def. 13) } \subseteq R_{x,y} = \text{Swap(id}_{\text{dom } R}(R, x, y)) \times \text{dom } R \times \text{id}_{\{x\}} \times (\text{succ } y) \cup (\text{succ } x) \times \{y\}.
\]

We now state the proposition

(59) \( c \in \text{succ } b \setminus a \) if \( a \subseteq c \subseteq b \).

We adopt the following convention: \( T \) is a non empty array of \( O \) and \( p, q, r, s \) are elements of \( \text{dom } T \).

We now state a number of propositions:

(60) \( \text{succ } q \setminus p \subseteq \text{dom } T \).

(61) \( \text{dom } \subseteq T_{p,q} = \text{dom } T \times \text{dom } T \) and \( \text{rng } \subseteq T_{p,q} \subseteq \text{dom } T \times \text{dom } T \).

(62) If \( p \subseteq r \subseteq q \), then \( \subseteq T_{p,q}(p, r) = \{p, r\} \) and \( \subseteq T_{p,q}(r, q) = \{r, q\} \).

(63) If \( r \neq p \) and \( s \neq q \) and \( f = \text{Swap(id}_{\text{dom } T}(p, q) \), then \( \subseteq T_{p,q}(r, s) = \{f(r), f(s)\} \).

(64) If \( r \in p \) and \( f = \text{Swap(id}_{\text{dom } T}(p, q) \), then \( \subseteq T_{p,q}(r, q) = \{f(r), f(q)\} \) and \( \subseteq T_{p,q}(r, p) = \{f(r), f(p)\} \).

(65) If \( q \in r \) and \( f = \text{Swap(id}_{\text{dom } T}(p, q) \), then \( \subseteq T_{p,q}(p, r) = \{f(p), f(r)\} \) and \( \subseteq T_{p,q}(q, r) = \{f(q), f(r)\} \).

(66) If \( p \in q \), then \( \subseteq T_{p,q}(p, q) = \{p, q\} \).

(67) If \( p \in q \) and \( r \neq p \) and \( r \neq q \) and \( s \neq p \) and \( s \neq q \), then \( \subseteq T_{p,q}(r, s) = \{r, s\} \).

(68) If \( r \in p \) and \( p \in q \), then \( \subseteq T_{p,q}(r, p) = \{r, q\} \) and \( \subseteq T_{p,q}(r, q) = \{r, p\} \).

(69) If \( p \in s \) and \( s \in q \), then \( \subseteq T_{p,q}(p, s) = \{p, s\} \) and \( \subseteq T_{p,q}(s, q) = \{s, q\} \).

(70) If \( p \in q \) and \( q \in s \), then \( \subseteq T_{p,q}(p, s) = \{q, s\} \) and \( \subseteq T_{p,q}(q, s) = \{p, s\} \).

(71) If \( p \in q \), then \( \subseteq T_{p,q} \) \( \text{inversions } \text{Swap}(T, p, q) \text{ qua set} \) is one-to-one.

Let us consider \( O, R, x, y, z \). One can check that \( \subseteq R_{x,y} \) is relation-like.

7. Correctness of Sorting by Exchanging

We now state the proposition
(72) If \( \langle x, y \rangle \in \text{inversions } R \), then \((\leq_{x,y}^R)^0 \text{inversions } \text{Swap}(R, x, y) \subset \text{inversions } R\).

Let \( R \) be a finite function and let us consider \( x, y \). One can check that \( \text{Swap}(R, x, y) \) is finite.

One can prove the following two propositions:

(73) For every array \( R \) of \( O \) such that \( \langle x, y \rangle \in \text{inversions } R \) and \( \text{inversions } R \) is finite holds \( \text{inversions } \text{Swap}(R, x, y) \in \text{inversions } R \).

(74) For every finite array \( R \) of \( O \) such that \( \langle x, y \rangle \in \text{inversions } R \) holds \( \text{inversions } \text{Swap}(R, x, y) < \text{inversions } R \).

Let us consider \( O, R \). A non empty array is called a computation of \( R \) if it satisfies the conditions (Def. 14).

(Def. 14) (i) \( \text{It}(\text{base it}) = R \),

(ii) for every \( a \) such that \( a \in \text{dom } \text{it} \) holds \( \text{it}(a) \) is an array of \( O \), and

(iii) for every \( a \) such that \( a, \text{succ } a \in \text{dom } \text{it} \) there exist \( R, x, y \) such that \( \langle x, y \rangle \in \text{inversions } R \) and \( \text{it}(a) = R \) and \( \text{it}(\text{succ } a) = \text{Swap}(R, x, y) \).

Next we state the proposition

(75) \( \{a, R\} \) is a computation of \( R \).

Let us consider \( O, R, a \). Observe that there exists a computation of \( R \) which is \( a \)-based and finite.

Let us consider \( O, R, a \), let \( C \) be a computation of \( R \), and let us consider \( x \). Note that \( C(x) \) is segmental, function-like, and relation-like.

Let us consider \( O, R, a \), let \( C \) be a computation of \( R \), and let us consider \( x \). Observe that \( C(x) \) is the carrier of \( O \)-valued.

Let us consider \( O, R \) and let \( C \) be a computation of \( R \). Observe that last \( C \) is segmental, relation-like, and function-like.

Let us consider \( O, R \) and let \( C \) be a computation of \( R \). One can verify that last \( C \) is the carrier of \( O \)-valued.

Let us consider \( O, R \) and let \( C \) be a computation of \( R \). We say that \( C \) is complete if and only if:

(Def. 15) last \( C \) is ascending.

One can prove the following propositions:

(76) For every \( 0 \)-based computation \( C \) of \( R \) such that \( R \) is a finite finite array of \( O \) holds \( C \) is finite.

(77) Let \( C \) be a \( 0 \)-based computation of \( R \). Suppose \( R \) is a finite finite array of \( O \) and for every \( a \) such that \( \text{inversions } C(a) \neq \emptyset \) holds \( \text{succ } a \in \text{dom } C \). Then \( C \) is complete.

(78) Let \( C \) be a finite computation of \( R \). Then last \( C \) is a permutation of \( R \) and for every \( a \) such that \( a \in \text{dom } C \) holds \( C(a) \) is a permutation of \( R \).
8. Existence of Complete Computations

We now state three propositions:

(79) For every 0-based finite array $A$ of $X$ such that $A \neq \emptyset$ holds last $A \in X$.

(80) $\text{last}(x) = x$.

(81) For every 0-based finite array $A$ holds $\text{last}(A \cap \langle x \rangle) = x$.

Let $X$ be a set. Observe that every element of $X^\omega$ is $X$-valued.

The scheme $A$ deals with a unary functor $F$ yielding a set, a non empty set $A$, a set $B$, and a binary predicate $P$, and states that:

There exists a finite 0-based non empty array $f$ and there exists an element $k$ of $A$ such that

(i) $k = \text{last } f$,

(ii) $F(k) = \emptyset$,

(iii) $f(0) = B$, and

(iv) for every $a$ such that $\text{succ } a \in \text{dom } f$ there exist elements $x, y$ of $A$ such that $x = f(a)$ and $y = f(\text{succ } a)$ and $P[x, y]$ provided the following requirements are met:

- $B \in A$,
- $F(B)$ is finite, and
- For every element $x$ of $A$ such that $F(x) \neq \emptyset$ there exists an element $y$ of $A$ such that $P[x, y]$ and $F(y) \subset F(x)$.

In the sequel $A$ denotes an array and $B$ denotes a permutation of $A$.

The following proposition is true

(82) $B \in (\text{rng } A)_{\text{dom } A}$.

Let $A$ be a real-valued array. One can check that every permutation of $A$ is real-valued.

Let us consider $a$ and let $X$ be a non empty set. Note that every element of $X^a$ is transfinite sequence-like.

Let us consider $X$ and let $Y$ be a real-membered non empty set. Observe that every element of $Y^X$ is real-valued.

Let us consider $X$ and let $A$ be an array of $X$. One can verify that every permutation of $A$ is $X$-valued.

Let $X$ be a set, let $Z$ be a set, and let $Y$ be a subset of $Z$. One can verify that every element of $Y^X$ is $Z$-valued.

Next we state four propositions:

(83) Every $X$-defined $Y$-valued binary relation is a relation between $X$ and $Y$.

