Explicit and implicit methods for probabilistic common-cause failure analysis

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A B S T R A C T

The occurrence of a probabilistic common-cause failure (PCCF) in a system results in failures of multiple system components with different probabilities. A PCCF can be caused by external shocks or propagated failures originating from some components within the system. This paper proposes an explicit method and an implicit method to analyze the reliability of systems subject to internal or external PCCFs. Both methods can handle any arbitrary types of time-to-failure distributions for the system components. Both of the proposed methods are illustrated through detailed analyses of an example computer system. Applicability and advantages are also discussed and compared for the two methods.

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1. Introduction

Common cause failures (CCFs) are failures of multiple components due to a shared root cause or a common cause (CC). The presence of CCFs in a system tends to increase the joint failure probabilities and thus contributes greatly to the overall unreliability of the system [1,2]. Therefore, it is significant to incorporate their effects in the reliability modeling and evaluation of systems subject to CCFs.

CCFs can be caused by some external factors (also known as shocks), such as malicious attacks, computer viruses, human errors, or extreme environmental conditions (hurricane, floods, lightning strikes) [7–9]. They can also be caused by propagated failures originating from some components within the system [3–6]. For example, the destructive effect originating from a system component failure such as fire, overheating, short circuit, blackout, explosion may destroy or incapacitate other system components.

Components affected by the same CC form a common cause group (CCG). The effect from a CCF on its CCG can be deterministic or probabilistic. A deterministic CCF (DCCF) results in guaranteed failures of all components within the CCG; whereas a probabilistic CCF (PCCF) results in failures of different components within the CCG with different occurrence probabilities [10,11]. For a practical example of PCCFs, consider a system of multiple gas detectors installed in a production room [24]. These gas detectors can be purchased at different times and from different companies, and thus be resistant to different levels of humidity. A shared root cause of a potential PCCF event is the increased humidity in the production room. This cause may fail the different gas detectors installed at different locations of the room with different probabilities. To be different from the CCG in the DCCF, we refer the CCG in the PCCF as probabilistic CCG (PCCG) hereafter.

Considerable research efforts have been dedicated to analyzing systems subject to DCCFs. The existing approaches can be classified into explicit and implicit approaches. The basic idea of the explicit approaches is to evaluate an expanded system model which is established by modeling the occurrence of each CC as a basic event shared by all the components affected by this CC (i.e., all the components of its CCG) in the original system model [12–14]. The basic idea of the implicit approaches is to develop the system model without considering the effects of DCCFs first and then evaluate the system model including the contributions of DCCFs by some special treatments [15–18].

To the best of our knowledge, very few works have considered PCCFs. Ref. [19] presents a binomial failure rate model to address PCCFs. However, the model can only be used to analyze systems with s-identical and s-independent components with the same fixed failure probability given the occurrence of a CC. Ref. [11] proposes more general methods that allow non-identical system components and non-identical component failure probabilities in the case of a CC occurring. However, the methods of [11] have a restrictive assumption that the conditional failure events of
different components due to the same CC are s-independent. Moreover, they are only applicable to external CCs (not internal CCs). In this work, we propose both an explicit method and an implicit method to analyze the reliability of systems subject to PCCFs while relaxing the limitations of the existing methods. The proposed methods are applicable to both internal and external PCCFs. Also, they allow a component to belong to multiple PCCGs with different probabilities.

The remainder of the paper is organized as follows. Section 2 presents an overview of the problem to be addressed. Section 3 presents an illustrative example. Section 4 presents the proposed explicit PCCF analysis method with the example illustration. Section 5 presents the proposed implicit PCCF analysis method with the example illustration. Section 6 discusses and compares the two proposed methods. Section 7 gives conclusions and directions for future work.

2. System description and problem statement

The paper considers the problem of reliability evaluation of systems subject to internal or external PCCFs. The system consists of elements with different individual failure probabilities. Some elements can fail also as a result of different common causes, which can be associated with external factors and with failures of other system components. The probability of failure caused by any common cause event is known for any element. The system structure function, which determines the state of the entire system for any combination of the states of the elements is given.

Fault tree is used to represent the structure function of a system in this paper [20]. The PCCF behavior is modeled by a PCCF gate which is based on the functional dependency (FDEP) gate as shown in Fig. 1 [11]. The input of the PCCF gate represents the trigger event of a CC occurring, which can be either an external shock or failure of an internal system component in this work. One or more dependent events represent failures of components affected by the CC (i.e., components appearing in the PCCG), and they are forced to occur with certain (maybe different) probabilities when the trigger event occurs.

