INTEGRAL FUNCTIONALS AND THE GAP PROBLEM: SHARP BOUNDS FOR RELAXATION AND ENERGY CONCENTRATION

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Abstract. We consider integral functionals of the type
\[ F(u) := \int_\Omega f(x,u,Du) \, dx \]
exhibiting a gap between the coercivity and the growth exponent:
\[ L^{-1}|Du|^p \leq f(x,u,Du) \leq L(1 + |Du|^q) \quad 1 < p < q \quad 1 \leq L < +\infty. \]
We give lower semicontinuity results and conditions ensuring that the relaxed functional \( F \) is equal to \( \int_\Omega Qf(x,u,Du) \, dx \), where \( Qf \) denotes the usual quasi-convex envelope; our conditions are sharp. Indeed we also provide counterexamples where such an integral representation fails, showing that energy concentrations appear in the relaxation procedure leading to a measure representation of \( F \) with a non zero singular part, which is explicitly computed. The main point in our analysis is that such relaxation results depend in subtle way on the interaction between the ratio \( q/p \) and the degree of regularity of the integrand \( f \) with respect to the variable \( x \). Our results extend theorems for non-convex integrals due to Fonseca & Maly and Kristensen; the energies we treat are related to strongly anisotropic settings.

Key words. Relaxation, gap phenomenon, quasiconvexity, non-standard growth conditions.

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1. Introduction. In recent years there has been an increasing interest in variational integrals defined on Sobolev spaces and exhibiting a gap between the growth and coercivity exponents
\[ (1.1) \quad \int_\Omega f(x,u,Du) \, dx ; \quad L^{-1}|Du|^p \leq f(x,u,Du) \leq L(1 + |Du|^q) \quad L \geq 1 \]
where \( 1 < p < q < +\infty \), \( u : \Omega \to \mathbb{R}^N \) and \( \Omega \) is a domain in \( \mathbb{R}^n \). The main issues treated in this setting are concerned with the lower semicontinuity, relaxation and regularity of minimizers of such functionals. Therefore a great deal of analytical techniques has been developed; examples of papers devoted to such an issue are [1], [8], [24], [25], [33], [34] and [40]. In particular, in the paper [24] Fonseca & Malý addressed the issue of studying the relaxation and the lower semicontinuity of quasi-convex functionals satisfying (1.1) with \( f = f(Du) \). They succeeded in proving that
\[ (1.2) \quad \int_\Omega f(Du) \, dx \leq \liminf_k \int_\Omega f(Du_k) \, dx \]
for any sequence of functions \( u_k \in W^{1,q}(\Omega;\mathbb{R}^N) \) weakly converging to \( u \), \( u_k \rightharpoonup u \), in \( W^{1,p}(\Omega;\mathbb{R}^N) \); moreover they proved that the relaxed functional (when considered with respect to the weak topology of \( W^{1,p}(\Omega;\mathbb{R}^N) \)) is a Radon measure, say \( \mu_u \). Concerning this type of results, see also the work of Kristensen [33], [34] and [36]. The previous theorems are valid provided the gap between \( p \) and \( q \), measured in terms of the ratio \( q/p \), is not too large, depending on the dimension \( n \), i.e.
\[ (1.3) \quad \frac{q}{p} < \frac{n}{n-1}; \]

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see [24] and [37] for discussion on the optimality of (1.3); see also [40], [30]. Subsequently, in [8], Bouchitté, Fonseca & Maly also proved that the density of the absolutely continuous part (with respect to the Lebesgue measure) of \( \mu_u \) coincides with the quantity \( Q_f(Du) \), where \( Q_f(\cdot) \) denotes the quasi-convex envelope of \( f \) (see [14]). A main problem of the issue is, at this stage, saying something about the singular part of \( \mu_u \). In a more recent paper [1], Acerbi, Bouchitté & Fonseca examined the non-autonomous case \( f \equiv f(x, Du) \), analyzing the relaxed functional and proving, under the main assumption of convexity of the function \( z \mapsto f(x, z) \), that the existence of the singular part of the measure \( \mu_u \) is related to the presence of the Lavrentiev phenomenon that such functionals typically present, i.e., the impossibility to approximate in energy a given function \( u \in W^{1,p} \) with \( W^{1,q} \)-functions. In particular, they prove that, if there is no Lavrentiev phenomenon at \( u \), then there is no singular part of the measure \( \mu_u \). Note that the significance of the situation of the paper [1] (even if \( f \) is considered to be convex with respect to the gradient variable) lies in the combination of the facts that \( f \) both depends on \( x \) and exhibits a gap. Needless to say there is no Lavrentiev gap when one of the two previous conditions fails (by a well known convolution argument based on the convexity of \( f \) and Jensen inequality). This suggests that, when dealing with functionals as in (1.1), the presence of the \( x \) and, even worse, of both \( x \) and \( u \) determines a critical situation. In any case not much is known about the relaxed functional and the singular part of \( \mu_u \) in the general case (1.1), compare [9, Ch. 21] for a partial result. It is important to note that all the analysis in [1] is based on the convexity of \( f \). Let us explicitly remark that the techniques of the previous works do not apply to quasi-convex energy densities of the type \( f(x, Du) \) without imposing severe restrictions on the way the function \( f \) depends on \( x \).

The aim of this paper is to investigate such an issue concentrating on some classes of non-convex functionals as in (1.1) that will have to satisfy certain structure assumptions but that, nevertheless, will allow to consider large class of functionals not covered in the available literature. For ease of exposition we assume that

\[
F(u) \equiv F(u, \Omega) := \int_{\Omega} f(x, Du) \, dx
\]

and consider the relaxed functional (see also Remark 2.1 below)

\[
\overline{F}(u, \Omega) := \inf \{ \liminf_{k \to +\infty} F(u_k, \Omega) \mid \{u_k\} \subset W^{1,q}_{\text{loc}}(\Omega; \mathbb{R}^N), u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^N) \}.\]

The problem we address is: proving measure representation properties of the relaxed functional, representing its absolute continuous part and finally discovering whether or not in the relaxation procedure a singular part emerges. Moreover, it turns out to be relevant also the problem of finding explicit examples of singular parts of \( \mu_u \), when the Lavrentiev phenomenon does occur. In this direction very few results are available in the literature, see [12], [25], [26], [40], [42].

Due to the lack of a general theory, our analysis starts, and largely proceeds, considering some model examples. Let us consider the following relevant ones:

\[
F_1(u) := \int_{\Omega} |Du|^{p(x)} \, dx \quad F_2(u) := \int_{\Omega} (|Du|^p + a(x)|Du|^q) \, dx
\]

where \( p \leq p(x) \leq q \) and \( 0 \leq a(x) \leq L < +\infty \) are continuous functions.

What we are going to discover in the following is that, in such a situation, the form of the relaxed functional is linked to a subtle interplay between the gap of
the functional and the regularity of the energy density $f(x, Du)$ with respect to the variable $x$. Roughly speaking, and, for the sake of clarity, referring to $F_2$, we are going to show that the larger is the gap between $p$ and $q$, the higher is the regularity required on the function $f(x, \cdot)$. Indeed, we shall see that for any functional of the type in (1.1), that is controlled by $F_2$ in the sense

$$L^{-1}(|z|^p + a(x)|z|^q) \leq f(x, z) \leq L(|z|^p + a(x)|z|^q + 1), \quad L \geq 1$$

then the relaxed functional described in (1.5) is exactly

$$\int_\Omega Qf(x, Du) \, dx$$

provided the function $a(x)$ in $\alpha$-Hölder continuous and the following bound is satisfied:

$$\frac{q}{p} \leq \frac{n + \alpha}{n}.$$ (1.7)

Therefore no energy concentration appears in the relaxation procedure. This condition must clearly be compared to the one appearing in (1.3): the difference is that the regularity of $f$ with respect to the variable $x$ comes into the play via the exponent $\alpha$. Now, though this bound may appear of technical nature (at least looking at the proof) the interesting thing is that it actually turns out to be sharp: indeed we build a functional, which is exactly $F_2$ for a particular choice of the function $a(x)$, for which the relaxation process does not lead to Radon measure, but rather to a Borel measure, in the form of an infinite Dirac mass concentrated in one point. This can be done as soon as the bound in (1.7) is violated; note that this counterexample can be obtained already in the scalar case $N = 1$ and in the case of convex integrals. A similar situation occurs when considering the relaxation problem for functional $F_1$, where another condition, in some sense similar to (1.7), involving the oscillations and the regularity of the exponent function $p(x)$ must be considered; see (5.6) below and Sec. 8.

