Fractals, coherent states and self-similarity induced noncommutative geometry

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The self-similarity properties of fractals are studied in the framework of the theory of entire analytical functions and the \( q \)-deformed algebra of coherent states. Self-similar structures are related to dissipation and to noncommutative geometry in the plane. The examples of the Koch curve and logarithmic spiral are considered in detail. It is suggested that the dynamical formation of fractals originates from the coherent boson condensation induced by the generators of the squeezed coherent states, whose (fractal) geometrical properties thus become manifest. The macroscopic nature of fractals appears to emerge from microscopic coherent local deformation processes.

Keywords: self-similarity; fractals; squeezed coherent states; noncommutative geometry; dissipation

I. INTRODUCTION

Recently, self-similarity properties of deterministic fractals have been studied in the framework of the theory of entire analytical functions and their functional realization in terms of the \( q \)-deformed algebra of squeezed coherent states has been discussed [1]. These studies are further pursued in the present paper. Self-similar structures are related to the notion of dissipative quantum interference phase and noncommutative geometry. The results are framed in the research line which has shown that macroscopic quantum systems or “extended objects”, such as kinks, vortices, monopoles, crystal dislocations, domain walls, and other so-called “defects” in condensed matter physics are generated by coherent quantum condensation processes at the microscopic level [2–4]. The theoretical description so obtained has been quite successful in explaining many experimental observations, e.g., in superconductors, crystals, ferromagnets, etc. [4]. This motivates the present attempt to frame in such a dynamical description also self-similar structures, such as fractals (to the extent in which fractals are considered under the point of view of self-similarity, considering that in some sense self-similarity is the most important property of fractals (p. 150 in Ref. [5])). Such an attempt is also in agreement with and supported by recent observations [6] showing that when a crystal is bent (submitted to deforming stress actions), the so produced crystal defects in the lattice (dislocations) form, at low temperature, self-similar fractal patterns, which thus appear as the result of non-homogeneous coherent phonon (boson) condensation [2, 3] and provide an example of “emergence of fractal dislocation structures” [6] in nonequilibrium (dissipative) systems. The possibility of relating some properties of fractals to coherent states and dissipative systems is per se of great theoretical and practical interest due to the ubiquitous presence of fractals in nature [5, 7] and the relevance of coherent states and dissipative systems in a wide range of physical phenomena.

The measure of lengths in fractals, the Hausdorff measure, the fractal “mass”, random fractals, and other fractal properties are not discussed in this paper. The discussion is limited to the self-similarity properties of deterministic fractals (which are generated iteratively according to a prescribed recipe), with specific references to the examples of the Koch curve and the logarithmic spiral (the conclusions can be extended to other examples, such as the Sierpinski gasket and carpet, the Cantor set, etc.). The self-similarity relation to dissipation and (squeezed) coherent states is presented in Section II and to noncommutative geometry in Section III. Section IV is devoted to conclusions. The Appendix contains some comments on the golden spiral and the Fibonacci progression.

II. SELF-SIMILARITY, DISSIPATION AND COHERENT STATES

The relation between self-similarity properties of Koch curve and \( q \)-deformed coherent states can be summarized as follows. In a standard fashion [7] (see also [1, 8]), let the \( n \)-th step or stage of the Koch curve construction be denoted by \( u_{n,q}(\alpha) \), with \( \alpha = 4 \) and \( q = 1/3d \). Setting the starting stage \( u_0 = 1 \), one has [1, 8]

\[
    u_{n,q}(\alpha) = (q\alpha)^n = 1, \quad \text{for any } n,
\]

\( * \) Dedicated to Francesco Guerra on the occasion of his seventieth birthday.

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from which the fractal dimension \( d = \ln 4 / \ln 3 \approx 1.2619 \), also called the self-similarity dimension [5], is obtained. It has to be stressed that self-similarity is properly defined only in the \( n \to \infty \) limit. By considering in full generality the complex \( \alpha \)-plane, and putting \( q = e^{-d \theta} \), Eq. (1) is written as \( d \theta = \ln \alpha \) and the functions \( u_{n,q}(\alpha) \) are, apart the normalization factor \( 1 / \sqrt{n!} \), nothing but the restriction to real \( q \alpha \) of the functions

\[
u_{n,q}(\alpha) = \frac{(q \alpha)^n}{\sqrt{n!}}, \quad n \in \mathbb{N}_+, \quad q \alpha \in \mathbb{C},
\]

