In this paper, we study bifurcation structures of period-adding phenomena in an internal wave model that is a mathematical model for ocean internal waves. It has been suggested that chaotic solutions observed in the internal wave model may be related to the universal property of the energy spectra of ocean internal waves. In numerical bifurcation analyses of the internal wave model, we illustrate bifurcation routes to chaos and parameter regions where chaotic behavior is observed. Furthermore, it is found that the chaotic solutions are related to the period-adding sequence, that is, successive generations of periodic solutions with longer periods as a control parameter is changed. Considering the period-adding sequence as successive local bifurcations, we discuss a mechanism of the phenomena from the viewpoint of bifurcation analysis. We also consider similarity between period-adding phenomena in the internal wave model and ones in the Lorenz model.

**Keywords**: Period-adding phenomena; internal wave model; bifurcations.

1. Introduction

It has been known that ocean internal waves can be found in the sea, not on the sea surface [Garett & Munk, 1975]. They have very long periods and very large amplitudes compared with the surface waves, and the order of the period ranges from tens of minutes to tens of hours and that of the amplitude from tens of meters to hundreds of meters. The study of the mechanism generating the waves is important in order to investigate the propagation of the energy and the deep ocean mixing. The mechanism of ocean internal waves has been examined experimentally with the recent development of the measuring devices. A striking property of ocean internal waves suggested by the experimental data is the universality of the energy spectra, namely, broadband characteristics of the spectra, which are independent of the location in the sea [Wunsh, 1975]. Although several studies have investigated various energy resources in order to clarify the mechanism, the universality of the energy spectra has not been fully clarified. On the other hand, it is suggested that the nonlinear interactions in ocean internal waves may be an essential factor of the universality, because it can diffuse the external force and recover the same state as before [Hibiya et al., 1998]. In particular, it is possible that the usual state corresponds to the chaotic attractor of internal waves, since the energy spectrum of chaos generally shows broadband characteristics.

The mechanism of ocean internal waves may be understood through a mathematical model with nonlinear dynamics. Based on such an idea, Abarbanel [1983] has derived a simple interesting mathematical model of ocean internal waves, which we refer to as the internal wave model, from the governing equations for fluid motion using the Boussinesq approximation. The derivation of the internal wave
model is similar to that of the Lorenz model. Besides, both models have a symmetric property with respect to the variable coordinates. Abarbanel [1983] had observed chaotic solutions at certain parameter values in the internal wave model. However, it is still an open problem to investigate bifurcation processes leading to chaotic solutions and find the parameter regions where chaotic solutions are observed in detail. The aim of this paper is to solve such remaining problems mainly by numerical bifurcation analysis. We study not only the chaotic solutions but also several phenomena on periodic solutions including a homoclinic bifurcation and the period-adding sequence.

The period-adding sequence means successive generations of periodic solutions with longer periods when the value of a control parameter is changed. In case that the period-adding sequence includes chaotic states between periodic states in a 1-parameter bifurcation diagram, the phenomenon is called the alternating periodic–chaotic sequence [Aihara et al., 1985]. These phenomena are interesting in terms of bifurcation phenomena because they can be considered as successive local bifurcations. They have been widely reported in a great variety of experiments and mathematical models in various fields such as biological neurons, electric circuits, and chemical reacting systems. Here, we give a brief summary of studies related to the period-adding sequence.