But let us give an outlook on the content of this paper. To be general, we shall treat functionals like the one in (1.1) and satisfying the following additional structure assumption:

$$L^{-1}\psi(x, |z|) \leq f(x, z) \leq L(\psi(x, |z|) + 1)$$ (1.8)

where $\psi(x, |z|)$ is a suitable convex function with $(p, q)$ growth (with respect to $z$), typical examples being the energy densities of the functionals $F_1$ and $F_2$; see Remark 3.1 below. Therefore, we shall not deal with typical examples of quasi-convex energy densities such as $|z|^p + |\det z|$ as considered, for example, in [40], [25], [24]. In order to prove the integral representation, a key point will be certain continuity estimates on the maximal function with respect to the function $\psi(x, |z|)$ and the density of smooth maps in energy, see Sec. 4 and Sec. 5. This is the point where bounds as in (1.7) come into the play. Then we proceed building in Sec. 7 and Sec. 8 the counterexamples proving the sharpness of our assumptions. It is worth pointing out that all the counterexamples we work out are developed in the scalar case ($N = 1$).

Finally, let us say that for the sake of brevity we confine our analysis to integral functional of the type in (1.1), which already incorporate all the technical and applicative significance of the present issues; the same results can be extended without serious additional efforts to integrands of the type $f \equiv f(x, u, Du)$. 
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2. Notation and preliminary results. In the sequel \( \Omega \) is always a fixed open subset of \( \mathbb{R}^n \) and \( A \) is the family of its open subsets; if \( A, B \in A \), by \( A \subset B \) we mean that the closure \( \overline{A} \) of \( A \) is a compact set contained in \( B \), and by \( A_0 \) we denote the class of all \( A \in A \) such that \( A \subset \Omega \). Also, \( B_r(x) \) denotes the ball of radius \( r > 0 \) centered at \( x \in \mathbb{R}^n \) and \( B_r := B_r(0) \). We will denote \( L^p(\Omega; \mathbb{R}^n) \) and \( W^{1,p}(\Omega; \mathbb{R}^N) \), \( p \geq 1 \), the standard Lebesgue and Sobolev spaces of functions \( u : \Omega \to \mathbb{R}^N \); for the sake of brevity these spaces will be also denoted omitting the dependence on the target space, e.g.: \( W^{1,p}(\Omega), L^p(\Omega) \) and so on. As customary, in the rest of the paper \( c \) will denote an unspecified positive constant, possibly varying from line to line; the relevant connections will be emphasized when needed while more peculiar occurrences will be stressed by \( c_1, c_2, \ldots, \) etc. We will consider non-negative variational functionals \( F : L^1(\Omega; \mathbb{R}^N) \to [0, +\infty] \) of the type

\[
F(u) = \begin{cases} 
\int_{\Omega} f(x, Du(x)) \, dx & \text{if } u \in C^1(\Omega; \mathbb{R}^N) \\
+\infty & \text{elsewhere on } L^1(\Omega; \mathbb{R}^N)
\end{cases}
\]

where \( f : \Omega \times \mathbb{R}^n \times \mathbb{R}^N \to [0, +\infty) \) is a Borel measurable function satisfying a non-standard growth condition, see (3.1) and (3.2). We are interested in the study of the relaxed functional of \( F \) with respect to the strong \( L^1(\Omega; \mathbb{R}^N) \) convergence, i.e., the lower semicontinuous envelope of \( F \) with respect to the \( L^1(\Omega; \mathbb{R}^N) \) topology. To show measure property and integral representation of the relaxed functional we make use of the localization method, which consists in considering at the same time the dependence on the function and on the open set. To this aim, we will work with non-negative variational functionals \( F : L^1(\Omega; \mathbb{R}^N) \times A \to [0, +\infty] \) of the form

\[
F(u, A) := \begin{cases} 
\int_A f(x, Du(x)) \, dx & \text{if } u \in C^1(A; \mathbb{R}^N) \\
+\infty & \text{elsewhere on } L^1(\Omega; \mathbb{R}^N)
\end{cases}
\]

for any open set \( A \in A \). Also, for every \( A \in A \), we denote by \( \overline{F}(\cdot, A) \) the relaxed functional of \( F(\cdot, A) \) with respect to the strong \( L^1(\Omega; \mathbb{R}^N) \) convergence, given for all \( u \in L^1(\Omega; \mathbb{R}^N) \) by

\[
\overline{F}(u, A) := \inf \{ \liminf_{k \to +\infty} F(u_k, A) \mid \{ u_k \} \subset L^1(\Omega; \mathbb{R}^N), u_k \to u \text{ in } L^1(\Omega; \mathbb{R}^N) \}.
\]

Remark 2.1. Since each sequence \( \{ u_k \} \subset L^1(A; \mathbb{R}^N) \) converging to \( u \) strongly in \( L^1(A; \mathbb{R}^N) \) can be extended to a sequence \( L^1(\Omega; \mathbb{R}^N) \)-converging to \( u \), if \( \overline{F}(u, A) < +\infty \) by (2.1) we have

\[
\overline{F}(u, A) = \inf \{ \liminf_{k \to +\infty} \int_A f(x, Du_k(x)) \, dx \mid \{ u_k \} \subset C^1(A; \mathbb{R}^N), u_k \to u \text{ in } L^1(A; \mathbb{R}^N) \}.
\]
We explicitly remark that whenever the function $f$ satisfies the following $(p, q)$-growth condition:

$$
L^{-1}|z|^p \leq f(x, z) \leq L(1 + |z|^q), \quad 1 < p < q < +\infty \quad 1 \leq L
$$

then the previous relaxed functional coincides with the following one, analyzed in [1] [8] [24]:

$$
\mathcal{F}(u, A) = \inf \left\{ \liminf_{k \to +\infty} \int_A f(x, Du_k(x)) \, dx \mid \{u_k\} \subset W^{1,q}_{\text{loc}}(A; \mathbb{R}^N), u_k \rightharpoonup u \text{ in } W^{1,p}(A; \mathbb{R}^N) \right\}.
$$

To show the measure property we recall some well known facts about set functions.

**Definition 2.2.** A function $\alpha : A \to [0, +\infty]$ is called an increasing set function if $\alpha(\emptyset) = 0$ and $\alpha(A) \leq \alpha(B)$ if $A \subseteq B$. An increasing set function $\alpha$ is said to be subadditive if

$$
\alpha(A \cup B) \leq \alpha(A) + \alpha(B)
$$

for all $A, B \in A$, and it is said to be superadditive if

$$
\alpha(A \cup B) \geq \alpha(A) + \alpha(B)
$$

for all $A, B \in A$ with $A \cap B = \emptyset$; finally $\alpha$ is said to be inner regular if for all $A \in A$

$$
\alpha(A) = \sup \{\alpha(B) \mid B \in A, \ B \subset A\}.
$$

**Remark 2.3.** Since $f \geq 0$, then $\mathcal{F}(u, \cdot)$ is an increasing set function for every $u \in L^1(\Omega; \mathbb{R}^N)$. Moreover, by definition of relaxation one directly obtains that $\mathcal{F}(u, \cdot)$ is superadditive. Finally, we denote by $\mathcal{F}_{-}(u, \cdot)$ the inner regular envelope of $\mathcal{F}(u, \cdot)$, given by

$$
\mathcal{F}_{-}(u, C) := \sup \{\mathcal{F}(u, B) \mid B \in A, \ B \subset C\}
$$

for every $C \in A$, so that $\mathcal{F}(u, \cdot)$ is inner regular if $\mathcal{F}(u, \cdot) \equiv \mathcal{F}_{-}(u, \cdot)$ on $A$. We will apply the following criterion due to De Giorgi-Letta [18], compare also [9, 10.2].