(84) For every finite ordinal number $a$ and for every $x$ such that $x \in a$ holds $x = 0$ or there exists $b$ such that $x = \text{succ } b$.

(85) For every 0-based finite non empty array $A$ of $O$ holds there exists a 0-based computation of $A$ which is complete.
(86) For every 0-based finite non empty array $A$ of $O$ holds there exists a permutation of $A$ which is ascending.

Let us consider $O$ and let $A$ be a 0-based finite array of $O$. Note that there exists a permutation of $A$ which is ascending.

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Linear Transformations of Euclidean Topological Spaces

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Summary. We introduce linear transformations of Euclidean topological spaces given by a transformation matrix. Next, we prove selected properties and basic arithmetic operations on these linear transformations. Finally, we show that a linear transformation given by an invertible matrix is a homeomorphism.

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The papers [2], [12], [6], [26], [7], [8], [30], [21], [22], [23], [15], [31], [29], [19], [24], [3], [4], [9], [16], [5], [20], [18], [1], [14], [28], [13], [10], [25], [27], [11], and [17] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we adopt the following convention: $X, Y$ denote sets, $n, m, k, i$ denote natural numbers, $r$ denotes a real number, $R$ denotes an element of $\mathbb{R}_F$, $K$ denotes a field, $f, f_1, f_2, g_1, g_2$ denote finite sequences, $r_1, r_2, r_3$ denote real-valued finite sequences, $c_1, c_2$ denote complex-valued finite sequences, and $F$ denotes a function.

Let us consider $X, Y$ and let $F$ be a positive yielding partial function from $X$ to $\mathbb{R}$. Observe that $F|Y$ is positive yielding.

Let us consider $X, Y$ and let $F$ be a negative yielding partial function from $X$ to $\mathbb{R}$. Note that $F|Y$ is negative yielding.

Let us consider $X, Y$ and let $F$ be a non-positive yielding partial function from $X$ to $\mathbb{R}$. Note that $F|Y$ is non-positive yielding.
Let us consider $X$, $Y$ and let $F$ be a non-negative yielding partial function from $X$ to $\mathbb{R}$. One can verify that $F|Y$ is non-negative yielding.

Let us consider $r_1$. Note that $\sqrt{r_1}$ is finite sequence-like.

Let us consider $r_1$. The functor $\oplus r_1$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

(Def. 1) $\oplus r_1 = r_1$.

Let $p$ be a finite sequence of elements of $\mathbb{R}$. The functor $\oplus p$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

(Def. 2) $\oplus p = p$.

Next we state several propositions:

(1) $(\oplus r_2) + \oplus r_3 = r_2 + r_3$.

(2) $\sqrt{r_2} \cdot \sqrt{r_3} = \sqrt{r_2} \cdot \sqrt{r_3}$.

(3) $\sqrt{\langle r \rangle} = \langle \sqrt{r} \rangle$.

(4) $\sqrt{|r|} = |r|$.

(5) If $r_1$ is non-negative yielding, then $\sqrt{\sum r_1} \leq \sum \sqrt{r_1}$.

(6) There exists $X$ such that $X \subseteq \text{dom } F$ and $\text{rng } F = \text{rng } (F|X)$ and $F|X$ is one-to-one.

Let us consider $c_1$, $X$. Note that $c_1 - X$ is complex-valued.

Let us consider $r_1$, $X$. One can verify that $r_1 - X$ is real-valued.

Let $c_1$ be a complex-valued finite subsequence. One can verify that $\text{Seq } c_1$ is complex-valued.

Let $r_1$ be a real-valued finite subsequence. One can verify that $\text{Seq } r_1$ is real-valued.

The following propositions are true:

(7) For every permutation $P$ of $\text{dom } f$ such that $f_1 = f \cdot P$ there exists a permutation $Q$ of $\text{dom } (f - X)$ such that $f_1 - X = (f - X) \cdot Q$.

(8) For every permutation $P$ of $\text{dom } c_1$ such that $c_2 = c_1 \cdot P$ holds $\sum (c_2 - X) = \sum (c_1 - X)$.

(9) Let $f$, $f_1$ be finite subsequences and $P$ be a permutation of $\text{dom } f$. If $f_1 = f \cdot P$, then there exists a permutation $Q$ of $\text{dom } \text{Seq } (f_1|P^{-1}(X))$ such that $\text{Seq } (f|X) = \text{Seq } (f_1|P^{-1}(X)) \cdot Q$.

(10) Let $c_1$, $c_2$ be complex-valued finite subsequences and $P$ be a permutation of $\text{dom } c_1$. If $c_2 = c_1 \cdot P$, then $\sum \text{Seq } (c_1|X) = \sum \text{Seq } (c_2|P^{-1}(X))$.

(11) Let $f$ be a finite subsequence and $n$ be an element of $\mathbb{N}$. If for every $i$ holds $i + n \in X$ iff $i \in Y$, then $\text{Shift }^n f|X = \text{Shift }^n (f|Y)$.

(12) There exists a subset $Y$ of $\mathbb{N}$ such that $\text{Seq } ((f_1 \sim f_2)|X) = (\text{Seq } (f_1|X)) \sim \text{Seq } (f_2|Y)$ and for every $n$ such that $n > 0$ holds $n \in Y$ iff $n + \text{len } f_1 \in X \cap \text{dom } (f_1 \sim f_2)$. 

(13) If \( \text{len } g_1 = \text{len } f_1 \) and \( \text{len } g_2 \leq \text{len } f_2 \), then \( \text{Seq}(f_1 \upharpoonright f_2)(g_1 \upharpoonright g_2)^{-1}(X) = (\text{Seq}(f_1 | g_1^{-1}(X)) \upharpoonright \text{Seq}(f_2 | g_2^{-1}(X))). \)

(14) Let \( D \) be a non empty set and \( M \) be a matrix over \( D \) of dimension \( n \times m \). Then \( M - X \) is a matrix over \( D \) of dimension \( n - 1 \cdot (M^{-1}(X)) \times m \).

(15) Let \( F \) be a function from \( \text{Seg } n \) into \( \text{Seg } n \), \( D \) be a non empty set, \( M \) be a matrix over \( D \) of dimension \( n \times m \), and given \( i \). If \( i \in \text{Seg width } M \), then \( (M \cdot F) \upharpoonright i = M \upharpoonright i \cdot F \).

(16) Let \( A \) be a matrix over \( K \) of dimension \( n \times m \). Suppose \( \text{rk}(A) = n \). Then there exists a matrix \( B \) over \( K \) of dimension \( m - i \cdot n \times m \) such that \( \text{rk}(A \cup B) = m \).

(17) Let \( A \) be a matrix over \( K \) of dimension \( n \times m \). Suppose \( \text{rk}(A) = m \). Then there exists a matrix \( B \) over \( K \) of dimension \( n \times n - i \cdot m \) such that \( \text{rk}(A \cup B) = n \).

For simplicity, we adopt the following rules: \( f, f_1, f_2 \) denote \( n \)-element real-valued finite sequences, \( p, p_1, p_2 \) denote points of \( \mathcal{E}_F^n \), \( M, M_1, M_2 \) denote matrices over \( \mathbb{R}_F \) of dimension \( n \times m \), and \( A, B \) denote square matrices over \( \mathbb{R}_F \) of dimension \( n \).

2. Linear Transformations of Euclidean Topological Spaces Given by a Transformation Matrix

Let us consider \( n, m, M \). The functor \( \text{Mx2Tran } M \) yields a function from \( \mathcal{E}_F^n \) into \( \mathcal{E}_F^n \) and is defined by:

(Def. 3)(i) \( \text{(Mx2Tran } M \}(f) = \text{Line(LineVec2Mx}(^0f) \cdot M, 1) \) if \( n \neq 0 \),

(ii) \( \text{(Mx2Tran } M \}(f) = 0_{\mathcal{E}_F^n}, \) otherwise.

Let us consider \( n, m, M \) and let \( x \) be a set. Note that \( (\text{Mx2Tran } M \}(x) \) is function-like and relation-like and \( (\text{Mx2Tran } M \}(x) \) is real-valued and finite sequence-like.

Let us consider \( n, m, M, f \). Note that \( (\text{Mx2Tran } M \}(f) \) is \( m \)-element.