which form a basis in the space \( \mathcal{F} \) of the entire analytic functions. Thus, the study of the fractal properties is carried on in \( \mathcal{F} \), by restricting, at the end, the conclusions to real \( q \alpha \), \( q \alpha \to \text{Re}(q \alpha) \) [1, 8]. The connection between fractal self-similarity properties and coherent states is then readily established since one realizes that \( \mathcal{F} \) is the vector space providing the so-called Fock-Bargmann representation of the Weyl–Heisenberg algebra [9] and the frame where the (Glauber) coherent states are described. By setting \( q = e^\xi \), \( \zeta \in \mathbb{C} \), the \( q \)-deformed algebraic structure is obtained by introducing the finite difference operator \( D_q \), also called the \( q \)-derivative operator [10]. By applying \( q^n \) to the coherent state \( |a \rangle \) \( (a|\alpha) = a|\alpha \rangle \), with \( a \) the annihilator operator), \( N \equiv \alpha d/\alpha \), one finds that

\[
q^N|\alpha\rangle = |q\alpha\rangle = \exp \left( -\frac{|q\alpha|^2}{2} \right) \sum_{n=0}^{\infty} \frac{(q\alpha)^n}{\sqrt{n!}} |n\rangle.
\]

The \( n \)-th iteration stage of the fractal is “seen” by applying \( (a)^n \) to \( |q\alpha\rangle \) and restricting to real \( q\alpha \)

\[
(qa)|\alpha\rangle|q\alpha\rangle = (q\alpha)^{n_q} = u_{n,q}(\alpha), \quad qa \to \text{Re}(qa).
\]

The operator \( (a)^n \) thus acts as a “magnifying” lens [1, 7, 8]. Thus the fractal \( n \)-th stage of iteration, with \( n = 0, 1, 2, \ldots, \infty \), is represented, in a one-to-one correspondence, by the \( n \)-th term in the coherent state series Eq. (3). The operator \( q^N \) is called the fractal operator [1, 8]. Note that \( |q\alpha\rangle \) is actually a squeezed coherent state [11, 12]. \( \zeta = \ln q \) is the squeezing parameter and \( q^N \) acts in \( \mathcal{F} \) as the squeezing operator. In conclusion, the self-similarity properties of the Koch curve (and other fractals) can be described in terms of the coherent state squeezing transformation.

These results can be extended also to the logarithmic spiral. Its defining equation in polar coordinates \( (r, \theta) \) is [5, 13]:

\[
r = r_0 e^{d \theta},
\]

with \( r_0 \) and \( d \) arbitrary real constants and \( r_0 > 0 \), whose representation is the straight line of slope \( d \) in a log-log plot with abscissa \( \theta = \ln e^d \):

\[
d \theta = \ln \frac{r}{r_0}.
\]

The constancy of the angular coefficient \( \tan^{-1} d \) signals the self-similarity property of the logarithmic spiral: rescaling \( \theta \to n \theta \) affects \( r/r_0 \) by the power \( (r/r_0)^n \). Thus, we may proceed again like in the Koch curve case and show the relation to squeezed coherent states. The result also holds for the specific form of the logarithmic spiral called the golden spiral (see Appendix A). The golden spiral and its relation to the Fibonacci progression is of great interest since the Fibonacci progression appears in many phenomena, ranging from botany to physiological and functional properties in living systems, as the “natural” expression in which they manifest themselves. Even in linguistics, some phenomena have shown Fibonacci progressions [14]. Consider now the parametric equations of the spiral:

\[
x = r(\theta) \cos \theta = r_0 e^{d \theta} \cos \theta,
\]

\[
y = r(\theta) \sin \theta = r_0 e^{d \theta} \sin \theta.
\]

In the complex \( z \)-plane, the point \( z = x + iy = r_0 e^{d \theta} e^{i \theta} \) on the spiral is fully specified only when the sign of \( d \theta \) is assigned. Due to the completeness of the (hyperbolic) basis \( \{e^{-d \theta}, e^{+d \theta}\} \), both the factors \( q = e^{\pm d \theta} \) must be considered. It is indeed interesting that in nature in many instances the so-called direct \( (q > 1) \) and indirect \( (q < 1) \) spirals are both realized in the same system (perhaps the most well known systems where this happens are found in phyllotaxis studies) The points \( z_1 = r_0 e^{-d \theta} e^{-i \theta} \) and \( z_2 = r_0 e^{+d \theta} e^{+i \theta} \) are then considered, where for convenience (see below) opposite signs for the imaginary exponent \( i \theta \) have been chosen. By using the parametrization \( \theta = \theta(t) \), \( z_1 \) and \( z_2 \) are easily shown to solve the equations