**Theorem 2.4.** (Measure property criterion) Let $\alpha : A \to [0, +\infty]$ be an increasing set function. Then the following statements are equivalent:

(i) $\alpha$ is the trace on $A$ of a Borel measure on $\Omega$;
(ii) $\alpha$ is subadditive, superadditive and inner regular;
(iii) the set function $\tilde{\alpha}(E) := \inf \{\alpha(A) \mid A \in A, \ E \subset A\}$ defines a Borel measure on $\Omega$.

We recall a celebrated lower semicontinuity result first obtained by De Giorgi [17], and due to Ioffe [32] in the following general form:

**Theorem 2.5.** ($L^1$-semicontinuity) Let $A$ be a bounded open set of $\mathbb{R}^n$ and let $g : A \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to [0, +\infty)$ be a Carathéodory function such that $g(x, u, \cdot)$ is convex for every $u \in \mathbb{R}^N$ and for a.e. $x \in A$. Then the functional

$$
G(u) := \int_A g(x, u(x), Du(x)) \, dx
$$

is lower semicontinuous on $W^{1,1}(A; \mathbb{R}^N)$ with respect to the weak convergence in $W^{1,1}(A; \mathbb{R}^N)$. 
We end this section by stating an elementary lemma which is a version of De Giorgi’s slicing argument.

**Lemma 2.6. (Slicing lemma revisited)** Let \( \{ f_k \} \) be sequence of non negative functions in \( L^1(B_1) \) with

\[
\sup_k \int_{B_1} f_k \, dx \leq M < +\infty.
\]

Then, fixed \( 0 < s < t < 1 \), for every \( \epsilon > 0 \) there exist \( N = N(\epsilon, M) \), an integer \( 1 \leq h \leq N \) and a (not relabelled) subsequence \( \{ f_k \} \) such that:

\[
\sup_k \int_{A_h} f_k \, dx \leq \epsilon,
\]

where, for \( i \in \{ 0, 1, 2, \ldots, N - 1 \} \),

\[
A_i := B_{s_{i+1}} \setminus B_{s_i} \quad \text{and} \quad s_i := s + \frac{t - s}{N} i.
\]

**Proof.** Choose \( N \) in such a way that \( N\epsilon > M \). It follows that for each \( k \in \mathbb{N} \) there exists \( i \equiv i(k) \) such that:

\[
\int_{A_{i(k)}} f_k \, dx \leq \frac{M}{N},
\]

the assertion follows via a standard compactness argument. \( \square \)

**3. Measure property of the relaxed functional.** In this section we consider non-negative variational functionals \( F : L^1(\Omega; \mathbb{R}^N) \times \mathcal{A} \rightarrow [0, +\infty] \) of the form (2.1) for any open set \( A \in \mathcal{A} \), where \( f : \Omega \times \mathbb{R}^{N \times n} \rightarrow [0, +\infty) \) is a Borel measurable function satisfying a nonstandard growth condition of the form

\[
\alpha \psi(x, |z|) \leq f(x, z) \leq b(x) + \beta \psi(x, |z|)
\]

for a.e. \( x \in \Omega \) and all \( z \in \mathbb{R}^{N \times n} \). Also, for every \( A \in \mathcal{A} \), we denote by \( \overline{F}(\cdot, A) \) the relaxed functional of \( F(\cdot, A) \) with respect to the strong \( L^1(\Omega; \mathbb{R}^N) \) convergence, given for all \( u \in L^1(\Omega; \mathbb{R}^N) \) by (2.2).

Here \( 0 < \alpha \leq \beta < +\infty \), \( b(x) \) is a non-negative function in \( L^1(\Omega) \) and \( \psi : \Omega \times [0, +\infty) \rightarrow [0, +\infty) \) is a suitable Borel function satisfying the following properties:

(i) \( t \mapsto \psi(x, t) \) is non decreasing and convex for a.e. \( x \in \Omega \), with \( \psi(x, 0) = 0 \);

(ii) for every open set \( A \in \mathcal{A}_0 \) there exist \( 1 < c = c(A) < +\infty \) and \( 1 < p = p(A) \leq q = q(A) < +\infty \) such that for a.e. \( x \in A \) we have

\[
\begin{align*}
\psi(x, t^p) &\leq c(t^q + 1) \quad \forall t \geq 0, \\
\psi(x, \lambda t) &\leq c \max\{\lambda^q, \lambda^p\} \psi(x, t) \quad \forall t \geq 0, \quad \lambda \geq 0, \\
\psi(x, t_1 + t_2) &\leq c 2^{q-1} (\psi(x, t_1) + \psi(x, t_2)) \quad \forall t_1, t_2 \geq 0.
\end{align*}
\]

**Remark 3.1.** Note that the third property in (3.2) follows from the second one and from the convexity of \( \psi(x, \cdot) \). Moreover, by monotonicity and convexity of \( \psi(x, \cdot) \) it follows that \( z \mapsto \psi(x, |z|) \) is convex for a.e. \( x \in \Omega \). Therefore our analysis of functionals with a gap in the sense of (1.1) is confined to those special functionals with an energy density satisfying (3.1); these also satisfy (1.1) in view of (i), for a suitable choice of \((p, q)\). Observe that the second property in (3.2) is a sort of \( \Delta_2 \) condition for the function \( t \mapsto \psi(x, t) \), uniform with respect to \( x \).
We introduce the following classes of measurable functions in $L^1(A;\mathbb{R}^N)$ and $W^{1,1}(A;\mathbb{R}^N)$, for every $A \in \mathcal{A}$:

\begin{align*}
L^\psi(A;\mathbb{R}^N) &:= \{ u \in L^1(A;\mathbb{R}^N) : \psi(x,|u(x)|) \in L^1(A) \} \\
W^\psi(A;\mathbb{R}^N) &:= \{ u \in W^{1,1}(A;\mathbb{R}^N) : u \in L^\psi(A;\mathbb{R}^N), \ Du \in L^\psi(A;\mathbb{R}^{N \times n}) \} \\
W^\psi_{\text{loc}}(A;\mathbb{R}^N) &:= \{ u \in L^1(A;\mathbb{R}^N) : u|_B \in W^\psi(B;\mathbb{R}^N) \ \forall B \in \mathcal{A}, B \subset \subset A \}.
\end{align*}

Note that by definition of $\psi$, these are all convex sets; by (3.2) one infers that $W^\psi_{\text{loc}}(A;\mathbb{R}^N)$ is a vector space. We remark that if $A \in \mathcal{A}_0$ these spaces, when equipped with a suitable norm via a suitable Jague function and under certain assumptions, become Banach spaces known as Orlicz-Musielak spaces; these are currently the object of intensive investigation (see for instance, [43], [22], [19], [20], [31]).