The following assumptions are used in the proposed methods:

- The component failure event caused by a CC and individual failure event for a component are s-independent.
- Failure cascading and loops are not considered, that is, the failure of a dependent component for a PCCF gate cannot trigger another PCCF gate.

3. An illustrative example

Fig. 2 illustrates a computer system consisting of two processors (P1 and P2), two buses (B1 and B2), input/output (I/O), and three memory units (M1, M2, and M3). The function of the system requires at least one of the two processors, at least one of the two buses, at least two of the three memory units, and the I/O be operating correctly. Fig. 3 illustrates the system fault tree model.

As shown in Fig. 3, the system is subjected to two external and s-independent CCs: CC1 and CC2. The occurrence of CC1 affects processor P1 and memory unit M1. The occurrence of CC2 affects processor P1, bus B1 and memory unit M2. Thus, the two PCCGs are PCCG1 = {P1, M1} and PCCG2 = {P1, B1, M2}.

The following parameter values are used in the subsequent analysis using the proposed methods:

- Local or individual failure probabilities of components: \( q_{P1} = q_{M2} = q_{B1} = q_{B2} = q_{I/O} = q_{M1} = q_{M2} = q_{M3} = 0.01 \).
- Occurrence probabilities of CCs: \( p_{CC1} = p_{CC2} = 0.001 \).
- Conditional component failure probabilities conditioned on the occurrence of related CC: \( q_{P1} = 0.2, q_{M1} = 0.5, q_{P2} = 0.3, q_{B2} = 0.4, q_{M2} = 0.6 \). For example, \( q_{1P1} \) represents the conditional failure probability of processor P1 given that CC1 occurs. In this example, processor P1 is affected by both CCs but with different probabilities.

![Fig. 1. The PCCF gate.](image)

![Fig. 2. The example computer system.](image)
4. Proposed explicit algorithm

This section presents the explicit method to handle PCCFs in the system reliability analysis, followed by the illustration using the example described in Section 3.

4.1. Explicit method

The explicit algorithm is described as the following two-step process: (1) establish an expanded fault tree to include the effect of PCCFs, and (2) evaluate the expanded fault tree to compute the system reliability. These two steps are elaborated as follows:

**Step 1:** Establish the expanded fault tree. Independent pseudo-nodes representing the component failure events triggered by CCs are added to the original system fault tree model excluding the PCCF gates. Specifically, if a component \( X \) appears in \( n \) PCCGs, that is, the component is affected by \( n \) CCs (\( CC_1, CC_2, \ldots, CC_n \)), \( n \) pseudo-nodes (\( X_1, X_2, \ldots, X_n \)) are added for this component, where \( X_i \) represents the failure event of component \( X \) caused by the \( i \)-th CC.

Since a component fails either locally or following any CC affecting this component, the total failure behavior of the component can be represented by logical expression

\[
X_{TF} = (CC_1X_1 + CC_2X_2 + \cdots + CC_nX_n) + X
\]  

(1)

Fig. 4 illustrates the fault tree model representing (1). The occurrence probability of \( X_i \) (denoted by \( q_{X_i} \)) is the conditional failure probability of \( X \) given that \( CC_i \) occurs. Since a PCCG may include more than one component, a CC event \( CC_i \) may appear more than once in the expanded fault tree model. For internal CC, the \( CC_i \) is simply the local failure event of the trigger component. After building the sub-fault tree models as in Fig. 4 for all the dependent components appearing in PCCGs, the expanded fault tree for the entire system can be established by using them to replace the corresponding component local failure events in the original fault tree model.

**Step 2:** Evaluate the expanded fault tree. The expanded fault tree can be evaluated using any traditional fault tree reliability analysis approach, such as inclusion/exclusion or sum-of-disjoint products based on cut/path-sets [20], or binary decision diagram (BDD)-based methods [6]. In this work, the BDD-based method is applied to evaluate the system reliability. The basic steps of the BDD-based method are summarized below.