\[
m \ddot{z}_1 + \gamma \dot{z}_1 + \kappa z_1 = 0,
\]

\[
m \ddot{z}_2 - \gamma \dot{z}_2 + \kappa z_2 = 0.
\]
respectively, provided the relation

$$\theta(t) = \frac{\gamma}{2m}t = \frac{\Gamma}{d}t$$

(9)

holds (up to an arbitrary additive constant $c$ which we set equal to zero). We see that $\theta(T) = 2\pi$ at $T = 2\pi d/\Gamma$. In Eqs. (8) “dot” denotes derivative with respect to $t$; $m$, $\gamma$, and $\kappa$ are positive real constants. The notations $\Gamma \equiv \gamma / 2m$ and $\Omega^2 = (1/m)(\kappa - \gamma^2 / 4m) = \Gamma^2 / d^2$, with $\kappa > \gamma^2 / 4m$, also are used. In conclusion, the parametric expressions $z_1(t)$ and $z_2(t)$ for the logarithmic spiral are $z_1(t) = r_0 e^{-i\Omega t} e^{-t^2}$ and $z_2(t) = r_0 e^{i\Omega t} e^{t^2}$, solutions of Eqs. (8a) and (8b). At $t = mT$, $z_1 = r_0 (e^{-2\pi d/m})^m$, $z_2 = r_0 (e^{2\pi d/m})^m$, with the integer $m = 1, 2, 3, \ldots$

These considerations and Eqs. (8) suggest to us that one can interpret the parameter $t$ as the time parameter. As a result, we have that the time-evolution of the system of direct and indirect spirals is described by the system of equations (8a) and (8b) for the damped and amplified harmonic oscillator. The spiral “angular velocity” is given by $|d\theta/dt| = |\Gamma/d|$. Note that the oscillator $z_1$ is an open (non-hamiltonian) system and it is well known [15] that in order to set up the canonical formalism one needs to double the degrees of freedom of the system by introducing its time-reversed image $z_2$: we thus see that the physical meaning of the mentioned mathematical necessity to consider both the elements of the basis $\{e^{-d\theta}, e^{-d\theta}\}$ is that one needs to consider the closed system $(z_1, z_2)$ [15] since then we are able to set up the canonical formalism. The closed system Lagrangian is given by

$$L = m \dot{z}_1 \dot{z}_2 + \frac{1}{2} \gamma (z_1 \dot{z}_2 - \dot{z}_1 z_2) - \kappa z_1 z_2,$$

(10)

from which Eqs. (8a) and (8b)) are both derived. The canonical momenta are

$$p_{z_1} = \frac{\partial L}{\partial \dot{z}_1} = m \dot{z}_1 - \frac{1}{2} \gamma z_2, \quad p_{z_2} = \frac{\partial L}{\partial \dot{z}_2} = m \dot{z}_2 + \frac{1}{2} \gamma z_1.$$

(11)

It is worth stressing that without closing the system $z_1$ with the system $z_2$ there would be no possibility to define the momenta $p_{z_i}$, $i = 1, 2$, since the same Lagrangian $L$ would not exist. The time evolution of the “two copies” $(z_1, z_2)$ of the $z$-coordinate can be viewed as the forward in time path and the backward in time path in the phase space $(z, p_z)$, respectively. We thus arrive at a crucial point in our discussion: it is known indeed that as far as $z_1(t) \neq z_2(t)$ the system exhibits quantum behavior and quantum interference takes place [3, 16–18]. By following Schwinger [19], this can be explicitly proven, for example, by considering the double slit experiment in quantum mechanics [3, 16–18]. In the quantum mechanical formalism of the Wigner function and density matrix it is required to consider doubling the system coordinates, from one coordinate $z(t)$ describing motion in time to two coordinates, $z_1(t)$ going forward in time and $z_2(t)$ going backward in time [16, 17]. In order to have quantum interference, the forward in time action $A(z - z_1)$ must be different from the backward in time action $A(z - z_2)$. The classical behavior of the system is obtained if the two paths can be identified, i.e. $z_1(t) \approx z_2(t) \approx z_{\text{classical}}(t)$. When the forward in time and backward in time motions are (at the same time) unequal $z_1(t) \neq z_2(t)$, then the system behaves in a quantum mechanical fashion. Of course, when $z$ is actually measured there is only one classical $z_{\text{classical}}$. This picture also agrees with ’t Hooft conjecture, which states that, provided some specific energy conditions are met and some constraints are imposed, classical, deterministic systems presenting loss of information (dissipation) might behave according to a quantum evolution [20, 21]. In the present case, this means that the logarithmic spiral and its time-reversed double (the direct and the indirect spiral) manifest themselves as macroscopic quantum systems.