**Definition 3.2.** We say that $W^\psi(\Omega;\mathbb{R}^N)$ satisfies a Sobolev type property if for any function $u \in L^1(\Omega;\mathbb{R}^N)$ and every open set $A \in \mathcal{A}_0$ with Lipschitz boundary such that $\int_A \psi(x,|Du|) \, dx < +\infty$ we have

$$\int_B \psi(x,|u(x)|) \, dx \leq C \left( \int_B \psi(x,|Du(x)|) \, dx + \int_B |u(x)| \, dx \right)^\beta \quad \forall B \in \mathcal{A}_0, \ B \subset \subset A$$

where $C, \beta \in [1, +\infty)$ are constants, possibly depending on $A$. Moreover, we say that $W^\psi(\Omega;\mathbb{R}^N)$ satisfies a Rellich’s type property if, for every function $u \in L^1_{\text{loc}}(\Omega;\mathbb{R}^N)$, every open set $A \in \mathcal{A}_0$ with Lipschitz boundary and every $\{u_j\} \subset W^{1,1}(A;\mathbb{R}^N)$ with $u_j \rightharpoonup u$ strongly in $L^1(A;\mathbb{R}^N)$ and $\sup_j \int_A \psi(x,|Du_j|) \, dx < +\infty$, we have

$$\lim_{j \to +\infty} \int_A \psi(x,|u_j - u|) \, dx = 0.$$

In particular, if $W^\psi(\Omega;\mathbb{R}^N)$ satisfies a Sobolev type property we easily obtain for every $A \in \mathcal{A}$

\begin{equation}
W^\psi_{\text{loc}}(A;\mathbb{R}^N) = \{ u \in L^1(A;\mathbb{R}^N) : Du \in L^\psi(B;\mathbb{R}^{N \times n}) \ \forall B \in \mathcal{A}, B \subset \subset A \}.
\end{equation}

In this section we prove the following

**Theorem 3.3. (Measure Property)** Let $F : L^1(\Omega;\mathbb{R}^N) \times \mathcal{A} \to [0, +\infty]$ be as in (2.1), with $f$ being as in (3.1), and $\psi : \Omega \times [0, +\infty) \to [0, +\infty)$ satisfying (i) and (ii) above. Suppose that $W^\psi(\Omega;\mathbb{R}^N)$ satisfies a Sobolev and a Rellich’s type property. Then, for every function $u \in L^1(\Omega;\mathbb{R}^N)$, the functional $\mathcal{F}(u, \cdot)$ is the trace on $\mathcal{A}$ of a Borel measure on $\Omega$.

**Example 3.4.** Of course, $\psi(x,|z|) := |z|^p$, $p > 1$, verifies Theorem 3.3. We outline here two important classes of convex functions satisfying the hypotheses of Theorem 3.3. The first one is the case of dependence on $x$ on the growth exponent, i.e.,

\begin{equation}
\psi(x,|z|) := |z|^{p(x)},
\end{equation}

where $p : \Omega \to (1, +\infty)$ is any fixed continuous function with $p(x) > 1$ for every $x \in \Omega$. It is easy to show that $|z|^{p(x)}$ satisfies (3.2), since for every $A \in \mathcal{A}_0$ we have $1 < p(A) \equiv \inf_A p(x) \leq \sup_A p(x) \equiv q(A) < +\infty$. Moreover in [13] it is shown that $|z|^{p(x)}$ satisfies both a Sobolev and a Rellich’s type property. The second example is

\begin{equation}
\psi(x,|z|) := |z|^p + a(x) |z|^q,
\end{equation}
where $1 < p \leq q < +\infty$ and $a(x) \in L^\infty(\Omega)$, with $a(x) \geq 0$. Of course, the Sobolev and Rellich's type property hold if $q \leq p^*$, where $p^*$ is the Sobolev conjugate of $p$, i.e., $p^* = np/(n-p)$ if $p < n$, $p^* = +\infty$ if $p \geq n$.

Before proving Theorem 3.3, we give some preliminary results. The following lemma is a straightforward consequence of the previous definitions and Theorem 2.5.

**Lemma 3.5.** Under the hypotheses of Theorem 3.3, let $A \in \mathcal{A}_0$ and $u$ be a function in $L^1(\Omega; \mathbb{R}^N)$ such that $\overline{F}(u, A) < +\infty$. Then $u \in W_0^\psi(\Omega; \mathbb{R}^N)$ and

$$
\int_A \psi(x, |Du|) \, dx \leq \frac{1}{\alpha} \overline{F}(u, A) < +\infty.
$$

Let us now recall that if $A', A$ are open sets in $\mathcal{A}$, with $A' \subset \subset A$, a cut-off function between $A'$ and $A$ is a smooth function $\phi \in C_0^\infty(\Omega)$ with $\text{spt} \phi \subset A$, $0 \leq \phi \leq 1$ and $\phi \equiv 1$ on $A'$.

Due to growth condition (3.1), we now obtain the following fundamental $L^\psi$ estimate. The proof is a readaptation of [9, 12.2], keeping into account the new growth conditions dictated by (3.2). We omit it for the sake of brevity.

**Lemma 3.6. (Fundamental Estimate)** Under the hypotheses of Theorem 3.3, for all open sets $A', A' \in \mathcal{A}$, with $A' \subset \subset A$, and for every $\sigma > 0$, there exists a constant $M_\sigma > 0$ such that for every $u, v \in L^1(\Omega; \mathbb{R}^N)$ there exists a cut-off function $\phi$ between $A'$ and $A$ such that

$$
F(\phi u + (1-\phi)v, A' \cup B) \leq (1 + \sigma)(F(u, A) + F(v, B)) + M_\sigma \int_{A' \cap B} \psi(x, |u - v|) \, dx + \sigma.
$$

By using Rellich's type property and the fundamental estimate above, and following arguments from [42], it is possible to prove a weak subadditivity property for the set function $\overline{F}(w, \cdot)$.

**Lemma 3.7. (Weak Subadditivity)** Under the hypotheses of Theorem 3.3, for every $w \in L^1(\Omega; \mathbb{R}^N)$ we have

$$
\overline{F}(w, A' \cup B) \leq \overline{F}(w, A) + \overline{F}(w, B)
$$

for every $A', A \in \mathcal{A}$, with $A' \subset \subset A$, and every $B \in \mathcal{A}$ such that $B$ has Lipschitz boundary.

We are now going to give the

**Proof of Theorem 3.3. Step 1: the case $f(x, Du) := \psi(x, |Du|)$.**

Define now $\Psi : L^1(\Omega; \mathbb{R}^N) \times \mathcal{A} \to [0, +\infty]$ by

$$
\Psi(u, A) := \begin{cases} 
\int_A \psi(x, |Du(x)|) \, dx & \text{if } u \in C^1(A; \mathbb{R}^N) \\
+\infty & \text{elsewhere on } L^1(\Omega; \mathbb{R}^N)
\end{cases}
$$

and let $\overline{\Psi}(\cdot, A)$ be the $L^1(\Omega)$-lower semicontinuous envelope of $\Psi(\cdot, A)$ for every $A \in \mathcal{A}$. Finally, let $\overline{\Psi}_-(u, \cdot)$ be the inner regular envelope of $\overline{\Psi}(u, \cdot)$, see (2.3), i.e., for every $u \in L^1(\Omega; \mathbb{R}^N)$

$$
\overline{\Psi}_-(u, C) := \sup\{\overline{\Psi}(u, B) \mid B \in \mathcal{A}, B \subset \subset C\}, \quad C \in \mathcal{A}.
$$

Making use of a convexity argument we are able to prove inner regularity. We omit the details of the proof of (3.17) and (3.18) and refer to [42, Prop. 3.1] for a similar computation (see also Remark 2.3).
Proposition 3.8. (Inner Regularity) Let $f(x, z) := \psi(x, |z|)$ and $\Psi : L^1(\Omega; \mathbb{R}^N) \times A \rightarrow [0, +\infty]$ be given by (3.8). Then for every $u \in L^1(\Omega; \mathbb{R}^N)$ the increasing set function $\overline{\Psi}(u, \cdot)$ is inner regular, i.e., for every $C \in A$
\begin{equation}
(3.10) \quad \overline{\Psi}(u, C) = \overline{\Psi}(u, C),
\end{equation}
where $\overline{\Psi}(u, C)$ is given by (3.9).