1) Order input component state variables using a variable ordering heuristic.
2) Generate BDD for each single phase using the manipulation rules of equation (38.5) in [6], which is reviewed as follows.

\[
G \odot H = \text{ite}(x, G_1 \odot H_1, G_2 \odot H_2) \quad \text{index}(x) = \text{index}(y)
\]

\[
\text{ite}(x, G_1 \odot H_1, G_2 \odot H_2) \quad \text{index}(x) < \text{index}(y)
\]

\[
\text{ite}(y, G_1 \odot H_1, G_2 \odot H_2) \quad \text{index}(x) > \text{index}(y)
\]

where \( \odot \) represents the logical operation of AND or OR, \( G \) and \( H \) represent two Boolean expressions encoding two sub-BDDs. The index represents the order of the Boolean variable generated in step 1. The above rules are used for combining the two sub-BDD models represented by \( G \) and \( H \) into one BDD model. To apply these rules, the orderings of two root nodes (i.e., \( x \) for \( G \) and \( y \) for \( H \)) are compared. If \( x \) and \( y \) have the same index (i.e., they belong to the same system component), the logic operation is applied to their child nodes; otherwise, the variable with a smaller index becomes the root node of the new combined BDD model and the logic operation is applied to each child of
the node with the smaller index and the other sub-BDD model as a whole. The rules are recursively applied for logic operations between sub-expressions/sub-BDDs \((G_i, H_i)\) until one of them becomes a constant expression ‘0’ or ‘1’. Then, Boolean algebra \((1+x=1, 0+x=x, 1\cdot x=x, 0\cdot x=0)\) is applied to simplify the representation.

3) Evaluate the generated BDD to obtain the system unreliability. Specifically, each path from the root node of the system BDD to the sink node ‘1’ represents a disjoint combination of component failures and non-failures that can lead to the entire system failure. The system unreliability is thus given as the sum of probabilities for all the paths from the root to sink node ‘1’.

4.2. Illustrative example

In this section, we analyze the example computer system of Section 3 using the proposed explicit method.

Step 1: The expanded fault tree is shown in Fig. 5, which is obtained by replacing the local failure events of \(P_1, B_1, M_1,\) and \(M_2\) subject to PCCFs in the original fault tree of Fig. 3 (excluding the two PCCF gates) with sub-fault tree models in the form of Fig. 4.

Step 2: Evaluate the expanded fault tree in Fig. 5 using the BDD-based method. Fig. 6 illustrates the BDD model generated using the ordering of I/O, \(CC_1, P_{11}, CC_2, P_{12}, P_1, B_{12}, B_1, B_2, M_{11}, M_1, M_{22}, M_2, M_5\). To avoid many cross-lines in the BDD, the isomorphic sub-BDDs rooted at shaded nodes are shown once with other appearance of them showing the root nodes only. This BDD model contains 35 non-sink nodes. By evaluating the BDD model, we obtain the unreliability of the example computer system with PCCFs as 0.010523.

5. Proposed implicit method

This section presents the implicit method to handle PCCFs in the system reliability analysis, followed by the illustration using the example described in Section 3.

5.1. Implicit method

The implicit algorithm is described as the following five-step process.

Step 1: Construct an event space that involves all combinations of occurrence and non-occurrence of CCs and then evaluate occurrence probability of each combination. Assume \(m\) CCs occur in the system. We construct an event space that consists of \(2^m\) disjoint events. Each event, called a probabilistic common-cause event (PCCE), is a combination of occurrence and non-occurrence of the \(m\) CCs, as follows:

\[
PCCE_1 = CC_1 \land CC_2 \land \cdots \land CC_m
\]

\[
PCCE_2 = CC_1 \land CC_2 \land \cdots \land CC_m
\]

\[\vdots\]

\[
PCCE_{2^m} = CC_1 \land CC_2 \land \cdots \land CC_m.
\]

For example, \(PCCE_i\) represents the event that all the CCs do not occur, \(PCCE_2\) represents the event that \(CC_1\) occurs while other CCs do not. In general \(PCCE_s\) \((0 \leq s \leq 2^m - 1)\) represents the event when \(CC_k\) happens if \(\text{mod}_{2}^{s} \left\lfloor \frac{s}{2^k} \right\rfloor = 1\) and does not happen otherwise. Let \(Pr(PCCE_i)\) denote the occurrence probability of \(PCCE_i\), then we have \(\sum_{i=1}^{2^m} Pr(PCCE_i) = 1\). Note that the explicit method presented in Section 4 can handle only \(s\)-independent CCs. On the contrary, the implicit method can handle mutually exclusive, \(s\)-independent, or \(s\)-dependent CCs. One needs only to know the probabilities of all possible CCs.