The canonical commutators $[z_1, p_{z_1}] = i \hbar = [z_2, p_{z_2}], \quad [z_1, z_2] = 0 = [p_{z_1}, p_{z_2}]$, and the corresponding sets of annihilation operators are introduced in a standard fashion as

$$a \equiv \left( \frac{1}{2\hbar\Omega} \right)^{\frac{1}{2}} \left( \frac{p_{z_1}}{\sqrt{m}} - i\sqrt{m}\Omega z_1 \right), \quad b \equiv \left( \frac{1}{2\hbar\Omega} \right)^{\frac{1}{2}} \left( \frac{p_{z_2}}{\sqrt{m}} - i\sqrt{m}\Omega z_2 \right),$$

(12)

with the respective creation operators $a^\dagger$ and $b^\dagger$. $\{a, a^\dagger\} = 1 = \{b, b^\dagger\}, \quad \{a, b\} = 0 = \{a, b^\dagger\}$. Performing the linear canonical transformation $A \equiv \frac{1}{\sqrt{2}}(a + b), \quad B \equiv \frac{1}{\sqrt{2}}(a - b)$, with commutation relations $[A, A^\dagger] = 1 = [B, B^\dagger], \quad [A, B] = 0 = [A, B^\dagger]$, from Eq. (10) the Hamiltonian $H = H_0 + H_I$ is obtained [15], with

$$H_0 = \hbar \Omega (A^\dagger A - B^\dagger B) = 2\hbar Q, \quad H_I = i\hbar \Gamma (A^\dagger B^\dagger - AB) = -2\hbar \Gamma J_2,$$

(13)

which is called the fractal Hamiltonian and where $C$ and $J_2$ are, respectively, the Casimir operator and one of the three generators of the SU(1,1) group associated with the system of quantum oscillators $A$ and $B$. We also have $[H_0, H_I] = 0$. The eigenvalue of $H_0$ is the constant (conserved) quantity $\hbar \Omega (n_A - n_B)$. The ground state (the vacuum) is $|0\rangle \equiv |n_A = 0, n_B = 0\rangle$, such that $A|0\rangle = 0 = B|0\rangle$, and $H_0|0\rangle = 0$; its time evolution is

$$|0(t)\rangle = \exp \left( -i \frac{H_I}{\hbar} t \right) |0\rangle = \frac{1}{\cosh(\Gamma t)} \exp \left( \tanh(\Gamma t) J_+ \right) |0\rangle,$$

(14)
with \( J_+ \equiv A^\dagger B^\dagger \), namely by a two-mode \( SU(1,1) \) generalized coherent state \([9, 15]\) produced by condensation of couples of (entangled) \( A \) and \( B \) modes: \( AB^n \), \( n = 0, 1, 2, \ldots \infty \). At every time \( t \), \( |0(t)\rangle \) has unit norm, \( \langle 0(t)|0(t)\rangle = 1 \), however as \( t \to \infty \), the asymptotic state turns out to be orthogonal to the initial state \( |0\rangle \):

\[
\lim_{t \to \infty} \langle 0(t)|0\rangle = \lim_{t \to \infty} \exp (-\ln \cosh(\Gamma t)) \to 0 ,
\]

which expresses the decay (dissipation) of the vacuum under the time evolution operator \( \mathcal{U}(t) \equiv \exp (-i\mathcal{H}_t/\hbar) \) (for large \( t \), Eq. (15) gives the ratio \( r_0/\mathcal{r}(t) \) (cf. Eqs. (6) and (9)) and the limit consistently expresses the (unbounded) growth of \( \mathcal{r}(t) \)). The state \( |0(t)\rangle \) is known to be a thermal state \([15]\) and its time evolution may be written as \([3, 15]\):

\[
|0(t)\rangle = \exp \left( -\frac{1}{2}S_A(t) \right) \exp (A^\dagger B^\dagger)|0\rangle = \exp \left( -\frac{1}{2}S_B(t) \right) \exp (A^\dagger B^\dagger)|0\rangle ,
\]

where \( S_A(t) \) and \( S_B(t) \) have the same formal expressions (with \( B \) and \( B^\dagger \) replacing \( A \) and \( A^\dagger \)):

\[
S_A(t) \equiv -\left\{ A^\dagger A \ln \sinh^2(\Gamma t) - AA^\dagger \ln \cosh^2(\Gamma t) \right\} ,
\]