Proof. By the monotonicity of $\overline{\Psi}(u, \cdot)$, it suffices to show that “$\leq$” holds in (3.10), in case $\overline{\Psi}(u, C) < +\infty$. To this aim, for every $\epsilon > 0$ and $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, let $A^j \in A_0$ be such that $A^j \subset A^{j+1} \subset C$, $A^j$ has Lipschitz boundary so that $|\partial A^j| = 0$, $C = \cup_j A_j$ and
\begin{equation}
(3.11) \quad \overline{\Psi}(u, C) - \epsilon 2^{-j} \leq \overline{\Psi}(u, A^j) \leq \overline{\Psi}(u, C) \quad \forall j \in \mathbb{N}_0.
\end{equation}
For every $j \in \mathbb{N}_0$ let $\{u^j_h\} \subset L^1(\Omega)$, obviously depending also on $\epsilon$, be such that
\begin{equation}
(3.12) \quad \lim_{h \rightarrow +\infty} \|u^j_h - u\|_{L^1(\Omega)} = 0 \quad \text{and} \quad \overline{\Psi}(u, A^j) = \liminf_{h \rightarrow +\infty} \overline{\Psi}(u^j_h, A^j) < +\infty.
\end{equation}
Possibly passing to a subsequence, we can suppose that $u^j_h \rightarrow u$ a.e. on $\Omega$,
\begin{equation}
(3.13) \quad \lim_{h \rightarrow +\infty} \int_{A^j} \psi(x, |Du^j_h|) \, dx < +\infty,
\end{equation}
\{u^j_{h|A^j}\} \subset C^1(A^j)$ and that the lower limit in (3.12) is a limit. Then, by the Rellich’s type property (Definition 3.2) and (3.12)
\begin{equation}
(3.14) \quad 0 \leq \phi_j(x) \leq 1, \quad \sum_{j=0}^{+\infty} \phi_j(x) = 1 \quad \forall x \in C.
\end{equation}
For every $j \in \mathbb{N}$, let $h(j) \in \mathbb{N}$ to be chosen later, set $v_j := u^j_{h(j)}$ and
\begin{equation}
(3.15) \quad u^\epsilon(x) := \sum_{j=1}^{+\infty} \phi_{j-1}(x) v_j(x), \quad x \in C.
\end{equation}
Note that, since $v_j_{|A^j} \subset C^1(A^j)$, we have that $\phi_{j-1}(x) v_j(x) \in C^1_0(C)$ for every $j \in \mathbb{N}$. Moreover, since every $x \in C$ in a neighborhood contained at most in the union of three sets of the type $A^{j+1} \setminus \overline{A}^{j-1}$, for every $x \in C$ the infinite sum in the right-hand side of (3.15) reduces to a finite one, hence $u^\epsilon \in C^1(C)$ for every $\epsilon > 0$. Taking $w_{\epsilon} := u$ in $\Omega \setminus C$, for every $\epsilon \in [0, 1]$ the function $t w_{\epsilon}$ belongs to $L^1(\Omega)$ and by (3.14)
\begin{equation}
(3.16) \quad \|t w_{\epsilon} - u\|_{L^1(\Omega)} \leq t \sum_{j=1}^{+\infty} \int_{A^j} |u^j_{h(j)} - u| \, dx + (1 - t) \|u\|_{L^1(\Omega)}.
\end{equation}
Now, it is possible to choose the sequence \( \{h(j)\} \) so that (3.16)
\[
(3.17) \quad tw_{\epsilon} \to u \quad \text{in } L^1(\Omega) \quad \text{as } \epsilon \to 0^+ \quad \text{and} \quad t \to 1^-.
\]
Moreover, taking account of the convexity of \( z \to \psi(x,|z|) \), since \( 0 \leq \phi_{j-1} \leq 1 \) and the sum in (3.15) is locally finite, arguing as in [42, Prop. 3.1], by (3.11), (3.12) and (3.13) we can choose \( \{h(j)\} \) so that for any \( t \in [0,1] \) we also have
\[
(3.18) \quad \int_C \psi(x,|t Dw|) \, dx \leq \psi_-(u, C) + 5\epsilon + (1 - t) \epsilon < +\infty.
\]
In particular, since \( tw_{\epsilon} \in C^1(C) \), by (3.8) we have \( \psi(t w_{\epsilon}, C) = \int_C \psi(x, |t Dw_{\epsilon}|) \, dx \) and hence (3.19) follows by letting again \( \epsilon \to 0^+ \) and \( t \to 1^- \).

Finally, as \( \epsilon \to 0^+ \) and \( t \to 1^- \), by (3.18) and (3.17) we obtain that \( \psi(u, C) \leq \psi_-(u, C) \) and hence the assertion.

Now, since the increasing set function \( \psi(u, \cdot) \) is inner regular, and \( \psi(u, \cdot) \) is superadditive, thanks to Theorem 2.4 we obtain measure property of \( \psi(u, \cdot) \), for every \( u \in L^1(\Omega; \mathbb{R}^N) \), if we show that \( \psi(u, \cdot) \) is subadditive.

**Proposition 3.9.** (Subadditivity) For every \( w \in L^1(\Omega; \mathbb{R}^N) \) we have
\[
(3.19) \quad \psi(w, A \cup B) \leq \psi(w, A) + \psi(w, B) \quad \forall A, B \in \mathcal{A}.
\]

**Proof.** By inner regularity (Proposition 3.8), it is well known that weak subadditivity (Lemma 3.7 with \( F = \psi \)) yields (3.19) for any \( A, B \in \mathcal{A} \), provided \( B \) has Lipschitz boundary. In fact, for any \( C \in \mathcal{A} \) with \( C \subset A \cup B \), by enlarging a bit the subset \( C \setminus \overline{B} \), we can find \( A' \subset A \) such that \( C \subset A' \cup B \), which yields, by (3.7), \( \psi(w, C) \leq \psi(w, A' \cup B) \leq \psi(w, A) + \psi(w, B) \), and hence (3.19), by inner regularity, letting \( C \nearrow A \cup B \). Finally, to prove (3.19) for any \( B \in \mathcal{A} \), in case \( \psi(w, A \cup B) < +\infty \), for each small \( \epsilon > 0 \) take \( C \subset A \cup B \) such that by inner regularity \( \psi(w, C) \geq \psi(w, A \cup B) - \epsilon \). We can find an open set \( \tilde{B} \in \mathcal{A} \) with \( C \setminus \overline{A} \subset \tilde{B} \subset B \) and such that \( \tilde{B} \) has Lipschitz boundary. Then since \( C \subset A \cup \tilde{B} \)
\[
\psi(w, A \cup B) \leq \psi(w, C) + \epsilon \\
\leq \psi(w, A \cup \tilde{B}) + \epsilon \\
\leq \psi(w, A) + \psi(w, \tilde{B}) + \epsilon \\
\leq \psi(w, A) + \psi(w, B) + \epsilon
\]
and hence we obtain (3.19), letting \( \epsilon \to 0^+ \). In case \( \psi(w, A \cup B) = +\infty \), take \( C \subset A \cup B \) with \( \psi(w, C) > 1/\epsilon \), so that arguing as before
\[
\epsilon^{-1} \leq \psi(w, C) \leq \psi(w, A \cup \tilde{B}) \leq \psi(w, A) + \psi(w, \tilde{B}) \leq \psi(w, A) + \psi(w, B)
\]
and hence (3.19) follows by letting again \( \epsilon \to 0^+ \).