Step 2: Evaluate the total failure probability for each component subject to PCCFs under each PCCE. Let \(q_x\) be the local failure probability of component X and \(q_iX\) be the conditional failure probability of component X given that \(CC_i\) occurs. If component X is affected by \(k\) CCs \((CC_1, CC_2, \ldots, CC_k)\) under \(PCCE_j\), the total failure probability for component X under \(PCCE_j\), denoted by \(Q_jX\), is

\[
Q_jX = 1 - (1 - q_X) \prod_{i=1}^{k} (1 - q_iX).
\]

Step 3: Build the reliability model of the system without considering effect of PCCFs. Similar to the explicit method,
the BDD-based method is used. For external CCs, only one BDD model is established based on the original fault tree. However, for internal CCs, the reduced fault tree model should be built first under each PCCE since the occurrence (or non-occurrence) of an internal CC under a PCCE also implies the failure occurrence (or non-occurrence) in the corresponding system component. Therefore, the local failure event of the component in the original fault tree should be replaced with logical ‘1’ (or ‘0’) when building the reduced fault tree under the PCCE. Then a BDD model is built for each reduced fault tree.

Step 4: Evaluate BDD models using total component failure probabilities under each PCCE. Let \( \Pr(\text{System fails} \mid \text{PCCE}_i) \) be the conditional system failure probability given that \( \text{PCCE}_i \) occurs. It is computed by evaluating the BDD model(s) established in Step 3 using the component total failure probabilities obtained in Step 2. For components not affected by \( \text{PCCE}_i \), their local failure probabilities are simply used in the corresponding BDD evaluation.

Step 5: Evaluate the system unreliability considering PCCFs using the total probability law as

\[
UR = \sum_{i=1}^{2^n} \left[ \Pr(\text{System fails} \mid \text{PCCE}_i) \cdot \Pr(\text{PCCE}_i) \right]
\]

5.2. Illustrative example

In this section, we analyze the example computer system of Section 3 using the proposed implicit method.

Step 1: Since there are two external CCs in the example computer system, an event space that consists of \( 2^2 = 4 \) PCCEs is constructed. The four PCCEs are \( \text{PCCE}_1 = \overline{CC_1} \cap \overline{CC_2} \); \( \text{PCCE}_2 = CC_1 \cap \overline{CC_2} \); \( \text{PCCE}_3 = \overline{CC_1} \cap CC_2 \); \( \text{PCCE}_4 = CC_1 \cap CC_2 \).

Because the two CCs are \( s \)-independent, the occurrence probabilities of the four PCCEs can be computed as

\[
\begin{align*}
\Pr(\text{PCCE}_1) &= (1 - p_{CC_1})(1 - p_{CC_2}) = 0.998001, \\
\Pr(\text{PCCE}_2) &= p_{CC_1}(1 - p_{CC_2}) = 0.000999, \\
\Pr(\text{PCCE}_3) &= (1 - p_{CC_1})p_{CC_2} = 0.000999, \\
\Pr(\text{PCCE}_4) &= p_{CC_1}p_{CC_2} = 0.000001.
\end{align*}
\]

Step 2: Evaluate the total failure probability for each component subject to PCCFs (i.e., \( P_1, M_1, B_1, \) and \( M_2 \)) under each PCCE. \( \text{PCCE}_1 \) is the event that no CC happens at all. Therefore, no components are subject to PCCFs under \( \text{PCCE}_1 \). \( \text{PCCE}_2 \) is the event when only \( CC_1 \) happens. Under this event, components \( P_1 \) and \( M_1 \) can fail due to the occurrence of \( CC_1 \). According to (3), the total failure probability for component \( P_1 \) is \( Q_{2, P_1} = 1 - (1 - q_{P_1})(1 - q_{MP_1}) = 0.208 \).

Similarly, we can obtain the total failure probabilities for all the components subject to PCCFs under each PCCE, which are listed in Table 1. In Table 1, ‘-’ represents that the component is not affected by any CC under the corresponding PCCE and the total component failure probability is simply the local failure probability.

Step 3: Build the BDD model of the system without considering the effect of PCCFs. Based on the original fault tree model (Fig. 3 excluding the PCCF gates), we generate the BDD model for the computer system without considering the effect of PCCFs, as shown in Fig. 7.

Step 4: Evaluate the BDD model in Fig. 7 under each PCCE. Using the total failure probabilities in Table 1 and component local failure probabilities given in Section 3, we can calculate the conditional system failure probability conditioned on the occurrence of each PCCE by evaluating the BDD model in Fig. 7 as

\[
\begin{align*}
\Pr(\text{System fails} \mid \text{PCCE}_1) &= 0.010493, \\
\Pr(\text{System fails} \mid \text{PCCE}_2) &= 0.022134.
\end{align*}
\]
Step 5: According to (4), evaluate the system unreliability considering PCCFs using the total probability law as

\[ UR = \sum_{i=1}^{4} \Pr(\text{System fails}|\text{PCCE}_i) \cdot \Pr(\text{PCCE}_i) = 0.010523. \]

This result matches the system unreliability result obtained using the explicit method in Section 4.2.