Since \( A \)'s and \( B \)'s commute, one simply writes \( S \) for either \( S_A \) or \( S_B \). \( S \) in Eq. (17) is recognized to be the entropy for the dissipative system \([15]\) (see also \([2, 3]\)). It is remarkable that the same operator \( S \) that controls the irreversibility of time evolution (breaking of time-reversal symmetry) associated with the choice of a privileged direction in time evolution (time arrow), also defines formally the entropy operator. The breakdown of time-reversal symmetry characteristic of dissipation is clearly manifest in the formation process of fractals; in the case of the logarithmic spiral the breakdown of time-reversal symmetry is associated with the chirality of the spiral: the indirect (right-handed) spiral is the time-reversed, distinct, image of the direct (left-handed) spiral. The Hamiltonian \( \mathcal{H} \) then turns out to be the fractal free energy for the coherent boson condensation process out of which the fractal is formed. By identifying \( \mathcal{H}_0/\hbar = 2 \Omega C \) with the “internal energy” \( U \) and \( 2J_2/\hbar \) with the entropy \( S \), from Eq. (13) and the defining equation for the temperature \( T \) (putting \( k_B = 1 \)), we have \( \partial S/\partial U = 1/T \) and obtain \( T = h \Gamma \). Thus, \( \mathcal{H}/\hbar \) represents the free energy \( \mathcal{F} = U - TS \), and the heat contribution in \( \mathcal{F} \) is given by \( 2\Gamma J_2 \). Consistently, we also have \( \partial \mathcal{F}/\partial T|_0 = -2J_2/\hbar \). It is also interesting that the temperature \( T = h \Gamma \) is actually proportional to the background zero point energy: \( h \Gamma \propto h \Omega/2 \) \([3, 18, 21]\). One can also show that the Planck distribution for the \( A \) and \( B \) modes is obtained by extremizing the free energy functional \([2, 3, 15, 18]\).

So far for simplicity the problem has been tackled within the framework of quantum mechanics. However, it is necessary to stress that the correct mathematical framework to study quantum dissipation is the one of quantum field theory (QFT) \([3, 15]\), where one considers an infinite number of degrees of freedom. The realization of the logarithmic spiral (and in general of fractals) in the many cases it is observed in nature involves indeed an infinite number of elementary degrees of freedom, as it always happens in many-body physics.

The operator \( \mathcal{U}(t) \) written in terms of the \( a \) and \( b \) operators is \([3, 11]\)

\[
\mathcal{U}(t) = \exp \left( -\frac{\Gamma t}{2} \left( (a^2 - a^\dagger^2) - (b^2 - b^\dagger^2) \right) \right) ,
\]

and it is recognized to be the two mode squeezing generator with squeezing parameter \( \zeta = -\Gamma t \). The \( SU(1,1) \) generalized coherent state (14) is thus a squeezed state. In the case of the Koch curve, by doubling the degrees of freedom in the operator \( q^N \) one obtains the “doubled” exponential operator

\[
\left( \langle \tilde{c}^2 - c^\dagger^2 \rangle - \langle \tilde{c}^\dagger^2 - c^2 \rangle \right) = -2 \left( C^\dagger D^\dagger - CD \right) .
\]

where \( \tilde{c} \) and \( c^\dagger \) denote the doubled degrees of freedom and \( C \equiv \frac{1}{\sqrt{2}}(c + \tilde{c}) \), \( D \equiv \frac{1}{\sqrt{2}}(c - \tilde{c}) \). The fractal operator has thus the same form of the dissipative time evolution operator \( \exp (-i\mathcal{H}_t/\hbar) \) and the description in terms of generalized \( SU(1,1) \) coherent state is recovered also in the case of the Koch curve.

Of course, one can obtain the relation (19) by starting since the begin from the analysis of the structure of Eq. (1); use of \( q = e^{-i\theta} \), with \( d \) the fractal dimension, allows us to write the self-similarity equation \( q_0 \alpha = 1 \) in polar coordinates as \( u = u_0 \alpha e^{i\theta} \), which is similar to Eq. (5). Then we may proceed in a similar fashion as we did for the logarithmic spiral, writing the parametric equations for the fractal in the \( z \)-plane, considering the closed system (\( z_1, z_2 \)), and so on. This leads us again to the fractal Hamiltonian (similar to the one in Eq.(13)) and fractal free energy and to \( SU(1,1) \) generalized coherent state. The quantum dynamical scheme here presented seems therefore underlying the morphogenesis processes which manifest themselves in the macroscopic appearances (forms) of the fractals.
III. NONCOMMUTATIVE GEOMETRY

The above results lead us to consider the noncommutative geometry in the plane. The noncommutative geometry notion is here referred to in the sense analyzed in Ref. [22] (see also [23]), namely considering that a quantum interference phase (of the Aharonov-Bohm type) can be shown [3, 18, 22] to be associated with the noncommutative plane, a phenomenon which appears to be related to dissipation and squeezing. There is an even deeper level of analysis which relates such noncommutative features to the noncommutativity of the Hopf algebra characterizing \( q \)-deformed algebraic structures of the type considered in this paper in connection with \( q \)-deformed coherent states. For brevity, such a level of analysis will not be presented here (see [3, 11, 24]).