In biological neurons, the period-adding sequence has been experimentally observed in the giant axons of squid [Aihara et al., 1985] and the neural pacemaker [Ren et al., 1997]. It has also been found in mathematical models describing the response of neurons such as the periodically forced Hodgkin–Huxley equations [Hodgkin & Huxley, 1952; Aihara et al., 1984], a three-variable excitable cell model [Chay, 1985], the modified Bonhoeffer–van der Pol (MBVP) equations [Tsumoto et al., 1999], and the excitatory neural relaxation oscillators [Coombes & Osbaldestin, 2000]. In artificial neural networks, the Nagumo–Sato model shows the period-adding sequence, and the chaotic neuron model [Aihara et al., 1990] generates an alternating periodic-chaotic sequence. These 1D maps provide an intuitive understanding of the mechanism of the phenomena, that is, the staircase-like structure of the trajectory. The period-adding sequence has been known to be one of rich behaviors in electric circuits such as the neon tube circuit [Kennedy & Chua, 1986; Levi, 1990] and the negative-resistance device [Pei et al., 1986; Yasuda & Hoh, 1994]. Levi [1990] has considered a simple circle map derived from the neon tube relaxation circuit and discussed geometrical properties of the dynamics of the map. In both systems, the V–I characteristics are considered to be essential factors for the period-adding sequences. Sanjuan [1996] has found the period-adding and the alternating periodic-chaotic sequence in the cubic van der Pol oscillator and investigated the cascade of saddle-node bifurcations. Other examples of systems with period-adding phenomena include the Belousov–Zhabotinsky reaction [Tomita & Tsuda, 1979, 1980a; Pikovsky, 1981] that is well known as a remarkable example of chemical self-sustained oscillation, the homopolar disk dynamo [Sachdev & Sarathy, 1995], the peroxidase–oxidase reaction [Hauser et al., 1997], and the Lorenz model [Tomita & Tsuda, 1980b]. On the other hand, some studies have examined the mechanism of period-adding phenomena using simple 1D maps. In particular, Kaneko [1982, 1983] has numerically found scaling properties on successive saddle-node bifurcations using a simple circle map. However, the mechanisms of various period-adding phenomena have not been fully understood.

This paper is organized as follows. First of all, several mathematical properties of the internal wave model are described and the stabilities of equilibria are theoretically investigated. Secondly, bifurcations of periodic solutions including the period-adding sequence are numerically analyzed. The bifurcation diagrams of periodic solutions show that chaotic solutions emerge due to the period-doubling cascade of asymmetric periodic solutions. Furthermore, a 1D structure of the nonlinear dynamics in the period-adding sequence is extracted using the Lorenz plot that is constructed from the time series of chaotic solutions. Finally, the similarity between the bifurcation structures in the internal wave model and those in the Lorenz model is considered.

2. Bifurcations of the Internal Wave Model

2.1. The internal wave model

It has been suggested that chaotic behavior of ocean internal waves may be related to the universality of the energy spectra of the waves [Abarbanel, 1983].
Therefore, it is an important problem to investigate routes to chaos and clarify regions where chaos is generated using a mathematical model for internal waves. We use the following equations for internal waves, which were derived by Abarbanel [1983]:

\[
\begin{align*}
\frac{dx}{dt} &= -x + y - e \sin w, \\
\frac{dy}{dt} &= \frac{c}{a^2} x - \frac{d}{a} y - xz, \\
\frac{dz}{dt} &= b \frac{z}{a} + xy, \\
\frac{dw}{dt} &= x,
\end{align*}
\]

where \(x\) corresponds to the velocity, \(y\) the energy related to a dominant mode, \(z\) the energy related to another dominant mode, \(w\) the frequency of the density fluctuation, \(a\) the viscosity, \(b\) the diffusion coefficient related to a dominant mode, \(c\) the energy injected from the sea surface, \(d\) the diffusion coefficient related to another dominant mode, and \(e\) the amplitude of the density fluctuations. Here \(x, y, z,\) and \(w\) are state variables, and \(a, b, c, d,\) and \(e\) are positive parameters. Note that \(sin w\) is the only term associated with \(w\) in the right-hand sides of Eq. (1). If we put \(w' = w \pmod{2\pi},\) sin \(w' = \sin w\) for \(\forall w.\) Thus, we define \(w\) on the interval \([0, 2\pi]\) and assume \(w = w \pmod{2\pi}\) in this paper. In the bifurcation analysis, \(c\) and \(d\) are chosen as control parameters since they are dominant for the dynamics of the system, and the other parameters are fixed as \(a = b = e = 1.0\) according to Abarbanel [1983].

Equation (1) is very similar to the Lorenz model because both models are derived from the governing equations for fluid motion and through similar process of simplification and approximation. As a result, these models have a symmetric property with respect to the variable coordinates and show period-adding phenomena. Later, we compare the bifurcation structures of period-adding phenomena in the internal wave model with those in the Lorenz model.

2.2. Basic mathematical properties

It is easy to see that the internal wave model (1) has a symmetric property, that is, it is invariant under the following transformation:

\[(x, y, z, w) \rightarrow (-x, -y, z, -w),\]  

which persists for all values of the system parameters. This symmetry characterizes the topological aspect of the dynamics of the system. For example, a stable periodic orbit, which is symmetric with respect to the origin, may lose its stability due to the symmetry breaking, that is, the pitchfork bifurcation.