We now state a number of propositions:

(18) If \( 1 \leq i \leq m \) and \( n \neq 0 \), then \( (\text{Mx2Tran } M \}(f) \) \( (i) = (^0f) \cdot M \upharpoonright i \).

(19) \( \text{lenMX2FinS} (I_R^{K \times n}) = n \).

(20) Let \( B_1 \) be an ordered basis of the \( n \)-dimension vector space over \( \mathbb{R}_F \) and \( B_2 \) be an ordered basis of the \( m \)-dimension vector space over \( \mathbb{R}_F \). Suppose \( B_1 = \text{MX2FinS}(I_R^{n \times n}) \) and \( B_2 = \text{MX2FinS}(I_R^{m \times m}) \). Let \( M_1 \) be a matrix over \( \mathbb{R}_F \) of dimension \( \text{len } B_1 \times \text{len } B_2 \). If \( M_1 = M \), then \( \text{Mx2Tran } M = \text{Mx2Tran}(M_1, B_1, B_2) \).

(21) Let \( P \) be a permutation of \( \text{Seg } n \). Then \( (\text{Mx2Tran } M \}(f) = (\text{Mx2Tran}(M \cdot P))(f \cdot P) \) and \( f \cdot P \) is a \( n \)-element \( n \)-element finite sequence of elements of \( \mathbb{R} \).
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The following propositions are true:

(39) $\text{Mx2Tran}(M)$ is one-to-one iff $\text{rk}(M) = n$.

(40) $\text{Mx2Tran}(A)$ is one-to-one iff $\text{Det}(A) \neq 0_{\mathbb{R}^F}$.

(41) $\text{Mx2Tran}(M)$ is onto iff $\text{rk}(M) = m$.

(42) $\text{Mx2Tran}(A)$ is onto iff $\text{Det}(A) \neq 0_{\mathbb{R}^F}$.

(43) For all $A$, $B$ such that $\text{Det}(A) \neq 0_{\mathbb{R}^F}$ holds $\text{Mx2Tran}(A)^{-1} = \text{Mx2Tran}(B)$ if $A^{-1} = B$. 

3. Selected Properties of the Mx2Tran Operator

The following propositions are true:

(39) $\text{Mx2Tran}(M)(f_1 + f_2) = (\text{Mx2Tran}(M)(f_1) + (\text{Mx2Tran}(M)(f_2)$.

(40) $\text{Mx2Tran}(M)(r \cdot f) = r \cdot (\text{Mx2Tran}(M)(f)$.

(41) $\text{Mx2Tran}(M)(f_1 - f_2) = (\text{Mx2Tran}(M)(f_1) - (\text{Mx2Tran}(M)(f_2)$.

(42) $\text{Mx2Tran}(M)(M_1 + M_2)(f) = (\text{Mx2Tran}(M_1)(f) + (\text{Mx2Tran}(M_2)(f)$.

(43) $\text{Mx2Tran}(M)(R \cdot M)(f) = (\text{Mx2Tran}(R \cdot M)(f)$.

(44) $\text{Mx2Tran}(M)(p_1 + p_2) = (\text{Mx2Tran}(M)(p_1) + (\text{Mx2Tran}(M)(p_2)$.

(45) $\text{Mx2Tran}(M)(p_1 - p_2) = (\text{Mx2Tran}(M)(p_1) - (\text{Mx2Tran}(M)(p_2)$.

(46) $\text{Mx2Tran}(M)(0_{E^F}) = 0_{E^F}$.

(47) For every subset $A$ of $E^F$ holds $(\text{Mx2Tran}(M)^{\circ}(p + A) = (\text{Mx2Tran}(M)(p) + (\text{Mx2Tran}(M)^{\circ}A$.

(48) For every subset $A$ of $E^F$ holds $(\text{Mx2Tran}(M)^{-1}((\text{Mx2Tran}(M)(p) + A) = p + (\text{Mx2Tran}(M)^{-1}(A$.

(49) Let $A$ be a matrix over $\mathbb{R}_F$ of dimension $n \times m$ and $B$ be a matrix over $\mathbb{R}_F$ of dimension width $A \times k$. If if $n = 0$, then $m = 0$ and if $m = 0$, then $k = 0$, then $\text{Mx2Tran}(B \cdot \text{Mx2Tran}(A = \text{Mx2Tran}(A \cdot B$.

(50) $\text{Mx2Tran}(I_{E^F}) = \text{id}_{E^F}$.

(51) If $\text{Mx2Tran}(M_1 = \text{Mx2Tran}(M_2$, then $M_1 = M_2$.

(52) Let $A$ be a matrix over $\mathbb{R}_F$ of dimension $n \times m$ and $B$ be a matrix over $\mathbb{R}_F$ of dimension $k \times m$. Then $(\text{Mx2Tran}(A \cap B))(f \cap (k \mapsto 0))) = (\text{Mx2Tran}(A)(f)$ and $(\text{Mx2Tran}(B \cap A))(f \cap (k \mapsto 0)) = (\text{Mx2Tran}(A)(f)$.

(53) Let $A$ be a matrix over $\mathbb{R}_F$ of dimension $n \times m$, $B$ be a matrix over $\mathbb{R}_F$ of dimension $k \times m$, and $g$ be a $k$-element real-valued finite sequence. Then $(\text{Mx2Tran}(A \cap B))(f \cap g) = (\text{Mx2Tran}(A)(f) + (\text{Mx2Tran}(B)(g$.

(54) Let $A$ be a matrix over $\mathbb{R}_F$ of dimension $n \times k$ and $B$ be a matrix over $\mathbb{R}_F$ of dimension $n \times m$ such that if $n = 0$, then $k + m = 0$. Then $(\text{Mx2Tran}(A \cap B))(f) = (\text{Mx2Tran}(A)(f) \cap (\text{Mx2Tran}(B)(f$.

(55) $(\text{Mx2Tran}(I_{E^F}^{m \times n}(n))(f) | n = f$. 


(44) There exists a $m$-element finite sequence $L$ of elements of $\mathbb{R}$ such that for every $i$ such that $i \in \text{dom } L$ holds $L(i) = |\partial((M_{\square},i))|$ and for every $f$ holds $|(M_{\text{x2Tran}}M)(f)| \leq \sum L \cdot |f|$. 

(45) There exists a real number $L$ such that $L > 0$ and for every $f$ holds $|(M_{\text{x2Tran}}M)(f)| \leq L \cdot |f|$. 

(46) If $\text{rk}(M) = n$, then there exists a real number $L$ such that $L > 0$ and for every $f$ holds $|f| \leq L \cdot |(M_{\text{x2Tran}}M)(f)|$. 

(47) $M_{\text{x2Tran}}M$ is continuous.

Let us consider $n, K$. One can verify that there exists a square matrix over $K$ of dimension $n$ which is invertible.

Let us consider $n$ and let $A$ be an invertible square matrix over $\mathbb{R}_F$ of dimension $n$. One can verify that $M_{\text{x2Tran}}A$ is homeomorphism.

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Linear Transformations of Euclidean Topological Spaces. Part II

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Summary. We prove a number of theorems concerning various notions used in the theory of continuity of barycentric coordinates.

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The papers [2], [9], [4], [5], [6], [14], [10], [25], [13], [16], [3], [7], [12], [1], [24], [15], [21], [23], [19], [17], [8], [11], [22], [20], and [18] provide the terminology and notation for this paper.

1. Correspondence Between Euclidean Topological Space and Vector Space over $\mathbb{R}_F$

For simplicity, we adopt the following rules: $X$ denotes a set, $n, m, k$ denote natural numbers, $K$ denotes a field, $f$ denotes a $n$-element real-valued finite sequence, and $M$ denotes a matrix over $\mathbb{R}_F$ of dimension $n \times m$.

The following propositions are true:

1. $X$ is a linear combination of the $n$-dimension vector space over $\mathbb{R}_F$ if and only if $X$ is a linear combination of $\mathcal{E}_T^n$.

2. Let $L_2$ be a linear combination of the $n$-dimension vector space over $\mathbb{R}_F$ and $L_1$ be a linear combination of $\mathcal{E}_T^n$. If $L_1 = L_2$, then the support of $L_1$ is the support of $L_2$.