**Step 2: measure property of \( F(u, \cdot) \).**
Consider now any Borel function \( f \) as in Theorem 3.3. We first prove the following

**Proposition 3.10.** (Inner Regularity) For every \( w \in L^1(\Omega; \mathbb{R}^N) \), \( F(w, \cdot) \) is an inner regular set function.

**Proof.** Since \( F(w, \cdot) \) is an increasing set function, if \( F_-(w, \cdot) \) is defined by (2.3), it suffices to prove that
\[
(3.20) \quad F(w, C) \leq F_-(w, C)
\]
for every fixed open set \( C \in \mathcal{A} \) and every function \( w \in L^1(\Omega) \) such that \( \mathcal{F}_-(w, C) < +\infty \). To this aim note that growth condition (3.1) yields the estimate

\[
\alpha \overline{\Psi}(w, A) \leq \overline{\mathcal{F}}(w, A) \leq \int_A b(x) \, dx + \beta \overline{\mathcal{F}}(w, A)
\]

(3.21)

for every \( w \in L^1(\Omega) \) and \( A \in \mathcal{A} \), where \( \Psi \) is given by (3.8), and the same estimate with \( \overline{\Psi}_- \) and \( \overline{\mathcal{F}}_- \), respectively, instead of \( \overline{\Psi} \) and \( \overline{\mathcal{F}} \) in (3.21). In particular, by the monotonicity and the inner regularity of \( \overline{\Psi}(w, \cdot) \), see Proposition 3.8,

\[
\overline{\Psi}(w, A) \leq \overline{\Psi}(w, C) = \overline{\mathcal{F}}_-(w, C) \leq \frac{1}{\alpha} \overline{\mathcal{F}}_-(w, C) < +\infty
\]

(3.22)

for every \( A \in \mathcal{A} \) with \( A \subset C \). For every \( \epsilon > 0 \) we can choose an open set \( A_\epsilon \in \mathcal{A} \) with Lipschitz boundary and such that \( A_\epsilon \subset C \) so that, by inner regularity of \( \overline{\Psi}(w, \cdot) \) and absolute continuity of \( b(x) \in L^1(\Omega) \),

\[
\overline{\Psi}(w, C) \leq \overline{\Psi}(w, A_\epsilon) + \epsilon \quad \text{and} \quad 0 \leq \int_{C \setminus A_\epsilon} b(x) \, dx \leq \epsilon.
\]

(3.23)

Also, let \( B_\epsilon := C \setminus \overline{A_\epsilon} \in \mathcal{A} \), so that if \( \overline{\Psi}(w, \cdot) \) is the Borel measure given by the extension of \( \overline{\Psi}(w, \cdot) \) to \( \Omega \) (see (iii) in Theorem 2.4), by (3.23) we have

\[
\overline{\Psi}(w, B_\epsilon) = \overline{\Psi}(w, C) - \overline{\Psi}(w, A_\epsilon) \leq \overline{\Psi}(w, C) - \overline{\Psi}(w, A_\epsilon) \leq \epsilon.
\]

(3.24)

Moreover there exists a sequence \( \{v_j\} \subset L^1(\Omega) \), converging to \( w \) in \( L^1(\Omega) \), such that \( v_jB_\epsilon \in C^1(B_\epsilon) \) for every \( j \) and

\[
\overline{\Psi}(w, B_\epsilon) = \lim_{j \to +\infty} \int_{B_\epsilon} \psi(x, |Du_j|) \, dx < +\infty.
\]

(3.25)

In particular, by (3.1), (3.23) and (3.25)

\[
\liminf_{j \to +\infty} F(v_j, B_\epsilon) \leq \int_{C \setminus A_\epsilon} b(x) \, dx + \beta \lim_{j \to +\infty} \Psi(v_j, B_\epsilon) \leq \epsilon + \beta \overline{\mathcal{F}}(w, B_\epsilon).
\]

(3.26)

Choose now \( A', A \in A_0 \), such that \( A \) has Lipschitz boundary and \( A_\epsilon \subset A' \subset \subset A \subset \subset C \). Since \( \mathcal{F}(w, A) < +\infty \), there exists a sequence \( \{u_j\} \subset L^1(\Omega) \), converging to \( w \) in \( L^1(\Omega) \), such that \( u_jA \in C^1(A) \) for every \( j \) and

\[
\mathcal{F}(w, A) = \lim_{j \to +\infty} \int_A f(x, Du_j) \, dx < +\infty.
\]

(3.27)

By the fundamental estimate (Lemma 3.6) applied with \( u_j \) on \( A \) and \( v_j \) on \( B_\epsilon \), for any \( \sigma > 0 \) we can find \( M_\sigma > 0 \) and a sequence \( \{\phi_j\} \) of smooth cut-off functions between \( A' \) and \( A \) such that

\[
F(w_j, A' \cup B_\epsilon) \leq (1 + \sigma)(F(u_j, A) + F(v_j, B_\epsilon)) + M_\sigma \int_{A \cap B_\epsilon} \psi(x, |u_j - v_j|) \, dx + \sigma,
\]

(3.28)

where \( w_j := \phi_j u_j + (1 - \phi_j)v_j \). By (3.25), (3.27) and (3.1) we have

\[
\sup_j \int_{A \cap B_\epsilon} (\psi(x, |Du_j|) + \psi(x, |Dv_j|)) \, dx < +\infty.
\]
Moreover, \( A \cap B_x = A \setminus \overline{A}_x \in \mathcal{A}_0 \), whereas \( A_x \subseteq A \) and hence \( A \cap B_x \) has Lipschitz boundary given by the disjoint union \( \partial A \cup \partial A_x \). Then, by the Rellich’s type property (Definition 3.2) and (3.2) we conclude that

\[
\lim_{j \to +\infty} \int_{A \cap B_x} \psi(x, |u_j - v_j|) \, dx = 0 .
\]

Then, since \( w_j = \phi_j u_j + (1 - \phi_j) v_j \to w \) in \( L^1(\Omega) \), by (3.28), (3.27) and (3.26) we obtain

\[
\mathcal{F}(w, A' \cup B_x) \leq \liminf_{j \to +\infty} F(w_j, A' \cup B_x) \leq (1 + \sigma)(\mathcal{F}(w, A) + \epsilon + \beta \mathcal{W}(w, B_x)) + \sigma .
\]

Finally, since \( B_x = C \setminus \overline{A}_x \) yields \( A' \cup B_x = C \), taking \( \epsilon > 0 \) small so that \( \epsilon(1+\beta) \leq \sigma \), by (3.24) and (3.29)

\[
\mathcal{F}(w, C) \leq (1 + \sigma)(\mathcal{F}(w, A) + \sigma) + \sigma \leq (1 + \sigma)(\mathcal{F}_-(w, C) + \sigma) + \sigma
\]

and hence (3.20) holds by the arbitrariness of \( \sigma > 0 \). \( \square \)

Since we have just proved that \( \mathcal{F}(w, \cdot) \) is inner regular for every \( w \in L^1(\Omega; \mathbb{R}^N) \), arguing as in Proposition 3.9, by weak subadditivity (3.7) we obtain that \( \mathcal{F}(w, \cdot) \) is subadditive. Since \( \mathcal{F}(w, \cdot) \) is trivially superadditive, by Theorem 2.4 the proof of Theorem 3.3 is complete. \( \square \)

**4. Integral representation of the relaxed functional.** In this section we show that, under suitable hypotheses on the function \( \psi(x, t) \) defined in the previous section, the relaxed functional \( \mathcal{F}(u, A) \) obtained in Theorem 3.3 is of variational type.