This model is a dissipative system because the expansion rate of volume is negative for all values of the system parameters as follows:

\[
\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} + \frac{\partial w}{\partial w} = -(1 + \frac{b + d}{a}) < 0. \tag{3}
\]

Since the rate is constant over the state space, the volume decreases exponentially with time.

2.3. Stabilities of equilibria

Equation (1) has two equilibria:

\[x_0 = (0, 0, 0, 0), \tag{4}\]

\[x_1 = (0, 0, 0, \pi). \tag{5}\]

Under the parameter condition \(c < ad,\) \(x_0\) is proved to be asymptotically stable using the following Lyapunov function:

\[
V(x, y, z, w) = \alpha x^2 + \beta y^2 + \beta z^2 + 4ae \sin^2 \frac{w}{2}, \tag{6}
\]

where \(\alpha\) and \(\beta\) are positive. It is easily confirmed that \(V(x) > 0\) for \(\forall x \neq 0\) and \(V(x_0) = 0.\) Moreover, Eq. (6) gives

\[
\frac{dV}{dt}(x) = 2\alpha x \frac{dx}{dt} + 2\beta y \frac{dy}{dt} + 2\beta z \frac{dz}{dt} + 2ae \sin w \frac{dw}{dt} = -2\alpha \left\{ x - \left( \frac{1}{2} + \frac{\beta c}{2a^2} \right) y \right\}^2 - \frac{2b\beta}{a} z^2 \]

\[
-\frac{\alpha c^2}{4a^4} \left( \frac{\beta}{\alpha} - \frac{a^2(2ad - c)}{c^2} \right)^2 y^2 \]

\[
-\frac{aad}{c^2} (ad - c)y^2. \tag{7}
\]

Therefore, if \(c < ad,\) \(\frac{dV}{dt}(x) < 0\) for \(\forall x \neq 0, \forall \alpha > 0,\) and \(\forall \beta > 0.\)

It is theoretically shown that the Hopf bifurcation of the equilibrium point \(x_0\) occurs at

\[c = ad + \frac{a^3 e}{a + d}. \tag{8}\]

This relation can be obtained by analyzing the eigenvalues of the Jacobian matrix of Eq. (1) at \(x_0.\)
On the other hand, it will be shown that another equilibrium point $x_1$ is related to the homoclinic bifurcation of periodic solutions.

### 2.4. Bifurcations of periodic solutions

#### 2.4.1. 1-parameter bifurcation

Period-adding sequence and bifurcations of periodic solutions are investigated numerically. First, we fix the values of several system parameters in Eq. (1) as $a = b = e = 1.0$ and $d = 0.5$, and change $c$ as a control parameter. The qualitative change of periodic solutions with change of $c$ is discussed.

Figure 1(a) shows the return time associated with the cross-section $y = 0$ as $c$ is varied. The discontinuous jump of the return time succeeds and the return time increases globally with increase of $c$. This property suggests some bifurcations. We can see the period-adding sequence clearly in Fig. 1(b) that shows local maximum values of $x$. These figures are obtained with both increase and decrease of $c$ and, therefore, the hysteresis is confirmed. It implies that there are parameter ranges where two different periodic solutions coexist. Examples of the projection of periodic orbits to the $(x, y, z)$-space and the corresponding time series are shown in Fig. 2. In this figure, we call a periodic solution having $m$ small oscillations in a half period of waves an “order-$m$ periodic solution” for distinguishing different periodic solutions.

When $c$ is increased from a sufficiently small value, a stable order-1 periodic solution in Fig. 2(a) emerges through the Hopf bifurcation at the parameter value corresponding to the theoretical one in Eq. (8). With further increase of $c$, the stable solution loses its stability and trajectories are attracted to a stable order-2 periodic solution in Fig. 2(b). Such a period-adding phenomenon seems to succeed until the period increases to infinity. After the period-adding sequence accumulates, a pair of asymmetric periodic solutions appears as shown in Sec. 2.4.2.