3. Let $F$ be a finite sequence of elements of $\mathcal{E}_T^n$, $f_1$ be a function from $\mathcal{E}_T^n$ into $\mathbb{R}$, $F_1$ be a finite sequence of elements of the $n$-dimension vector space over $\mathbb{R}_F$, and $f_2$ be a function from the $n$-dimension vector space over $\mathbb{R}_F$ into $\mathbb{R}_F$. If $f_1 = f_2$ and $F = F_1$, then $f_1 F = f_2 F_1$. 

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(4) Let $F$ be a finite sequence of elements of $E^n_T$ and $F_1$ be a finite sequence of elements of the $n$-dimension vector space over $\mathbb{R}_F$. If $F_1 = F$, then $\sum F = \sum F_1$.

(5) Let $L_2$ be a linear combination of the $n$-dimension vector space over $\mathbb{R}_F$ and $L_1$ be a linear combination of $E^n_T$. If $L_1 = L_2$, then $\sum L_1 = \sum L_2$.

(6) Let $A_2$ be a subset of the $n$-dimension vector space over $\mathbb{R}_F$ and $A_1$ be a subset of $E^n_T$. If $A_2 = A_1$, then $\Omega_{\text{Lin}(A_1)} = \Omega_{\text{Lin}(A_2)}$.

(7) Let $A_2$ be a subset of the $n$-dimension vector space over $\mathbb{R}_F$ and $A_1$ be a subset of $E^n_T$. Suppose $A_2 = A_1$. Then $A_2$ is linearly independent if and only if $A_1$ is linearly independent.

(8) Let $V$ be a vector space over $K$, $W$ be a subspace of $V$, and $L$ be a linear combination of $V$. Then $L\upharpoonright \text{the carrier of } W$ is a linear combination of $W$.

(9) Let $V$ be a vector space over $K$, $A$ be a linearly independent subset of $V$, and $L_3$, $L_4$ be linear combinations of $V$. Suppose the support of $L_3 \subseteq A$ and the support of $L_4 \subseteq A$ and $\sum L_3 = \sum L_4$. Then $L_3 = L_4$.

(10) Let $V$ be a real linear space, $W$ be a subspace of $V$, and $L$ be a linear combination of $V$. Then $L\upharpoonright \text{the carrier of } W$ is a linear combination of $W$.

(11) Let $U$ be a subspace of the $n$-dimension vector space over $\mathbb{R}_F$ and $W$ be a subspace of $E^n_T$. Suppose $\Omega_U = \Omega_W$. Then $X$ is a linear combination of $U$ if and only if $X$ is a linear combination of $W$.

(12) Let $U$ be a subspace of the $n$-dimension vector space over $\mathbb{R}_F$, $W$ be a subspace of $E^n_T$, $L_5$ be a linear combination of $U$, and $L_6$ be a linear combination of $W$. If $L_5 = L_6$, then the support of $L_5 = \text{the support of } L_6$ and $\sum L_5 = \sum L_6$.

Let us consider $m$, $K$ and let $A$ be a subset of the $m$-dimension vector space over $K$. Observe that $\text{Lin}(A)$ is finite dimensional.

2. CORRESPONDENCE BETWEEN THE Mx2Tran OPERATOR AND DECOMPOSITION OF A VECTOR IN BASIS

One can prove the following propositions:

(13) If $\text{rk}(M) = n$, then $M$ is an ordered basis of $\text{Lin}(\text{lines}(M))$.

(14) Let $V$, $W$ be vector spaces over $K$, $T$ be a linear transformation from $V$ to $W$, $A$ be a subset of $V$, and $L$ be a linear combination of $A$. If $T\upharpoonright A$ is one-to-one, then $T(\sum L) = \sum(T^\otimes L)$.

(15) Let $S$ be a subset of $\text{Seg}\, n$. Suppose $M\upharpoonright S$ is one-to-one and $\text{rng}(M\upharpoonright S) = \text{lines}(M)$. Then there exists a linear combination $L$ of $\text{lines}(M)$ such that $\sum L = (\text{Mx2Tran}(M))(f)$ and for every $k$ such that $k \in S$ holds $L(\text{Line}(M, k)) = \sum \text{Seq}(f\upharpoonright M^{-1}(\{\text{Line}(M, k)\}))$. 
(16) Suppose $M$ is without repeated line. Then there exists a linear combination $L$ of lines$(M)$ such that $\sum L = (\text{Mx2Tran } M)(f)$ and for every $k$ such that $k \in \text{dom } f$ holds $L(\text{Line}(M, k)) = f(k)$.

(17) For every ordered basis $B$ of Lin(lines$(M)$) such that $B = M$ and for every element $M_1$ of Lin(lines$(M)$) such that $M_1 = (\text{Mx2Tran } M)(f)$ holds $M_1 \to B = f$.

(18) $\text{rng Mx2Tran } M = \Omega_{\text{Lin}(\text{lines}(M))}$.

(19) Let $F$ be an one-to-one finite sequence of elements of $\mathcal{E}_T^n$. Suppose $\text{rng } F$ is linearly independent. Then there exists a square matrix $M$ over $\mathbb{R}_F$ of dimension $n$ such that $M$ is invertible and $M \upharpoonright \text{len } F = F$.

(20) Let $A$, $B$ be linearly independent subsets of $\mathcal{E}_T^n$. Suppose $\text{rk}(M) = n$ holds $(\text{Mx2Tran } M)^{-1}(A)$ is linearly independent.

3. Preservation of Linear and Affine Independence of Vectors by the Mx2Tran Operator

Next we state several propositions:

(23) For every linearly independent subset $A$ of $\mathcal{E}_T^n$ such that $\text{rk}(M) = n$ holds $(\text{Mx2Tran } M)^{\circ} A$ is linearly independent.

(24) For every affinely-independent subset $A$ of $\mathcal{E}_T^n$ such that $\text{rk}(M) = n$ holds $(\text{Mx2Tran } M)^{\circ} A$ is affinely-independent.

(25) Let $A$ be an affinely-independent subset of $\mathcal{E}_T^n$. Suppose $\text{rk}(M) = n$. Let $v$ be an element of $\mathcal{E}_T^n$. If $v \in \text{Affin } A$, then $(\text{Mx2Tran } M)(v) \in \text{Affin } ((\text{Mx2Tran } M)^{\circ} A)$ and for every $f$ holds $(v \rightarrow A)(f) = ((\text{Mx2Tran } M)(v) \rightarrow (\text{Mx2Tran } M)^{\circ} A)(((\text{Mx2Tran } M)(f))$.

(26) For every linearly independent subset $A$ of $\mathcal{E}_T^n$ such that $\text{rk}(M) = n$ holds $(\text{Mx2Tran } M)^{-1}(A)$ is linearly independent.

(27) For every affinely-independent subset $A$ of $\mathcal{E}_T^n$ such that $\text{rk}(M) = n$ holds $(\text{Mx2Tran } M)^{-1}(A)$ is affinely-independent.
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The Axiomatization of Propositional Linear Time Temporal Logic

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Summary. The article introduces propositional linear time temporal logic as a formal system. Axioms and rules of derivation are defined. Soundness Theorem and Deduction Theorem are proved [9].

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The terminology and notation used in this paper have been introduced in the following papers: [10], [3], [4], [5], [8], [11], [13], [1], [2], [6], [12], and [7].

1. Preliminaries

In this paper a, b, c denote boolean numbers.

We now state three propositions:

(1) \((a \Rightarrow b \land c) \Rightarrow (a \Rightarrow b) = 1\).

(2) \((a \Rightarrow (b \Rightarrow c)) \Rightarrow (a \land b \Rightarrow c) = 1\).

(3) \((a \land b \Rightarrow c) \Rightarrow (a \Rightarrow (b \Rightarrow c)) = 1\).

2. The Language. Basic Operators. Further Operators as Abbreviations

We introduce the LTLB-WFF as a synonym of HP-WFF.

For simplicity, we adopt the following rules: \(r, s, A, B, C\) are elements of the LTLB-WFF, \(G\) is a subset of the LTLB-WFF, \(i, j, n\) are elements of \(\mathbb{N}\), and \(f_1, f_2\) are finite sequences of elements of the LTLB-WFF.
We introduce \( \bot_t \) as a synonym of VERUM.

Let us consider \( p, q \). We introduce \( p \cup_s q \) as a synonym of \( p \land q \).