Note that the analyses of the example computer system in Sections 4.2 and 5.2 are both based on the parameters given in Section 3, where all the probabilities are assumed to be constant values (independent of mission time). Thus the system unreliability results obtained using these constant values are also fixed and independent of mission time. As we discuss in Section 6.3, when components have different time-to-failure distributions that are dependent on time, the system unreliability results vary with mission time.

6. Discussions and comparisons

This section presents more details on handling mixed internal and external CCs, different statistical relationships between CCs, different time-to-failure distribution types for system components, and combined internal CCs. Effects of PCCFs on the system reliability results as well as the comparison between the two proposed methods are also presented.

6.1. Mixed internal and external CCs

The analysis of the example computer system in Section 3 illustrates how the proposed methods handle external CCs. If the system is subject to internal CCs, the reliability analysis process using the explicit method is the same as that presented in Section 4.1. However, for the implicit method, as mentioned in step 3 of Section 5.1, a reduced fault tree model has to be built first under each PCCE.

In this subsection, we study the same system in Fig. 3 but subject to both external and internal CCs. Specifically, CC1 is an internal CC (failure propagated from B2) and CC2 is an external CC. The two PCCGs are PCCG1 = {P1, M1} and PCCG2 = {P1, B1, M2}. The parameters are the same as those described in Section 3, except the occurrence probability of CC1 is the local failure probability of B2: \( p_{\text{CC1}} = 0.01 \). When B2 fails, the failure propagates to the components in PCCG1 with different probabilities as in Section 3: \( q_{P1} = 0.2, q_{M1} = 0.5 \).
6.1.1. Explicit method

For the explicit method, since the two PCCGs are the same as those for the illustrative example in Section 4.2, the expanded fault tree in this case is obtained by replacing node CC1 in Fig. 5 with node B2, as shown in Fig. 8.

Evaluating the fault tree model in Fig. 8, we can get the unreliability as 0.010628.

6.1.2. Implicit method

In this subsection, the implicit method proposed in Section 5.1 is applied step by step to analyze the case with mixed internal and external CCs.

Step 1: The four PCCEs are PCCE1 = \( \overline{CC_1} \cap \overline{CC_2} \); PCCE2 = \( CC_1 \cap \overline{CC_2} \); PCCE3 = \( \overline{CC_1} \cap CC_2 \); PCCE4 = \( CC_1 \cap CC_2 \).

The occurrence probabilities of the four PCCEs can be computed as

\[
\begin{align*}
Pr(\text{PCCE}_1) &= (1 - p_{CC1})(1 - p_{CC2}) = (1 - q_{B2})(1 - p_{CC2}) = 0.98901, \\
Pr(\text{PCCE}_2) &= p_{CC1}(1 - p_{CC2}) = q_{B2}(1 - p_{CC2}) = 0.00999, \\
Pr(\text{PCCE}_3) &= (1 - p_{CC1})p_{CC2} = (1 - q_{B2})p_{CC2} = 0.00099, \\
Pr(\text{PCCE}_4) &= p_{CC1}p_{CC2} = q_{B2}p_{CC2} = 0.00001.
\end{align*}
\]

Step 2: Evaluate the total failure probability for each component subject to PCCFs (i.e., P1, M1, B1, and M2) under each PCCE. Because parameters in this case are the same as those for the illustrative example of Section 3, the total failure probabilities for all the components subject to PCCFs under each PCCE are the same as those listed in Table 1.

For PCCE1 = \( \overline{CC_1} \cap \overline{CC_2} \) and PCCE3 = \( \overline{CC_1} \cap CC_2 \), the internal CC does not happen which implies that component B2 does not fail. Therefore, the reduced fault tree models under these two PCCEs are the same and can be obtained by replacing B2 in original fault tree of Fig. 3 (excluding the two PCCF gates) with constant '0'. Fig. 9 shows the reduced fault tree after Boolean reduction (\( B_1 \cap 1 = B_1 \)).

The BDD models for Fig. 9 and Fig. 10 are shown in Fig. 11 (a) and (b), respectively.