To understand the mechanism of the period-adding sequence, we analyze bifurcations of periodic solutions in Fig. 2 using the method proposed by Kawakami [1984] for tracing fixed points and bifurcation points on the Poincaré map of periodic solutions. The bifurcation structure of the order-$m$ periodic solution for $m \geq 2$ is schematically drawn in Fig. 3. The obtained 1-parameter bifurcation diagram is omitted here, since actually the branches resulting from the pitchfork bifurcation are stable in only a quite narrow parameter range. In the schematic diagram, we use notations for types of local bifurcations as follows: $\text{SN}^m$, $\text{PF}^m$, and $\text{PD}^m$ indicate the saddle-node, the pitchfork, and the period-doubling bifurcations of the order-$m$ periodic solution, respectively. As $c$ increases, the stable order-$m$ periodic solution emerges through $\text{SN}^m$ and loses its stability by subcritical $\text{PF}^m$, and asymmetric periodic solutions corresponding to the branches of $\text{PF}^m$ develop into chaotic solutions through the saddle-node bifurcation followed by the period-doubling cascade. If the Hopf bifurcation which generates the order-1 periodic solution replaces a saddle-node bifurcation, the schematic
Fig. 2. Examples of \((x, y, z)\)-projection of orbits (left) and the corresponding waveforms (right) of periodic solutions with \(a = b = e = 1.0\) and \(d = 0.5\): (a) order-1 \((c = 1.5)\), (b) order-2 \((c = 2.2)\), (c) order-3 \((c = 2.6)\), and (d) order-4 \((c = 2.75)\).
Fig. 3. Schematic bifurcation diagram of the order-$m$ periodic solutions for $m \geq 2$ in the internal wave model. Solid lines and dotted lines correspond to stable and unstable periodic solutions, respectively. SN$^m$, PF$^m$, and PD$^m$ indicate the saddle-node, the subcritical pitchfork, and the period-doubling bifurcation of the order-$m$ periodic solution, respectively.

The bifurcations of the internal wave model are very similar to those of the MBVP equations [Tsumoto et al., 1999]. In the MBVP equations, the order-$m$ periodic solution emerges through the saddle-node bifurcation and becomes unstable by the subcritical pitchfork bifurcation. Moreover, the coalescence of the asymmetric order-1 periodic solutions is also observed. However, it is unique in the internal wave model that each asymmetric periodic solution leads to chaotic solutions through successive period-doubling bifurcations.

2.4.2. 2-parameter bifurcation

Next, we study 2-parameter bifurcations when $c$ and $d$ are varied. The other parameters are the same as in the previous section. Figure 5 shows a bifurcation diagram in the $(d, c)$-plane. In the lower-right region satisfying $c < d$, the equilibrium point $x_0$ is asymptotically stable as we have already shown in Sec. 2.3 and no periodic solution exists. In the middle region above HF that indicates the Hopf bifurcation set of $x_0$, we can see islands of stable periodic solutions. With change of the parameter values, the island seems to succeed and accumulate by the period-adding sequence. In the upper region above PD that indicates the period-doubling bifurcation, a pair of asymmetric periodic solutions as shown in Fig. 6(a) exists stably. Since the period-

![Bifurcation Diagram](image-url)
Fig. 5. 2-parameter bifurcation diagram on the \((d, c)\)-parameter plane. SN, PF, PD, and HF indicate the saddle-node, the subcritical pitchfork, the period-doubling, and the Hopf bifurcation set, respectively. An equilibrium point \(x_0\) is asymptotically stable below the dotted line and no periodic solution exists. In the middle region of this figure, the succession of islands of stable periodic solutions is observed.

Fig. 6. Period-doubling cascade of a pair of asymmetric periodic solutions: (a) \((x, y, z)\)-projection of the orbits of a pair of asymmetric periodic solutions \((c = 2.836, d = 0.5)\), (b) the \((x, y)\)-projection of one of periodic solutions in (a), (c) the \((x, y)\)-projection of a periodic solution obtained through the period-doubling bifurcation of the periodic solution in (b) \((c = 2.835, d = 0.5)\), and (d) the \((x, y)\)-projection of a chaotic solution developed from the periodic solution in (c) through the period-doubling cascades \((c = 2.83326, d = 0.5)\).
Fig. 6. (Continued)