Let \((x_1, x_2)\) represent the coordinates of a point in a plane and suppose that they do not commute: \([x_1, x_2] = iL^2\). In order to understand the physical meaning of the geometric length scale \( L \), one introduces

\[
z = \frac{x_1 + ix_2}{L\sqrt{2}}, \quad z^* = \frac{x_1 - ix_2}{L\sqrt{2}},
\]

with non-vanishing commutator \([z, z^*] = 1\). Then, the Pythagoras’ distance \(\delta\) in the noncommutative plane is

\[
\delta^2 = x_1^2 + x_2^2 = L^2(2z^*z + 1).
\]

From the known properties of the oscillator destruction and creation operators \( z \) and \( z^* \), it follows that \(\delta\) is quantized in units of the length scale \( L \) according to

\[
\delta_n^2 = L^2(2n + 1) \quad \text{where} \quad n = 0, 1, 2, 3, \ldots.
\]

Thus, in the plane \((x_1, x_2)\) we have quantized “disks” of squared radius \(\delta_n^2\) and \( L \) gives the non-zero radius of the “smallest” of such disks (see also [25]). In the path integral formulation of quantum mechanics one may consider many possible paths with the same initial point and final point. The quantum interference phase \(\vartheta\) (of the Aharonov-Bohm type) between two of such paths can be shown [3, 18, 22] to be determined by \( L \) and the enclosed area \( A \), \(\vartheta = A/L^2\).

In the case of the Koch curve, one can show that [1, 8]

\[
c \rightarrow \frac{1}{\sqrt{2}}(x + ip_x) \equiv \hat{z}, \quad c^\dagger \rightarrow \frac{1}{\sqrt{2}}(x - ip_x) \equiv \hat{z}^\dagger, \quad [\hat{z}, \hat{z}^\dagger] = 1,
\]

where \( p_x = -i\partial/dx \). \( \hat{z}^\dagger \leftrightarrow a^\dagger \) and \( \hat{z} \leftrightarrow a \) are the usual creation and annihilation operators in the configuration representation. Under the action of the squeezing transformation, this leads us to obtain [1, 8]

\[
\hat{z} \rightarrow \hat{z}_q = \frac{1}{q\sqrt{2}}(x + iq^2p_x), \quad \hat{z}^\dagger \rightarrow \hat{z}_q^\dagger = \frac{1}{q\sqrt{2}}(x - iq^2p_x), \quad [\hat{z}_q, \hat{z}_q^\dagger] = 1.
\]

In the “deformed” phase space, the coordinates \(x_1 \equiv x\) and \(x_2 \equiv q^2p_x \) do not commute: \([x_1, x_2] = iq^2\) (adimensional units and \( h = 1 \) are used). Thus the \( q \)-deformation introduces noncommutative geometry in the \((x_1, x_2)\)-plane. We recognize that the \( q \)-deformation parameter (squeezing) plays the rôle of the noncommutative geometric length \( L \) and the remarks made above again apply. The noncommutative Pythagora’s theorem holds:

\[
\delta^2 = x_1^2 + x_2^2 = 2q^2(\hat{z}_q^\dagger\hat{z}_q + \frac{1}{2}),
\]

and in \( F \), in the limit \(\alpha \rightarrow Re(\alpha)\), \(\alpha \equiv x + iy\), the properties of creation and annihilation operators give

\[
\delta_n^2 = 2q^2(n + \frac{1}{2}), \quad n = 0, 1, 2, 3, \ldots.
\]

The unit scale in the noncommutative plane is now set by the \( q \)-deformation parameter. The quantum interference phase between two alternative paths in the plane is given by \(\vartheta = A/q^2\). The zero point uncertainty relation is also controlled by \( q, \Delta x_1 \Delta x_2 \geq (q^2/2)\).