Step 4: Evaluate the BDD models in Fig. 11 under each PCCE. Using the total failure probabilities under each PCCE in Table 1 and component local failure probabilities given in Section 3, we can calculate the conditional system failure probability conditioned on the occurrence of each PCCE by evaluating the BDD model in Fig. 11 as

\[
\begin{align*}
Pr(\text{System fails}|\text{PCCE}_1) &= 0.010394, \\
Pr(\text{System fails}|\text{PCCE}_2) &= 0.031816, \\
Pr(\text{System fails}|\text{PCCE}_3) &= 0.024941, \\
Pr(\text{System fails}|\text{PCCE}_4) &= 0.596052.
\end{align*}
\]

Step 5: According to (4), evaluate the system unreliability considering PCCFs using the total probability law as

\[
UR = \sum_{i=1}^{4} [Pr(\text{System fails}|\text{PCCE}_i) \cdot Pr(\text{PCCE}_i)] = 0.010628.
\]

This result matches the system unreliability result obtained using the explicit method in Section 6.1.1.

6.2. Different types of s-relationship between CCs

The two external CCs are assumed to be s-independent in the example computer system described in Section 3. When the statistical relationship between CCs exhibit other types (in particular, mutually exclusive or s-dependent), only the evaluation of \( Pr(\text{PCCE}_i) \) in step 1 of the implicit method is affected. Specifically, if the two external CCs are mutually exclusive, the occurrence probabilities of the four PCCEs are computed as

\[
\begin{align*}
Pr(\text{PCCE}_1) &= Pr(\overline{CC_1} \cap \overline{CC_2}) = 1 - Pr(CC_1) - Pr(CC_2) = 0.998, \\
Pr(\text{PCCE}_2) &= Pr(CC_1 \cap \overline{CC_2}) = Pr(CC_1) = 0.001, \\
Pr(\text{PCCE}_3) &= Pr(\overline{CC_1} \cap CC_2) = Pr(CC_2) = 0.001, \\
Pr(\text{PCCE}_4) &= Pr(CC_1 \cap CC_2) = 0.
\end{align*}
\]

Fig. 11. BDD models (a) the BDD model for Fig. 9 and (b) the BDD model for Fig. 10.
If the two external CCs are $s$-dependent, and the occurrence probabilities for these two CCs are $\Pr(CC_1) = 0.001$, $\Pr(CC_2|CC_1) = 0.003$, and $\Pr(CC_2|CC_T) = 0.002$, the occurrence probabilities of the four PCCEs are computed as

$$
\Pr(PCCE_1) = \Pr(CC_1 \cap CC_2) = \Pr(CC_2|CC_1) \Pr(CC_1) = [1 - \Pr(CC_2|CC_T) \Pr(CC_T)] = 0.997002,
$$

$$
\Pr(PCCE_2) = \Pr(CC_1 \cap CC_T) = \Pr(CC_1) \Pr(CC_T) = [1 - \Pr(CC_2|CC_1) \Pr(CC_1)] = 0.000997,
$$

$$
\Pr(PCCE_3) = \Pr(CC_T \cap CC_2) = \Pr(CC_T) \Pr(CC_2) = 0.001998,
$$

$$
\Pr(PCCE_4) = \Pr(CC_T \cap CC_T) = \Pr(CC_T) = 0.000003.
$$

### 6.3. Different types of distributions

The illustrative example in Section 3 assumes that all the components fail with fixed probabilities. However, the proposed explicit and implicit methods have no limitation on the time-to-failure distributions. We analyze the same system in Section 3 with different sets of parameters listed in Table 2. Three types of failure distributions are used: (1) fixed probability denoted by $q$; (2) exponential distribution with constant failure rate $\lambda$. In this case, $q(t) = 1 - e^{-\lambda t}$; and (3) Weibull distribution with scale parameter $\lambda_W$ and shape parameter $\alpha_W$. In this case, $q(t) = 1 - e^{-\lambda_W t^{\alpha_W}}$.

The system unreliability results analyzed using the proposed methods are shown in Table 3 and Fig. 12 using different mission times.

Sets 1–4 cover component parameters with mixed distributions. Using Set 2 parameters as a baseline, Set 1 represents a case with lower occurrence probabilities of CCs. Obviously, the system failure probability under Set 1 should be smaller than that under Set 2. In Set 3, the conditional failure probabilities conditioned on the occurrence of CCs are all set to 0, which implies a case without PCCFs. It is intuitive that the system failure probability under Set 3 should be lower than that under Set 2. The last set indicates another extreme case where all conditional failure probabilities conditioned on the occurrence of CCs are set to 1, implying that the system is subject to deterministic CCs. The system failure probability under Set 4 should be larger than that under Set 2, as shown by Fig. 12 and the numerical results in Table 3.