We now state the proposition

(4) For every \( A \) holds \( A = \bot_t \) or there exists \( n \) such that \( A = \text{prop}_n \) or there exist \( p, q \) such that \( A = p \Rightarrow q \) or there exist \( p, q \) such that \( A = p \cup_s q \).

Let us consider \( p \). The functor \( \neg p \) yielding an element of the LTLB-WFF is defined as follows:

(Def. 1) \( \neg p = p \Rightarrow \bot_t \).

The functor \( X p \) yielding an element of the LTLB-WFF is defined as follows:

(Def. 2) \( X p = \bot_t \cup_s p \).

The element \( \top_t \) of the LTLB-WFF is defined by:

(Def. 3) \( \top_t = \neg \bot_t \).

Let us consider \( p, q \). The functor \( p \&\& q \) yielding an element of the LTLB-WFF is defined by:

(Def. 4) \( p \&\& q = (p \Rightarrow (q \Rightarrow \bot_t)) \Rightarrow \bot_t \).

Let us consider \( p, q \). The functor \( p || q \) yields an element of the LTLB-WFF and is defined as follows:

(Def. 5) \( p || q = \neg (\neg p \&\& \neg q) \).

Let us consider \( p \). The functor \( G p \) yielding an element of the LTLB-WFF is defined as follows:

(Def. 6) \( G p = \neg (p || (\top_t \&\& (\top_t \cup_s \neg p))) \).

Let us consider \( p \). The functor \( F p \) yielding an element of the LTLB-WFF is defined as follows:

(Def. 7) \( F p = \neg G \neg p \).

Let us consider \( p, q \). The functor \( p \cup q \) yielding an element of the LTLB-WFF is defined by:

(Def. 8) \( p \cup q = q || (p \&\& (p \cup_s q)) \).

Let us consider \( p, q \). The functor \( p \Rightarrow q \) yields an element of the LTLB-WFF and is defined as follows:

(Def. 9) \( p \Rightarrow q = \neg (\neg p \cup \neg q) \).

3. The Semantics

The subset \( AP \) of the LTLB-WFF is defined as follows:

(Def. 10) For every set \( x \) holds \( x \in AP \) iff there exists an element \( n \) of \( \mathbb{N} \) such that \( x = \text{prop}_n \).

A LTL Model is a sequence of \( 2^{AP} \).

Let \( M \) be a LTL Model. The functor \( \text{SAT}_M \) yields a function from \( \mathbb{N} \times \) the LTLB-WFF into \( \text{Boolean} \) and is defined by the condition (Def. 11).
(Def. 11) Let given $n$. Then

(i) $\text{SAT}_M(\langle n, \bot \rangle) = 0$,

(ii) for every $k$ holds $\text{SAT}_M(\langle n, \text{prop} \rangle) = 1$ iff $\text{prop} k \in M(n)$, and

(iii) for all $p, q$ holds $\text{SAT}_M(\langle n, p \Rightarrow q \rangle) = \text{SAT}_M(\langle n, p \rangle) \Rightarrow \text{SAT}_M(\langle n, q \rangle)$ and $\text{SAT}_M(\langle n, pUq \rangle) = 1$ iff there exists $i$ such that $0 < i$ and $\text{SAT}_M(\langle n+i, q \rangle) = 1$ and for every $j$ such that $1 \leq j < i$ holds $\text{SAT}_M(\langle n+j, p \rangle) = 1$.

The following propositions are true:

(5) $\text{SAT}_M(\langle n, \neg A \rangle) = 1$ iff $\text{SAT}_M(\langle n, A \rangle) = 0$.

(6) $\text{SAT}_M(\langle n, \top \rangle) = 1$.

(7) $\text{SAT}_M(\langle n, A \& B \rangle) = 1$ iff $\text{SAT}_M(\langle n, A \rangle) = 1$ and $\text{SAT}_M(\langle n, B \rangle) = 1$.

(8) $\text{SAT}_M(\langle n, A \Rightarrow B \rangle) = 1$ iff $\text{SAT}_M(\langle n, A \rangle) = 1$ or $\text{SAT}_M(\langle n, B \rangle) = 1$.

(9) $\text{SAT}_M(\langle n, \neg H A \rangle) = \text{SAT}_M(\langle n+1, A \rangle)$.

(10) $\text{SAT}_M(\langle n, G A \rangle) = 1$ iff for every $i$ holds $\text{SAT}_M(\langle n+i, A \rangle) = 1$.

(11) $\text{SAT}_M(\langle n, F A \rangle) = 1$ iff there exists $i$ such that $\text{SAT}_M(\langle n+i, A \rangle) = 1$.

(12) $\text{SAT}_M(\langle n, pUq \rangle) = 1$ iff there exists $i$ such that $\text{SAT}_M(\langle n+i, q \rangle) = 1$ and for every $j$ such that $j < i$ holds $\text{SAT}_M(\langle n+j, p \rangle) = 1$.

(13) $\text{SAT}_M(\langle n, pRq \rangle) = 1$ if and only if one of the following conditions is satisfied:

(i) there exists $i$ such that $\text{SAT}_M(\langle n+i, p \rangle) = 1$ and for every $j$ such that $j \leq i$ holds $\text{SAT}_M(\langle n+j, q \rangle) = 1$, or

(ii) for every $i$ holds $\text{SAT}_M(\langle n+i, q \rangle) = 1$.

(14) $\text{SAT}_M(\langle n, \neg H B \rangle) = \text{SAT}_M(\langle n, H \neg B \rangle)$.

(15) $\text{SAT}_M(\langle n, \neg H B \Rightarrow H \neg B \rangle) = 1$.

(16) $\text{SAT}_M(\langle n, \neg H B \Rightarrow H \neg B \rangle) = 1$.

(17) $\text{SAT}_M(\langle n, H(B \Rightarrow C) \Rightarrow (H B \Rightarrow H C) \rangle) = 1$.

(18) $\text{SAT}_M(\langle n, G B \Rightarrow B \& H C \& H C \rangle) = 1$.

(19) $\text{SAT}_M(\langle n, B \cup C \Rightarrow H C \& H(B \& H C) \rangle) = 1$.

(20) $\text{SAT}_M(\langle n, H C \& H(B \& H C) \Rightarrow B \cup C \rangle) = 1$.

(21) $\text{SAT}_M(\langle n, B \cup C \Rightarrow H F C \rangle) = 1$.


Let us consider $M, p$. The predicate $M \models p$ is defined as follows:

(Def. 12) For every element $n$ of $\mathbb{N}$ holds $\text{SAT}_M(\langle n, p \rangle) = 1$.

Let us consider $M, F$. The predicate $M \models F$ is defined as follows:

(Def. 13) For every $p$ such that $p \in F$ holds $M \models p$. 
Let us consider $F, p$. The predicate $F \models p$ is defined as follows:

(Def. 14) For every $M$ such that $M \models F$ holds $M \models p$.

The following propositions are true:

(22) $M \models F$ and $M \models G$ iff $M \models F \cup G$.
(23) $M \models A$ iff $M \models \{A\}$.
(24) If $F \models A$ and $F \models A \Rightarrow B$, then $F \models B$.
(25) If $F \models A$, then $F \models \mathcal{X} A$.
(26) If $F \models A$, then $F \models \mathcal{G} A$.
(27) If $F \models A \Rightarrow B$ and $F \models A \Rightarrow \mathcal{X} A$, then $F \models A \Rightarrow \mathcal{G} B$.
(28) $\text{SAT}_{(M \uparrow i)}(\langle j, A \rangle) = \text{SAT}_M(\langle i + j, A \rangle)$.
(29) If $M \models F$, then $M \uparrow i \models F$.
(30) $F \cup \{A\} \models B$ iff $F \models \mathcal{G} A \Rightarrow B$.

Let $f$ be a function from the LTLB-WFF into Boolean. The functor $\text{VAL}_f$ yielding a function from the LTLB-WFF into Boolean is defined by:

(Def. 15) $\text{VAL}_f(\bot_t) = 0$ and $\text{VAL}_f(\text{prop}_n) = f(\text{prop}_n)$ and $\text{VAL}_f(A \Rightarrow B) = \text{VAL}_f(A) \Rightarrow \text{VAL}_f(B)$ and $\text{VAL}_f(A \mathcal{U} B) = f(A \mathcal{U} B)$.