**Definition 4.1.** We say that a sequence \( \{ u_j \} \subseteq W^\psi_{\text{loc}}(\Omega; \mathbb{R}^N) \) converges to \( u \in W^\psi_{\text{loc}}(\Omega; \mathbb{R}^N) \) strongly in \( W^\psi_{\text{loc}}(\Omega; \mathbb{R}^N) \) if for every \( A \in \mathcal{A}_0 \)

\[
\lim_{j \to +\infty} \int_A (\psi(x, |u_j - u|) + \psi(x, |Du_j - Du|)) \, dx = 0 .
\]

**Remark 4.2.** If \( u_j \rightharpoonup u \) in \( W^\psi_{\text{loc}}(\Omega; \mathbb{R}^N) \), then by (3.2) \( u_j \to u \) in \( L^1_{\text{loc}}(\Omega; \mathbb{R}^N) \). Moreover, \( \psi(x, |Du_j|) \to \psi(x, |Du|) \) in \( L^1_{\text{loc}}(\Omega) \) too. In fact, by the monotonicity of \( \psi(x, \cdot) \), for every \( A \in \mathcal{A}_0 \) we estimate

\[
\psi(x, |Du_j|) \leq c 2^{n-1} (\psi(x, |Du_j - Du|) + \psi(x, |Du|))
\]

for a.e. \( x \in A \), where \( c = c(A) \) and \( q = q(A) \) are given by (3.2), hence it suffices to apply the dominated convergence theorem.

**Definition 4.3.** If \( f \in L^1_{\text{loc}}(\Omega) \), define the maximal function \( M(f) \) by

\[
(Mf)(x) := \sup_{r > 0} (M_r f)(x) , \quad \text{where} \quad (M_r f)(x) := \frac{1}{|B_r(x)|} \int_{B_r(x) \cap \Omega} |f(y)| \, dy .
\]

**Definition 4.4.** We say that the function \( \psi \) enjoys the maximal property if for every bounded open set \( A \in \mathcal{A}_0 \) and every function \( f \in L^1(A) \) with \( \psi(x, |f(x)|) \in L^1(A) \) we have

\[
\int_A \psi(x, |(Mf)(x)|) \, dx \leq C \left( \int_A \psi(x, |f(x)|) \, dx + 1 \right)^{\beta}
\]
where \( C, \beta \in (1, +\infty) \) are positive constants possibly depending on \( n, A \) and \( \psi \).

**Definition 4.5.** We say that the function \( \psi(x, |z|) \) satisfies the density property if for every \( u \in W^\psi_{\text{loc}}(\Omega; \mathbb{R}^N) \) there exists a sequence of smooth functions \( \{u_j\} \subset C^\infty_0(\Omega; \mathbb{R}^N) \) such that \( u_j \to u \) in \( W^\psi_{\text{loc}}(\Omega; \mathbb{R}^N) \). If in addition \( u \in W^\psi(\Omega; \mathbb{R}^N) \), then we also require that \( u_j \to u \) in \( L^1(\Omega; \mathbb{R}^N) \).

Let us observe the following relation between the definitions given above.

**Proposition 4.6.** The maximal property implies the density property.

**Proof.** It is a consequence of Lebesgue dominated convergence theorem; we shall keep the notation introduced for Definition 4.3. Let \( \{\varphi_\epsilon\}_{\epsilon \in (0, 1)} \), be a family of standard mollifiers and let \( w \in W^\psi_{\text{loc}}(\Omega; \mathbb{R}^N) \) with \( w_\epsilon(x) := w * \varphi_\epsilon(x) \) for every \( x \in \Omega \) such that \( \text{dist}(x, \partial \Omega) \geq 2\epsilon \). Observe that by the very definition of convolution and maximal function it follows that \( |w_\epsilon(x)| \leq (Mw)(x) \) for every \( x \in A \) such that \( \text{dist}(x, \partial A) \geq 2\epsilon \).

Now take an increasing sequence of open subsets \( A_j \subset \Omega \) such that \( A_j \subset A_{j+1} \subset \Omega \)

and define a related sequence of cut-off functions \( \eta_j \in C^\infty_0(A_{j+1}) \) such that \( \eta_j \equiv 1 \) on \( A_j \) and finally we define \( u_j := \eta_j w_1/j \); clearly \( w_1/j \in C^\infty_0(\Omega) \). Now let \( A \subset \Omega \) be an open subset; there exists \( j_0 \in \mathbb{N} \) such that \( A \subset A_j \) whenever \( j \geq j_0 \). For such values of \( j \) we observe that by the fact that \( \psi \) is non-decreasing with respect to the last variable we find

\[
\int_A \psi(x, |w_1/j(x)|) \, dx \leq \int_A \psi(x, |(Mw)(x)|) \, dx \leq C \int_A (\psi(x, |w(x)|) + 1) \, dx
\]

\[
\int_A \psi(x, |Dw_1/j(x)|) \, dx \leq \int_A \psi(x, |(MDw)(x)|) \, dx \leq C \int_A (\psi(x, |Dw(x)|) + 1) \, dx
\]

where \( C \) depends also on \( \int_A \psi(x, |w(x)|) \, dx \) and \( \int_A \psi(x, |Dw(x)|) \, dx \), see Definition 4.4. Therefore since

\[
\psi(x, |w_1/j(x)|) \to \psi(x, |w(x)|) \quad \text{and} \quad \psi(x, |Dw_1/j(x)|) \to \psi(x, |Dw(x)|)
\]
a.e., by Remark 4.2 such a convergence also holds in \( L^1(A) \). Now we conclude using the third property in (3.2) as follows:

\[
\int_A \psi(x, |w_1/j(x) - w(x)|) \, dx \leq c \int_A \psi(x, |w_1/j(x)|) \, dx + c \int_A \psi(x, |w(x)|) \, dx
\]

\[
\int_A \psi(x, |Dw_1/j(x) - Dw(x)|) \, dx \leq c \int_A \psi(x, |Dw_1/j(x)|) \, dx + c \int_A \psi(x, |Dw(x)|) \, dx
\]

and the conclusion follows from a well-known variant of Lebesgue’s dominated convergence theorem. Finally, it easy to see that if \( w \in L^1(\Omega; \mathbb{R}^N) \) then \( w_1/j \to w \) in \( L^1(\Omega; \mathbb{R}^N) \).

Before stating the representation results, we recall that a Borel function \( \varphi : \Omega \times \mathbb{R}^n \times \mathbb{R}^{N \times n} \to \mathbb{R} \) is called quasi-convex in the sense of Morrey [41] if for a.e. \( x_0 \in \Omega \), every \( u_0 \in \mathbb{R}^n \), \( z_0 \in \mathbb{R}^{N \times n} \), every bounded open set \( A \subset \mathbb{R}^n \) and every function \( \phi \in C^1_0(A; \mathbb{R}^N) \) we have

\[
|A| \varphi(x_0, u_0, z_0) \leq \int_A \varphi(x_0, u_0, z_0 + D\phi(x)) \, dx.
\]

Moreover, the quasi-convex envelope \( Qf \) of a function \( f(x, u, z) \) is the greatest function \( \varphi(x, u, z) \) which is quasi-convex being less than or equal to \( f \) (see [14], [15]).
Theorem 4.7. Under the hypotheses of Theorem 3.3, suppose in particular that 
\( \psi(x,t) \) satisfies the density property, see Definition 4.5. Then for every \( A \in \mathcal{A} \) we have

\[
\mathcal{F}(u, A) = \begin{cases} 
\int_A \varphi(x, Du(x)) \, dx & \text{if } u \in W^0_\text{loc}(A; \mathbb{R}^N) \\
+\infty & \text{elsewhere on } L^1(\Omega; \mathbb{R}^N) 
\end{cases}
\]

where \( \varphi : \Omega \times \mathbb{R}^{N \times n} \to [0, +\infty) \) is a quasi-convex function satisfying growth condition

\[
(3.1)
\]

for a.e. \( x \in \Omega \) and all \( z \in \mathbb{R}^N \).