### 6.4. Combined internal CCs

In some cases, a component may fail due to the failure of multiple internal system components. Consider the illustrative example system in Fig. 13, which is subject to two independent internal CCs: failure of component A and failure of component B. If only one of the two CCs happens, no component is affected. If both CCs happen, component C fails with a certain probability. Both explicit and implicit methods are applicable to handling the combined internal CCs. The following parameters are assumed for analyzing the reliability of this example system using the proposed methods:

- Local failure probabilities of components: $q_A = q_B = q_C = 0.1$.
- Occurrence probabilities of CCs: $p_{CC1} = q_A = 0.1$, $p_{CC2} = q_B = 0.1$.
- Conditional component failure probabilities conditioned on the occurrence of related CC: $q_{1:2c} = 0.2$.

### Table 3

<table>
<thead>
<tr>
<th>Mission time (h)</th>
<th>Set 1</th>
<th>Set 2</th>
<th>Set 3</th>
<th>Set 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>50,000</td>
<td>0.459108</td>
<td>0.628332</td>
<td>0.414589</td>
<td>0.784148</td>
</tr>
<tr>
<td>100,000</td>
<td>0.675461</td>
<td>0.786116</td>
<td>0.645820</td>
<td>0.891543</td>
</tr>
<tr>
<td>150,000</td>
<td>0.804168</td>
<td>0.873580</td>
<td>0.785434</td>
<td>0.940654</td>
</tr>
</tbody>
</table>

**Fig. 12.** System unreliability vs. mission time.

### Table 2

<table>
<thead>
<tr>
<th>Set 1</th>
<th>Set 2</th>
<th>Set 3</th>
<th>Set 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>$\lambda = 1e^{-5}$</td>
<td>$\lambda = 1e^{-5}$</td>
<td>$\lambda = 1e^{-5}$</td>
</tr>
<tr>
<td>$q_1$</td>
<td>$q = 0.01$</td>
<td>$q = 0.01$</td>
<td>$q = 0.01$</td>
</tr>
<tr>
<td>$q_2$</td>
<td>$q = 1.5e^{-5}$</td>
<td>$q = 1e^{-5}$</td>
<td>$q = 1.5e^{-5}$</td>
</tr>
<tr>
<td>$p_{CC1}$</td>
<td>$p_{CC1} = 0.2e^{-4} \alpha_W = 2$</td>
<td>$p_{CC1} = 2e^{-4} \alpha_W = 2$</td>
<td>$p_{CC1} = 2e^{-4} \alpha_W = 2$</td>
</tr>
<tr>
<td>$p_{CC2}$</td>
<td>$p_{CC2} = 0.01$</td>
<td>$p_{CC2} = 0.01$</td>
<td>$p_{CC2} = 0.01$</td>
</tr>
<tr>
<td>$p_{CC3}$</td>
<td>$p_{CC3} = 1.5e^{-4} \alpha_W = 2$</td>
<td>$p_{CC3} = 1.5e^{-4} \alpha_W = 2$</td>
<td>$p_{CC3} = 1.5e^{-4} \alpha_W = 2$</td>
</tr>
<tr>
<td>$p_{CC4}$</td>
<td>$p_{CC4} = 0.01$</td>
<td>$p_{CC4} = 0.01$</td>
<td>$p_{CC4} = 0.01$</td>
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</tbody>
</table>

**Table 2**

<table>
<thead>
<tr>
<th>OCCurrence probability of components $p_{CC}$</th>
<th>$CC_1$</th>
<th>$CC_2$</th>
<th>$CC_3$</th>
<th>$CC_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_{1:2c}$ for $CC_1$</td>
<td>0.2</td>
<td>0.5</td>
<td>0.6</td>
<td></td>
</tr>
<tr>
<td>$q_{1:2c}$ for $CC_2$</td>
<td>0.3</td>
<td>0.4</td>
<td>0.6</td>
<td></td>
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</tbody>
</table>
6.4.1. Explicit method

The explicit method proposed in Section 4.1 is applied in two steps. Firstly, the expanded fault tree is built as shown in Fig. 14, where the occurrence probability of \( C_{A \cap B} \) is the conditional failure probability of \( C \) given that both the failure of \( A (CC_1) \) and the failure of \( B (CC_2) \) occur, i.e., \( q_{1+2C} = 0.2 \). Secondly, the expanded fault tree model in Fig. 14 is evaluated to obtain the unreliability as 0.0028.