Eq. (26) actually is the expression of the energy spectrum of the harmonic oscillator and we can write \((1/2)\delta_0^2 = q^2(n + 1/2) \equiv E_n, n = 0, 1, 2, 3, \ldots\), where \( E_n \) might be thought as the “energy” associated with the fractal \( n \)-stage.

In the case of the two mode description (of the logarithmic spiral as well as the Koch curve), the interference phase appears as a “dissipative interference phase” [16, 22], which provides a relation between dissipation and noncommutative geometry in the plane. In the \((z_1, z_2)\) plane, by using for simplicity the index notation \(+ \equiv 1\) and \(- \equiv 2\) and Eq. (11) for the momenta \( p_{z\pm} \), the forward in time and backward in time velocities \( v_{\pm} \) are given by

\[
v_{\pm} = \frac{1}{m}(p_{z\pm} \mp \frac{1}{2}\gamma z_{\pm}), \quad \text{with} \quad [v_+, v_-] = -i\frac{\gamma}{m^2}.
\]
Then, a canonical set of conjugate position coordinates \((\xi_+, \xi_-)\) may be defined by putting \(\xi_\pm = \mp(m/\gamma)v_\pm\), so that
\[
[\xi_+, \xi_-] = \frac{i}{\gamma}.
\] (28)

Equation (28) characterizes the noncommutative geometry in the plane \((z_+, z_-)\) [16, 22] (although in a different context see also [26]). Thus, since in the present case \(L^2 = 1/\gamma\), the quantum dissipative phase interference \(\vartheta = \mathcal{A}/L^2 = \mathcal{A}\gamma\) is associated with the two paths \(P_1\) and \(P_2\) in the noncommutative plane, provided \(z_+ \neq z_-\).

A final remark is that the map \(\mathcal{A} \rightarrow \mathcal{A}_1 \otimes \mathcal{A}_2\), which duplicates the algebra, is the Hopf coproduct map \(\mathcal{A} \rightarrow \mathcal{A} \otimes 1 + 1 \otimes \mathcal{A}\), and the Bogoliubov transformations of “angle” \(\Gamma t\) are obtained by convenient combinations of the deformed coproduct \(\Delta a_q^\gamma = a_q^{\gamma} \otimes q^{1/2} + q^{-1/2} \otimes a_q^{\gamma}\), where \(a_q^{\gamma}\) are the creation operators in the \(q\)-deformed Hopf algebra [3, 11, 18, 24]. These deformed coproduct maps are noncommutative and the \(q\)-deformation parameter is related to the coherent condensate content of the state \(|0(t)\rangle\). This sheds some light on the physical meaning of the relation between dissipation (which is at the origin of \(q\)-deformation), noncommutative geometry and the non-trivial topology of paths in the phase space [24, 27].

It has to be remarked also that, finite difference operators and \(q\)-algebras which are at the basis of squeezed states [11] are essential tools in the physics of discretized systems. Indeed, it has been recognized [11] that \(q\)-deformed algebras appear, independently on other coexisting mathematical properties, such as analyticity and differentiability, whenever some discreteness in space and/or time plays a role in the formalism, which is consistent with the case of the fractal self-similarity properties here considered which are related to a limiting process involving discrete structures.

IV. CONCLUSION

In conclusion, the relation above established between the self-similarity properties of fractals and coherent states introduces dynamical considerations in the study of fractals and their origin. Like in the case of extended objects in condensed matter physics [2–4], also self-similarity properties of fractals appear to be generated by the dynamical process controlling boson condensation at a microscopic level. Quantum dissipation, through quantum deformation, or squeezing, of coherent states appears at the root of the observed self-similarity properties at a macroscopic level.

To the extent in which fractals are considered under the point of view of self-similarity, fractals are examples of macroscopic quantum systems, in the specific sense that their macroscopic self-similarity properties cannot be derived without recurring to the underlying quantum dynamics controlled by the Hamiltonian (Eq. (18)), or, more specifically, by the fractal free energy. The fact that the entropy controls the time-evolution is consistent with the breakdown of time-reversal symmetry (arrow of time) which is clearly manifest in the formation process of fractals; it is for example related to the chirality in the logarithmic spiral where the indirect (right-handed) spiral is the time-reversed, but distinct, image of the direct (left-handed) spiral (or vice-versa). These considerations suggest that the quantum dynamics is at the basis of the morphogenesis processes responsible of the fractal macroscopic growth.

The links, which also have been discussed, between fractal self-similarity and noncommutative geometry may open interesting perspectives in many applications where quantum dissipation cannot be actually neglected, e.g. in condensed matter physics, in quantum computing.