Fig. 7. An enlargement of Fig. 5 showing successive bifurcation structures. SN$^m$, PF$^m$, and PD$^m$ indicate the saddle-node bifurcation set of symmetric order-$m$ periodic solutions, the subcritical pitchfork bifurcation set generating asymmetric order-$m$ periodic solutions, and the period-doubling bifurcation set of asymmetric order-$m$ periodic solutions in Fig. 3, respectively. Figure 1 is a 1-parameter bifurcation diagram along the dotted line in this figure.
To examine bifurcation structures of periodic solutions, we focus on the middle region of Fig. 5. Figure 7 shows an enlargement of Fig. 5. This figure clearly shows a sequence of bifurcation structures including several local bifurcation sets as shown in Fig. 3. In the parameter region between SN\(^m\) and PF\(^m\), symmetric order-\(m\) periodic solutions exist stably. We can see that the order-\(m\) and order-(\(m + 1\)) periodic solutions coexist in a part of the parameter region. Chaotic solutions developed from the order-\(m\) periodic solution are not observed in this figure; stable periodic solutions exist as attracting sets all over the parameter region.

Chaotic solutions can be observed in another enlarged bifurcation diagram of Fig. 8. It is confirmed that chaotic solutions developed from bifurcation structures of order-2, 3, 4, and 5 periodic solutions are observed in the regions A, B, C and D, respectively. Bifurcation sets corresponding to PD\(^m\) in Fig. 7 are omitted in Fig. 8, because we have seen that they are along those corresponding to PF\(^m\). Figure 9 shows examples of chaotic solutions in the regions A and D. The chaotic attractor observed by Abarbanel [1983] corresponds to that developed in the case \(m = 2\) of Fig. 3. On the other hand, PD in Fig. 8 indicates the period-doubling bifurcation set of asymmetric periodic solutions in Fig. 6(a). These two periodic solutions develop into chaotic solutions through the period-doubling cascades, respectively. The transition of one of the periodic orbits through successive period-doubling bifurcations is shown in Figs. 6(b)–6(d). The chaotic solution in Fig. 6(d) is observed below PD in Fig. 8. In this case, it depends on how to change control parameters whether the period-adding sequence includes chaotic states in addition to the periodic states. As expected from the bifurcation structure in Fig. 8, the alternating periodic–chaotic sequence is observed when \(c = 2.825\) and \(d\) is varied. Figures 10(a) and 10(b) show the return time associated with the cross-section \(y = 0\) and the
Fig. 9. Examples of \((x, y, z)\)-projection of orbits (left) and the corresponding wave forms (right) of chaotic solutions: (a) one developed from an order-2 periodic solution in region A in Fig. 8 \((c = 3.069, d = 1.0)\) and (b) one developed from an order-5 periodic solution in region D in Fig. 8 \((c = 2.82, d = 0.5)\).

Fig. 10. Bifurcation diagram showing an alternating periodic–chaotic sequence with variation of \(d\): (a) return time associated with the cross-section \(y = 0\) and (b) maximum values of \(x\).
1-parameter bifurcation diagram of maximum values of $x$, respectively.

3. Discussions

3.1. The Lorenz plot

This work investigates a mechanism of the period-adding sequence in the internal wave model. One of the effective ways to extract essential structures of continuous-time dynamical systems is to derive a 1D map using the Poincaré plot or the Lorenz plot. The Lorenz plot is based upon observation of the local maximum values of time series on a Poincaré section. Here we analyze a 1D structure of the period-adding sequence using the Lorenz plot because it was useful for the original Lorenz model. Figure 11(a) shows a Lorenz plot constructed using time series of a chaotic solution. Since the chaotic solution is developed from a periodic solution related to the period-adding sequence, a 1D structure that is essential to the period-adding sequence may be displayed in the Lorenz plot. If we suppose that the Lorenz plot gives a graph of a 1D map, then the trajectory of the 1D map goes through narrow channels by showing staircase-like steps near $x_{n+1} = x_n$ as shown in Fig. 11(b). Therefore, a period-adding phenomenon may correspond to the increase of the number of steps of the staircase. Such a mechanism is common to the asymmetric case including the circle map [Kaneko, 1982, 1983] and the piecewise linear map [Coombes & Osbaldestin, 2000]. The Lorenz plot in Fig. 11 suggests that the mechanism of the period-adding sequence in symmetric dynamical systems can be understood using a symmetric 1D map.