One can prove the following two propositions:

(31) For every function $f$ from the LTLB-WFF into Boolean and for all $p, q$ holds $\text{VAL}_f(p \& \& q) = \text{VAL}_f(p) \& \& \text{VAL}_f(q)$.
(32) Let $f$ be a function from the LTLB-WFF into Boolean. Suppose that for every set $B$ such that $B \in$ the LTLB-WFF holds $f(B) = \text{SAT}_M(\langle n, B \rangle)$.

Then $\text{VAL}_f(A) = \text{SAT}_M(\langle n, A \rangle)$.

Let us consider $p$. We say that $p$ is tautologically valid if and only if:

(Def. 16) For every function $f$ from the LTLB-WFF into Boolean holds $\text{VAL}_f(p) = 1$.

The following proposition is true

(33) If $A$ is tautologically valid, then $F \models A$.


Let $D$ be a set. We say that $D$ has LTL axioms if and only if the condition (Def. 17) is satisfied.

(Def. 17) Let given $p, q$. Then if $p$ is tautologically valid, then $p, \neg \mathcal{X} p \Rightarrow \mathcal{X} \neg p$, $\mathcal{X} \neg p \Rightarrow \neg \mathcal{X} p$, $\mathcal{X}(p \Rightarrow q) \Rightarrow (\mathcal{X} p \Rightarrow \mathcal{X} q)$, $\mathcal{G} p \Rightarrow p \& \& \mathcal{X} \mathcal{G} p$, $p \mathcal{U} s q \Rightarrow \mathcal{X} q || \mathcal{X}(p \& \&(p \mathcal{U} s q))$, $\mathcal{X} q || \mathcal{X}(p \& \&(p \mathcal{U} s q)) \Rightarrow p \mathcal{U} s q$, $p \mathcal{U} s q \Rightarrow \mathcal{X} \mathcal{F} q \in D$.

The subset $AX_{\text{LTL}}$ of the LTLB-WFF is defined as follows:
AX\textsubscript{LTL} has LTL axioms and for every subset \( D \) of the LTLB-WFF such that \( D \) has LTL axioms holds \( AX\textsubscript{LTL} \subseteq D \).

One can verify that \( AX\textsubscript{LTL} \) has LTL axioms.

One can prove the following propositions:

\begin{align*}
(34) & \quad p \Rightarrow (q \Rightarrow p) \in AX\textsubscript{LTL}.
(35) & \quad (p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \in AX\textsubscript{LTL}.
\end{align*}

Let us consider \( p, q \). The predicate \( NEX(p, q) \) is defined as follows:

\begin{align*}
(\text{Def. 19}) & \quad q = Xp.
\end{align*}

Let us consider \( r \). The predicate \( MP(p, q, r) \) is defined by:

\begin{align*}
(\text{Def. 20}) & \quad q = p \Rightarrow r.
\end{align*}

The predicate \( IND(p, q, r) \) is defined as follows:

\begin{align*}
(\text{Def. 21}) & \quad \text{There exist } A, B \text{ such that } p = A \Rightarrow B \text{ and } q = A \Rightarrow XA \text{ and } r = A \Rightarrow G\,B.
\end{align*}

One can check that \( AX\textsubscript{LTL} \) is non empty.

Let us consider \( A \). We say that \( A \) is LTL axiom 1 if and only if:

\begin{align*}
(\text{Def. 22}) & \quad \text{There exists } B \text{ such that } A = \neg X\,B \Rightarrow X\neg B.
\end{align*}

We say that \( A \) is LTL axiom 1a if and only if:

\begin{align*}
(\text{Def. 23}) & \quad \text{There exists } B \text{ such that } A = X\neg B \Rightarrow \neg X\,B.
\end{align*}

We say that \( A \) is LTL axiom 2 if and only if:

\begin{align*}
(\text{Def. 24}) & \quad \text{There exist } B, C \text{ such that } A = X(B \Rightarrow C) \Rightarrow (X\,B \Rightarrow X\,C).
\end{align*}

We say that \( A \) is LTL axiom 3 if and only if:

\begin{align*}
(\text{Def. 25}) & \quad \text{There exists } B \text{ such that } A = GB \Rightarrow B \& XG\,B.
\end{align*}

We say that \( A \) is LTL axiom 4 if and only if:

\begin{align*}
(\text{Def. 26}) & \quad \text{There exist } B, C \text{ such that } A = B\mathcal{U}C \Rightarrow X\mathcal{F}C.
\end{align*}

Next we state two propositions:

\begin{align*}
(36) & \quad \text{Every element of } AX\textsubscript{LTL} \text{ is tautologically valid, or LTL axiom 1, or LTL axiom 1a, or LTL axiom 2, or LTL axiom 3, or LTL axiom 4, or LTL axiom 5, or LTL axiom 6.}
\end{align*}

\begin{align*}
(37) & \quad \text{Suppose that } A \text{ is LTL axiom 1, or LTL axiom 1a, or LTL axiom 2, or LTL axiom 3, or LTL axiom 4, or LTL axiom 5, or LTL axiom 6. Then } F \models A.
\end{align*}

\begin{align*}
& \text{Let } i \text{ be a natural number and let us consider } f, X. \text{ The predicate } prc(f, X, i) \text{ is defined by the conditions } (\text{Def. 29}).
\end{align*}
(Def. 29)(i) \( f(i) \in AX_{\text{LTL}} \), or
   (ii) \( f(i) \in X \), or
   (iii) there exist natural numbers \( j, k \) such that \( 1 \leq j < i \) and \( 1 \leq k < i \) and
        \( \text{MP}(f_j, f_k, f_i) \) or \( \text{IND}(f_j, f_k, f_i) \), or
   (iv) there exists a natural number \( j \) such that \( 1 \leq j < i \) and \( \text{NEX}(f_j, f_i) \).

Let us consider \( X, p \). The predicate \( X \vdash p \) is defined by:
(Def. 30) There exists \( f \) such that \( f(\text{len} f) = p \) and \( 1 \leq \text{len} f \) and for every natural number \( i \) such that \( 1 \leq i \leq \text{len} f \) holds \( \text{prc}(f, X, i) \).

Next we state four propositions:
(38) Let \( i, n \) be natural numbers. Suppose \( n + \text{len} f \leq \text{len} f_2 \) and for every natural number \( k \) such that \( 1 \leq k \leq \text{len} f \) holds \( f(k) = f_2(k + n) \) and \( 1 \leq i \leq \text{len} f \). If \( \text{prc}(f, X, i) \), then \( \text{prc}(f_2, X, i + n) \).
(39) Suppose that
   (i) \( f_2 = f \prec f_1 \),
   (ii) \( 1 \leq \text{len} f \),
   (iii) \( 1 \leq \text{len} f_1 \),
   (iv) for every natural number \( i \) such that \( 1 \leq i \leq \text{len} f \) holds \( \text{prc}(f, X, i) \), and
   (v) for every natural number \( i \) such that \( 1 \leq i \leq \text{len} f_1 \) holds \( \text{prc}(f_1, X, i) \).

Let \( i \) be a natural number. If \( 1 \leq i \leq \text{len} f_2 \), then \( \text{prc}(f_2, X, i) \).
(40) Suppose \( f = f_1 \prec (p) \) and \( 1 \leq \text{len} f_1 \) and for every natural number \( i \) such that \( 1 \leq i \leq \text{len} f_1 \) holds \( \text{prc}(f_1, X, i) \) and \( \text{prc}(f, X, \text{len} f) \). Then for every natural number \( i \) such that \( 1 \leq i \leq \text{len} f \) holds \( \text{prc}(f, X, i) \) and \( X \vdash p \).
(41)\(^1\) If \( F \vdash A \), then \( F \models A \).