I finally recall that the \(q\)-deformation parameter acts as a label for the Weyl-Heisenberg representations of the canonical commutation relations. In the limit of infinite degrees of freedom (i.e. in QFT) different values of \(q\) label unitarily inequivalent representations [11, 27]. By tuning the value of the \(q\)-parameter one thus moves from a given representation to another one, unitarily inequivalent to the former one. Besides such a parameter one might also consider phase parameters and translation parameters characterizing (generalized) coherent states. By varying these parameters in a deterministic iterated function process, also referred to as multiple reproduction copy machine process, the Koch curve, for example, may be transformed into another fractal (e.g. into Barnsley’s fern [5]). In the frame here presented, these fractals are described by unitarily inequivalent representations in the limit of infinitely many degrees of freedom. Work in such a direction is still needed and is planned for the future. From a practical point of view, the results here presented may be used as a predictive tool: anytime experimental results lead to a straight line of given non-vanishing slope in a log-log plot, one may infer that a specific coherent state dynamics underlies the phenomenon under study. Such a kind of “theorem” has been positively confirmed in some applications in neuroscience [1, 8, 28]. Other applications are planned to be pursued in the future.

In view of the rôle played by coherent states in condensed matter physics, quantum optics and molecular biology, in elementary particle physics and cosmology, it is also interesting that it is possible to “reverse” the arguments presented in this paper. Indeed, the conclusion that fractal self-similarity properties may be described in terms of coherent states may be reversed in the statement that coherent states have (fractal) self-similarity properties, namely there is a “geometry” characterizing coherent states which exhibits self-similarity properties.

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APPENDIX A: THE GOLDEN SPIRAL AND THE FIBONACCI PROGRESSION

Suppose that in Eq. (5) at $\theta = \pi/2$ we have $r/r_0 = e^{i(\pi/2)} = \phi$, where $\phi$ denotes the golden ratio, $\phi = (1 + \sqrt{5})/2$. Then the logarithmic spiral is called the golden spiral [5] and we may put $d_g \equiv (ln \phi)/\pi$. We may put $d_g \equiv (ln \phi)/\pi$, where the subscript $g$ stays for golden. Thus, the polar equation for the golden spiral is $r_g(\theta) = r_0 e^{d_g \theta}$. The radius of the golden spiral grows in geometrical progression of ratio $\phi$ as $\theta$ grows of $\pi/2$: $r_g(\theta + n \pi/2) = r_0 e^{d_g (\theta + n \pi/2)} = r_0 e^{d_g \theta} \phi^n$ and $r_g,n \equiv r_g(\theta = n \pi/2) = r_0 \phi^n$, $n = 0, 1, 2, 3, \ldots$.

A good “approximate” construction of the golden spiral is obtained by using the so called Fibonacci tiling, obtained by drawing in a proper way [5] squares whose sides are in the Fibonacci progression, 1

\[ F_0 = 0, F_1 = 1 \]

... the geometric progression of ratio $\phi$ (golden ratio). See ref. [29] for a recent analysis of the Fibonacci generic number is $F_n = F_{n-1} + F_{n-2}$, with $F_0 = 0$, $F_1 = 1$. The Fibonacci spiral is then made from quarter-circles tangent to the interior of each square and it does not perfectly overlap with the golden spiral. The reason is that $F_n/F_{n-1} \to \phi$ in the $n \to \infty$ limit, but is not equal to $\phi$ for given finite $n$. The golden ratio $\phi$ and its “conjugate” $\psi = 1 - \phi = -1/\phi = (1 - \sqrt{5})/2$ are both solutions of the “quadratic formula”:

\[ x^2 - x - 1 = 0 \]  

(A1)

and of the recurrence equation $x^n - x^{n-1} - x^{n-2} = 0$, which, for $n = 2$, is the relation (A1). This is satisfied also by the geometric progression of ratio $\phi$ of the radii $r_{g,n} = r_0 \phi^n$ of the golden spiral. See ref. [29] for a recent analysis on $q$-groups and the Fibonacci progression. I remark that Eq. (A1) is the characteristic equation of the differential equation $\ddot{r} + \dot{r} - r = 0$, which admits as solution $r(t) = r_0 e^{(\omega - d)t} e^{-d \theta}$ with $\omega = \pm i \sqrt{5}/2$ and $\theta = -t/(2d) + c$, with $c$, $r_0$ and $d$ constants. By setting $c = 0$, $r(t) = r_0 e^{i \theta t} e^{-d t/2}$, i.e. $r_\phi(t) = r_0 e^{-\phi t}$ and $r_\psi(t) = r_0 e^{-\psi t}$.