3.2. Comparison with the Lorenz model

The internal wave model is very similar to the Lorenz model in terms of the original equations of fluid motion and the approximation. It has been reported that the period-adding sequence is also observed in the Lorenz model [Tomita & Tsuda, 1980b]. In this subsection, it is illustrated that the bifurcation structures of period-adding sequences in these models are almost the same except for the type of the pitchfork bifurcation.

The Lorenz model is given as follows:

\[
\begin{align*}
\frac{dx}{dt} &= -ax + ay, \\
\frac{dy}{dt} &= cx - y - xz, \\
\frac{dz}{dt} &= -bz + xy,
\end{align*}
\]

where $a$, $b$, and $c$ are positive parameters. Equation (9) has a symmetric property, that is, it is
invariant under the transformation \((x, y, z) \rightarrow (-x, -y, z)\). This property is similar to that of the internal wave model of Eq. (2). It has three equilibria of the origin, \(x_+ = (\sqrt{b(c-1)}, \sqrt{b(c-1)}, c-1)\), and \(x_- = (-\sqrt{b(c-1)}, -\sqrt{b(c-1)}, c-1)\) under the condition \(c > 1\) which we consider here. It is known that periodic solutions of the Lorenz model show the period-adding sequence [Tomita & Tsuda, 1980b]. According to this, we select \(a = 16.0\) and \(b = 4.0\), and change \(c\) as a control parameter.

Figure 12 shows examples of periodic solutions of the Lorenz model when \(c\) is varied. It is illustrated that the number of order of periodic solutions increases one by one with decrease of the value of \(c\). Figure 13 shows 1-parameter bifurcation structures of periodic solutions in Fig. 12. The regions of stable periodic solutions are narrower than those of chaotic states. If we decrease \(c\) from a sufficiently large value, a stable order-1 periodic solution loses its stability by the supercritical pitchfork bifurcation and then a pair of asymmetric periodic solutions corresponding to the pitchfork branches develop into chaotic solutions through the period-doubling cascade. The basins of these chaotic solutions coalesce and the attractor-merging crisis occurs near \(c = 310\). From detailed numerical bifurcation analyses, we can confirm that the order-\(m\) periodic solution in Fig. 12 bifurcates as shown in the schematic bifurcation diagram of Fig. 14 for \(m = 2, 3,\) and \(4\). Therefore, it is almost the same as that of the internal wave model in Fig. 3 except for the type of pitchfork bifurcation. Namely, the pitchfork bifurcation is supercritical in the Lorenz model, while subcritical in the internal wave model. Figure 13(a) shows the return time associated with the cross-section \(x = 0\). It seems that the period-adding phenomenon succeeds and accumulates near \(c = 30\) in this figure.

The bifurcation analysis implies that the essential bifurcation structures of the period-adding sequence for symmetric systems may be the successive local bifurcations including the pitchfork bifurcations.

4. Conclusions

We have considered bifurcations and period-adding phenomena of the internal wave model derived by Abarbanel [1983]. In the bifurcation analysis of the internal wave model, we have clarified detailed bifurcation structures of the period-adding sequence and the route to chaos. In this analysis, coexisting periodic and chaotic attractors are observed in a certain parameter area. Using 2-parameter bifurcation diagrams, we have provided an explanation that an alternating periodic–chaotic sequence...
Fig. 12. Examples of \((x, y, z)\)-projection of orbits (left) and the corresponding waveforms (right) of periodic solutions at \(a = 16.0\) and \(b = 4.0\) in the Lorenz model: (a) order-1 \((c = 525.0)\), (b) order-2 \((c = 245.0)\), (c) order-3 \((c = 137.5)\), and (d) order-4 \((c = 102.81)\).
Fig. 13. Bifurcation diagram showing an alternating periodic–chaotic sequence of the Lorenz model with change of $c$: (a) return time associated with the cross-section $x = 0$ and (b) maximum values of $x$.

appears depending on the way to change a control parameter. In addition, the Lorenz plot of a chaotic solution has shown a common structure to several 1D maps in which period-adding sequences have been found. Finally, comparing the bifurcation structures of the internal wave model with those of the Lorenz model, we have found that the bifurcations of order-$m$ periodic solutions of both models have similar structures except for the type of the pitchfork bifurcation.

It is our future problem to explain universality of period-adding phenomena and derive conditions under which they occur, particularly, in symmetric systems.
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