6. Derivation of Some Formulas. Deduction Theorem of LTL

We now state a number of propositions:
(42) If \( p \in AX_{\text{LTL}} \) or \( p \in X \), then \( X \vdash p \).
(43) If \( X \vdash p \) and \( X \vdash p \Rightarrow q \), then \( X \vdash q \).
(44) If \( X \vdash p \), then \( X \vdash \text{x} p \).
(45) If \( X \vdash p \Rightarrow q \) and \( X \vdash p \Rightarrow \text{x} p \), then \( X \vdash p \Rightarrow \text{G} q \).
(46) If \( X \vdash r \Rightarrow p \&\& q \), then \( X \vdash r \Rightarrow p \) and \( X \vdash r \Rightarrow q \).
(47) If \( X \vdash p \Rightarrow q \) and \( X \vdash q \Rightarrow r \), then \( X \vdash p \Rightarrow r \).
(48) If \( X \vdash p \Rightarrow (q \Rightarrow r) \), then \( X \vdash p \&\& q \Rightarrow r \).
(49) If \( X \vdash p \&\& q \Rightarrow r \), then \( X \vdash p \Rightarrow (q \Rightarrow r) \).
(50) If \( X \vdash p \&\& q \Rightarrow r \) and \( X \vdash p \Rightarrow s \), then \( X \vdash p \&\& q \Rightarrow s \&\& r \).

\(^1\)Soundness Theorem for LTL
(51) If $X \vdash p \Rightarrow (q \Rightarrow r)$ and $X \vdash r \Rightarrow s$, then $X \vdash p \Rightarrow (q \Rightarrow s)$.

(52) If $X \vdash p \Rightarrow q$, then $X \vdash \neg q \Rightarrow \neg p$.

(53) $X \vdash X(p \&\& q) \Rightarrow X(p \&\& q)$.

(54) If $F \vdash p$, then $F \vdash Gp$.

(55) If $p \Rightarrow q \in F$, then $F \cup \{p\} \vdash Gq$.

(56) If $F \vdash q$, then $F \cup \{p\} \vdash q$.

(57) If $F \cup \{p\} \vdash q$, then $F \vdash Gp \Rightarrow q$.

(58) If $F \vdash p \Rightarrow q$, then $F \cup \{p\} \vdash Gp \Rightarrow Gq$.

(59) If $F \vdash Gp \Rightarrow q$, then $F \cup \{p\} \vdash q$.

(60) $F \vdash G(p \Rightarrow q) \Rightarrow (Gp \Rightarrow Gq)$.

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\(^2\)Deduction Theorem of LTL
Banach Algebra of Bounded Complex-Valued Functionals

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Summary. In this article, we describe some basic properties of the Banach algebra which is constructed from all bounded complex-valued functionals.

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The notation and terminology used in this paper are introduced in the following articles: [2], [16], [9], [14], [7], [8], [3], [18], [17], [4], [19], [5], [15], [1], [20], [12], [11], [10], [21], [13], and [6].

Let $V$ be a complex algebra. A complex algebra is called a complex subalgebra of $V$ if it satisfies the conditions (Def. 1).

(Def. 1)(i) The carrier of it $\subseteq$ the carrier of $V$,
(ii) the addition of it = (the addition of $V$) $\upharpoonright$ (the carrier of it),
(iii) the multiplication of it = (the multiplication of $V$) $\upharpoonright$ (the carrier of it),
(iv) the external multiplication of it = (the external multiplication of $V$)$\upharpoonright$($\mathbb{C} \times$ the carrier of it),
(v) $1_{it} = 1_V$, and
(vi) $0_{it} = 0_V$.

We now state the proposition

(1) Let $X$ be a non empty set, $V$ be a complex algebra, $V_1$ be a non empty subset of $V$, $d_1, d_2$ be elements of $X$, $A$ be a binary operation on $X$, $M$ be a function from $X \times X$ into $X$, and $M_1$ be a function from $\mathbb{C} \times X$ into $X$. Suppose that $V_1 = X$ and $d_1 = 0_V$ and $d_2 = 1_V$ and $A = (the$ addition of $V) \upharpoonright (V_1)$ and $M = (the$ multiplication of $V) \upharpoonright (V_1)$ and $M_1 = (the$ external multiplication of $V)\upharpoonright(\mathbb{C} \times V_1)$ and $V_1$ has inverse. Then $\langle X, M, A, M_1, d_2, d_1 \rangle$ is a complex subalgebra of $V$.
Let $V$ be a complex algebra. One can check that there exists a complex subalgebra of $V$ which is strict.

Let $V$ be a complex algebra and let $V_1$ be a subset of $V$. We say that $V_1$ is $\mathbb{C}$-additively-linearly-closed if and only if:

(Def. 2) $V_1$ is additively and has inverse and for every complex number $a$ and for every element $v$ of $V$ such that $v \in V_1$ holds $a \cdot v \in V_1$.

Let $V$ be a complex algebra and let $V_1$ be a subset of $V$. Let us assume that $V_1$ is $\mathbb{C}$-additively-linearly-closed and non empty. The functor $\text{Mult}(V_1, V)$ yields a function from $\mathbb{C} \times V_1$ into $V_1$ and is defined by:

(Def. 3) $\text{Mult}(V_1, V) = (\text{the external multiplication of } V) | (\mathbb{C} \times V_1)$.

Let $X$ be a non empty set. The functor $\mathbb{C}\text{-BoundedFunctions} X$ yields a non empty subset of $\mathbb{C}\text{Algebra}(X)$ and is defined by:

(Def. 4) $\mathbb{C}\text{-BoundedFunctions} X = \{f : X \rightarrow \mathbb{C}: f|X \text{ is bounded}\}$.

Let $X$ be a non empty set. One can verify that $\mathbb{C}\text{Algebra}(X)$ is scalar unital.

Let $X$ be a non empty set. Note that $\mathbb{C}\text{-BoundedFunctions} X$ is $\mathbb{C}$-additively-linearly-closed and multiplicatively-closed.

Let $V$ be a complex algebra. Note that there exists a non empty subset of $V$ which is $\mathbb{C}$-additively-linearly-closed and multiplicatively-closed.

Let $V$ be a non empty $\text{CLS}$ structure. We say that $V$ is scalar-multiplication-cancelable if and only if:

(Def. 5) For every complex number $a$ and for every element $v$ of $V$ such that $a \cdot v = 0_V$ holds $a = 0$ or $v = 0_V$.

We now state two propositions:

(2) Let $V$ be a complex algebra and $V_1$ be a $\mathbb{C}$-additively-linearly-closed multiplicatively-closed non empty subset of $V$. Then $\langle V_1, \text{mult}(V_1, V), \text{Add}(V_1, V), \text{Mult}(V_1, V) \rangle$ is a complex subalgebra of $V$.

(3) Let $V$ be a complex algebra and $V_1$ be a complex subalgebra of $V$. Then

(i) for all elements $v_1, w_1$ of $V_1$ and for all elements $v, w$ of $V$ such that $v_1 = v$ and $w_1 = w$ holds $v_1 + w_1 = v + w$,

(ii) for all elements $v_1, w_1$ of $V_1$ and for all elements $v, w$ of $V$ such that $v_1 = v$ and $w_1 = w$ holds $v_1 \cdot w_1 = v \cdot w$,

(iii) for every element $v_1$ of $V_1$ and for every element $v$ of $V$ and for every complex number $a$ such that $v_1 = v$ holds $a \cdot v_1 = a \cdot v$,

(iv) $1_{(V_1)} = 1_V$, and

(v) $0_{(V_1)} = 0_V$.

Let $X$ be a non empty set. The $\mathbb{C}$-algebra of bounded functions of $X$ yielding a complex algebra is defined by:

(Def. 6) The $\mathbb{C}$-algebra of bounded functions of $X = (\mathbb{C}\text{-BoundedFunctions} X, \text{mult}(\mathbb{C}\text{-BoundedFunctions} X))$.

We now state the proposition.
(4) For every non empty set $X$ holds the $\mathbb{C}$-algebra of bounded functions of $X$ is a complex subalgebra of $\text{CAlg}(X)$.

Let $X$ be a non empty set. Note that the $\mathbb{C}$-algebra of bounded functions of $X$ is vector distributive and scalar unital.

One can prove the following propositions:

(5) Let $X$ be a non empty set, $F, G, H$ be vectors of the $\mathbb{C}$-algebra of bounded functions of $X$, and $f, g, h$ be functions from $X$ into $\mathbb{C}$. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F + G$ if and only if for every element $x$ of $X$ holds $h(x) = f(x) + g(x)$.

(6) Let $X$ be a non empty set, $a$ be a complex number, $F, G$ be vectors of the $\mathbb{C}$-algebra of bounded functions of $X$, and $f, g$ be functions from $X$ into $\mathbb{C}$. Suppose $f = F$ and $g = G$. Then $G = a \cdot F$ if and only if for every element $x$ of $X$ holds $g(x) = a \cdot f(x)$.