6.4.2. Implicit method

In this subsection, the implicit method proposed in Section 5.1 is applied as follows.

Step 1: The four PCCEs are \( PCCE_1 = CC_1 \cap CC_2 \); \( PCCE_2 = CC_1 \cap CC_2' \); \( PCCE_3 = CC_1' \cap CC_2 \); \( PCCE_4 = CC_1' \cap CC_2' \). Because the two internal CCs are \( s \)-independent, the occurrence probabilities of the four PCCEs can be computed as

\[
\begin{align*}
Pr(PCCE_1) &= (1-p_{CC_1})(1-p_{CC_2}) = (1-q_A)(1-q_B) = 0.81, \\
Pr(PCCE_2) &= p_{CC_1}(1-p_{CC_2}) = q_A(1-q_B) = 0.09, \\
Pr(PCCE_3) &= (1-p_{CC_1})p_{CC_2} = (1-q_A)q_B = 0.09, \\
Pr(PCCE_4) &= p_{CC_1}p_{CC_2} = q_Aq_B = 0.01.
\end{align*}
\]

Step 2: Evaluate the total failure probability for components subject to PCCEs under each PCCE. For \( PCCE_1, PCCE_2 \) and \( PCCE_3 \), no component is affected. For \( PCCE_4 \) where both component \( A \) and component \( B \) fail, component \( C \) is affected with probability 0.2. Therefore the total failure probability for component \( C \) is

\[
Q_{AC} = 1 - (1-q_C)(1-q_{1+2C}) = 0.28
\]

Step 3: Since the system is subject to internal CCs, the reduced fault tree is built under each PCCE as shown in Fig. 15.

Step 4: Evaluate the reduce fault trees in Fig. 15 using the total failure probabilities and obtain the conditional system failure probability conditioned on the occurrence of each PCCE:

\[
\begin{align*}
\Pr(\text{System fails} | PCCE_1) &= 0, \\
\Pr(\text{System fails} | PCCE_2) &= 0, \\
\Pr(\text{System fails} | PCCE_3) &= 0, \\
\Pr(\text{System fails} | PCCE_4) &= 0.28.
\end{align*}
\]

Step 5: According to (4), the system unreliability considering PCCFs is

\[
UR = \sum_{i=1}^{4} [\Pr(\text{System fails} | PCCE_i) \cdot \Pr(PCCE_i)] = 0.0028.
\]

This result matches the system unreliability result obtained using the explicit method in Section 6.4.1.

6.5. Comparisons of two methods

The two proposed methods for considering PCCFs are both combinatorial and applicable to arbitrary types of time-to-failure distributions. The explicit method is more straightforward and easy to follow involving only two steps. Both external and internal CCs can be handled following the same process in the explicit method. However, the explicit method becomes computationally inefficient for large-scale systems because it introduces a large number of pseudo-nodes to the expanded fault tree and the dependence among subtrees due to the common nodes has to be handled during evaluation. Another limitation of the proposed explicit method is that it is only applicable to systems subject to \( s \)-independent CCs.

The implicit method contains five steps and is not as straightforward as the explicit method. For internal CCs, the reduced fault tree under each PCCE has to be built first before the BDD construction, which is different from the process for external CCs. However, the implicit method needs no expansion on the system reliability model and the size of the BDD model constructed is much smaller than that in the explicit method (Fig. 7 containing 9 non-sink nodes vs. Fig. 6 containing 35 non-sink nodes). In addition, the implicit method is more flexible in handling the \( s \)-relationship among CCs, which can be \( s \)-independent, mutually exclusive, or \( s \)-dependent.

7. Conclusions and future work

In this paper, we propose both explicit and implicit methods for the reliability analysis of systems subject to external or internal PCCFs. Both methods are combinatorial and applicable to any arbitrary types of time-to-failure distributions for the system components. The explicit method is straightforward but not as efficient as the implicit method. The explicit method is limited to systems subject to \( s \)-independent CCs whereas the implicit method can handle various statistical relationships among CCs,
which can be s-independent, mutually exclusive, or s-dependent. Applications of both methods and effects of PCCFs on the system reliability results are illustrated through detailed analyses of an example computer system subject to PCCFs.

In the future, we will study PCCFs in systems with cascading failures or loops [21], and in phased-mission systems involving multiple, consecutive, and non-overlapping phases of operations [22]. Also we will consider the effect of PCCFs in competing failure analysis [23].

Acknowledgments

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References