(7) Let $X$ be a non empty set, $F, G, H$ be vectors of the $\mathbb{C}$-algebra of bounded functions of $X$, and $f, g, h$ be functions from $X$ into $\mathbb{C}$. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F \cdot G$ if and only if for every element $x$ of $X$ holds $h(x) = f(x) \cdot g(x)$.

(8) For every non empty set $X$ holds $0$ the $\mathbb{C}$-algebra of bounded functions of $X = X \mapsto \overrightarrow{0}$.

(9) For every non empty set $X$ holds $1$ the $\mathbb{C}$-algebra of bounded functions of $X = X \mapsto \overrightarrow{1}$. $\mathbb{C}$.

Let $X$ be a non empty set and let $F$ be a set. Let us assume that $F \in \text{C-BoundedFunctions}_X$. The functor modetrans$(F, X)$ yielding a function from $X$ into $\mathbb{C}$ is defined as follows:

(Def. 7) modetrans$(F, X) = F$ and modetrans$(F, X) \mid X$ is bounded.

Let $X$ be a non empty set and let $f$ be a function from $X$ into $\mathbb{C}$. The functor $\text{PreNorms}(f)$ yielding a non empty subset of $\mathbb{R}$ is defined by:

(Def. 8) $\text{PreNorms}(f) = \{|f(x)| : x \text{ ranges over elements of } X\}$.

We now state two propositions:

(10) For every non empty set $X$ and for every function $f$ from $X$ into $\mathbb{C}$ such that $f \mid X$ is bounded holds $\text{PreNorms}(f)$ is upper bounded.

(11) Let $X$ be a non empty set and $f$ be a function from $X$ into $\mathbb{C}$. Then $f \mid X$ is bounded if and only if $\text{PreNorms}(f)$ is upper bounded.

Let $X$ be a non empty set. The functor $\text{C-BoundedFunctionsNorm}_X$ yields a function from $\text{C-BoundedFunctions}_X$ into $\mathbb{R}$ and is defined by:

(Def. 9) For every set $x$ such that $x \in \text{C-BoundedFunctions}_X$ holds $(\text{C-BoundedFunctionsNorm}_X)(x) = \sup \text{PreNorms}(\text{modetrans}(x, X))$.

One can prove the following propositions:
(13) For every non empty set $X$ and for every function $f$ from $X$ into $\mathbb{C}$ such that $f|X$ is bounded holds $\text{modetrans}(f, X) = f$.

(14) For every non empty set $X$ and for every function $f$ from $X$ into $\mathbb{C}$ such that $f|X$ is bounded holds $(\text{C-BoundedFunctionsNorm} X)(f) = \sup \text{PreNorms}(f)$.

Let $X$ be a non empty set. The $\mathbb{C}$-normed algebra of bounded functions of $X$ yielding a normed complex algebra structure is defined by:

(Def. 10) The $\mathbb{C}$-normed algebra of bounded functions of $X = \langle \text{C-BoundedFunctions} X, \text{mult}(\text{C-BoundedFunctions} X, \text{CAlgebra}(X)), \text{Add}(\text{C-BoundedFunctions} X, \text{CAlgebra}(X)), \text{Mult}(\text{C-BoundedFunctions} X, \text{CAlgebra}(X)), \text{One}(\text{C-BoundedFunctions} X, \text{CAlgebra}(X)), \text{Zero}(\text{C-BoundedFunctions} X, \text{CAlgebra}(X)) \rangle$.

Let $X$ be a non empty set. One can check that the $\mathbb{C}$-normed algebra of bounded functions of $X$ is non empty.

Let $X$ be a non empty set. One can check that the $\mathbb{C}$-normed algebra of bounded functions of $X$ is unital.

The following propositions are true:

(15) Let $W$ be a normed complex algebra structure and $V$ be a complex algebra. Suppose $\langle$ the carrier of $W$, the multiplication of $W$, the addition of $W$, the external multiplication of $W$, the one of $W$, the zero of $W$ $\rangle = V$. Then $W$ is a complex algebra.

(16) For every non empty set $X$ holds the $\mathbb{C}$-normed algebra of bounded functions of $X$ is a complex algebra.

(17) Let $X$ be a non empty set and $F$ be a point of the $\mathbb{C}$-normed algebra of bounded functions of $X$. Then $(\text{Mult}(\text{C-BoundedFunctions} X, \text{CAlgebra}(X)))(1_{\mathbb{C}}, F) = F$.

(18) For every non empty set $X$ holds the $\mathbb{C}$-normed algebra of bounded functions of $X$ is a complex linear space.

(19) For every non empty set $X$ holds $X \rightarrow 0 = 0$ the $\mathbb{C}$-normed algebra of bounded functions of $X$.

(20) Let $X$ be a non empty set, $x$ be an element of $X$, $f$ be a function from $X$ into $\mathbb{C}$, and $F$ be a point of the $\mathbb{C}$-normed algebra of bounded functions of $X$. If $f = F$ and $f|X$ is bounded, then $|f(x)| \leq \|F\|$.

(21) For every non empty set $X$ and for every point $F$ of the $\mathbb{C}$-normed algebra of bounded functions of $X$ holds $0 \leq \|F\|$.

(22) Let $X$ be a non empty set and $F$ be a point of the $\mathbb{C}$-normed algebra of bounded functions of $X$. Suppose $F = 0$ the $\mathbb{C}$-normed algebra of bounded functions of $X$. Then $0 = \|F\|$.

(23) Let $X$ be a non empty set, $f$, $g$, $h$ be functions from $X$ into $\mathbb{C}$, and $F$, $G$, $H$ be points of the $\mathbb{C}$-normed algebra of bounded functions of $X$. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F + G$ if and only if for every element $x$ of $X$ holds $h(x) = f(x) + g(x)$.

\footnote{The proposition (12) has been removed.}
(24) Let $X$ be a non empty set, $a$ be a complex number, $f, g$ be functions from $X$ into $\mathbb{C}$, and $F, G$ be points of the $\mathbb{C}$-normed algebra of bounded functions of $X$. Suppose $f = F$ and $g = G$. Then $G = a \cdot F$ if and only if for every element $x$ of $X$ holds $g(x) = a \cdot f(x)$.

(25) Let $X$ be a non empty set, $f, g, h$ be functions from $X$ into $\mathbb{C}$, and $F, G, H$ be points of the $\mathbb{C}$-normed algebra of bounded functions of $X$. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F \cdot G$ if and only if for every element $x$ of $X$ holds $h(x) = f(x) \cdot g(x)$.

(26) Let $X$ be a non empty set, $a$ be a complex number, and $F, G$ be points of the $\mathbb{C}$-normed algebra of bounded functions of $X$. Then

(i) if $\|F\| = 0$, then $F = 0$ the $\mathbb{C}$-normed algebra of bounded functions of $X$;

(ii) if $F = 0$ the $\mathbb{C}$-normed algebra of bounded functions of $X$, then $\|F\| = 0$;

(iii) $\|a \cdot F\| = |a| \cdot \|F\|$, and

(iv) $\|F + G\| \leq \|F\| + \|G\|$.

Let $X$ be a non empty set. Observe that the $\mathbb{C}$-normed algebra of bounded functions of $X$ is right complementable, Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, scalar unital, discernible, reflexive, and complex normed space-like.

Next we state two propositions:

(27) Let $X$ be a non empty set, $f, g, h$ be functions from $X$ into $\mathbb{C}$, and $F, G, H$ be points of the $\mathbb{C}$-normed algebra of bounded functions of $X$. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F - G$ if and only if for every element $x$ of $X$ holds $h(x) = f(x) - g(x)$.

(28) Let $X$ be a non empty set and $s_1$ be a sequence of the $\mathbb{C}$-normed algebra of bounded functions of $X$. If $s_1$ is $\mathbb{C}$Cauchy, then $s_1$ is convergent.

Let $X$ be a non empty set. One can check that the $\mathbb{C}$-normed algebra of bounded functions of $X$ is complete.

Next we state the proposition

(29) For every non empty set $X$ holds the $\mathbb{C}$-normed algebra of bounded functions of $X$ is a complex Banach algebra.

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