Example 4.8. In case \( \psi(x,|z|) := |z|^p(x) \), let \( p : \Omega \to [1, +\infty) \) be a continuous function satisfying following local estimate about the modulus of continuity: \( \forall A \in \mathcal{A}_0 \)

\[
\exists \gamma > 0 : |p(x) - p(y)| \leq \frac{\gamma A}{|x - y|} \quad \forall x, y \in A, \quad 0 < |x - y| < \frac{1}{2}.
\]

Then, in Proposition 5.2 we show that \( \psi(x,|z|) \) satisfies the maximal property and therefore the density property (this result is actually contained in [19] and extended by us to a more general class of functions). As a consequence, Theorem 4.7 holds.

Similarly, in case \( \psi(x,|z|) := |z|^p + a(x)|z|^q \), suppose in particular that \( a(x) \) is a bounded non-negative Hölder continuous function in \( C^{0,\alpha}(\Omega) \), for some \( 0 < \alpha \leq 1 \), and

\[
1 < p \leq q \leq \frac{n + \alpha}{n - p}.
\]

Then from Propositions 5.1 it follows that the function \( |z|^p + a(x)|z|^q \) satisfies the maximal property and Theorem 4.7 holds also in this case.

In order to prove Theorem 4.7, we make use of the following readaptation of the classical integral representation theorem [11, Thm. 1.1] in the setting of \( W^0 \)-spaces.

Proposition 4.9. Suppose \( \psi(x,t) \) is as in Theorem 4.7. Let \( \mathcal{F} : L^1(\Omega; \mathbb{R}^N) \times \mathcal{A} \to [0, +\infty] \) satisfy the following conditions:

(i) (locality) \( \mathcal{F} \) is local, i.e., \( \mathcal{F}(u, A) = \mathcal{F}(v, A) \) for every \( A \in \mathcal{A} \) and \( u, v \in L^1(\Omega; \mathbb{R}^N) \) with \( u = v \) a.e. on \( A \);

(ii) (measure property) for all \( u \in L^1(\Omega; \mathbb{R}^N) \) the set function \( \mathcal{F}(u, \cdot) \) is increasing, and is the trace on \( A \) of a Borel measure;

(iii) (growth conditions) there exist \( \beta > 0 \) and \( b(x) \in L^1_\text{loc}(\Omega) \) such that

\[
0 \leq \mathcal{F}(u, A) \leq \int_A (b(x) + \beta \psi(x,|Du(x)||) \, dx
\]

for all \( u \in W^0(\Omega; \mathbb{R}^N) \) and \( A \in \mathcal{A} \);

(iv) (translation invariance in \( u \)) \( \mathcal{F}(u + c, A) = \mathcal{F}(u, A) \) for all \( u \in L^1(\Omega; \mathbb{R}^N) \), \( A \in \mathcal{A}, c \in \mathbb{R}^N \);

(v) (lower semicontinuity) \( \mathcal{F}(\cdot, A) \) is sequentially lower semicontinuous with respect to the strong convergence in \( L^1(\Omega; \mathbb{R}^N) \) for all \( A \in \mathcal{A} \).

Then there exists a Carathéodory function \( \varphi : \Omega \times \mathbb{R}^{N \times n} \to [0, +\infty) \) such that

\[
\mathcal{F}(u, A) = \int_A \varphi(x, Du(x)) \, dx
\]

for every \( A \in \mathcal{A} \) and for every \( u \in L^1(\Omega; \mathbb{R}^N) \) such that \( u|_A \in W^0_\text{loc}(A; \mathbb{R}^N) \); in addition, the function \( \varphi(x,\cdot) \) is quasi-convex in \( \mathbb{R}^{N \times n} \) for a.e. \( x \in \Omega \) and satisfies
the growth condition
\begin{equation}
0 \leq \varphi(x, z) \leq b(x) + \beta \psi(x, |z|)
\end{equation}
for a.e. \( x \in \Omega \) and all \( z \in \mathbb{R}^{N \times n} \).

\textbf{Proof.} We recall that a function \( u \in L^1(\Omega; \mathbb{R}^N) \) is piecewise affine in \( \Omega \) if there exists a countable family \( \{ \Omega_i \}_{i \in I} \) of disjoint open subsets of \( \Omega \) and a Borel subset \( N \) of \( \Omega \) with \( |N| = 0 \) such that \( \Omega = (\bigcup_{i \in I} \Omega_i) \cup N \) and \( u|_{\Omega_i} \) is affine on each \( \Omega_i \).

\textbf{Step 1:} following [16, Thm. 20.1] or [9, Thm. 9.1], we find a Carathéodory function \( \varphi \), satisfying (4.7), such that (4.6) holds for all \( A \in \mathcal{A} \) and all piecewise affine on \( u \in W^\psi(\Omega) \).

\textbf{Step 2:} \( \mathcal{F}(u, A) \leq \int_A \varphi(x, Du(x)) \, dx \) for \( u \in W^\psi(\Omega; \mathbb{R}^N) \) and \( A \in \mathcal{A} \).

By Step 1 and in particular by (4.7), we have that for every \( A' \in \mathcal{A}_0 \) the functional
\begin{equation}
\varphi \mapsto \int_{A'} \varphi(x, Du(x)) \, dx
\end{equation}
is continuous with respect to the \( W^\psi_{\text{loc}}(\Omega) \) convergence (Definition 4.1). Moreover, by the density property of \( \psi \), the following density result can be achieved via the approximation argument in [21, Ch. X, Prop. 2.1].

\textbf{Lemma 4.10.} If \( \psi \) satisfies the hypotheses of Theorem 4.7, then for every function \( u \in W^\psi(\Omega; \mathbb{R}^N) \) there exists a sequence \( \{ u_j \} \subset W^\psi(\Omega; \mathbb{R}^N) \) of functions which are piecewise affine on \( \Omega \) and such that \( u_j \rightharpoonup u \) both in \( L^1(\Omega; \mathbb{R}^N) \) and in \( W^\psi_{\text{loc}}(\Omega; \mathbb{R}^N) \).

Now, let \( u \in W^\psi(\Omega) \) and \( A \in \mathcal{A} \). By Lemma 4.10 there exists a sequence \( \{ u_j \} \) of functions in \( W^\psi(\Omega) \) which are piecewise affine on \( \Omega \) and such that \( u_j \rightharpoonup u \) in \( L^1(\Omega; \mathbb{R}^N) \) and in \( W^\psi_{\text{loc}}(\Omega; \mathbb{R}^N) \). Then by lower semicontinuity \( v) \) of \( \mathcal{F} \), Step 1 and the continuity of the functional (4.8) in \( W^\psi_{\text{loc}}(\Omega) \), we obtain for every \( A' \in \mathcal{A}_0, A' \subset A \),
\begin{equation}
\mathcal{F}(u, A') \leq \liminf_{j \to +\infty} \mathcal{F}(u_j, A') = \lim_{j \to +\infty} \int_{A'} \varphi(x, Du_j(x)) \, dx = \int_{A'} \varphi(x, Du(x)) \, dx.
\end{equation}
Since \( \mathcal{F}(u, \cdot) \) is a measure, taking the limit as \( A' \nearrow A \) we get by the monotone convergence theorem
\begin{equation}
\mathcal{F}(u, A) \leq \int_A \varphi(x, Du(x)) \, dx
\end{equation}
for every \( u \in W^\psi(\Omega) \) and \( A \in \mathcal{A} \).

\textbf{Step 3:} \( \mathcal{F}(u, A) = \int_A \varphi(x, Du(x)) \, dx \) for \( u \in W^\psi(\Omega; \mathbb{R}^N) \) and \( A \in \mathcal{A} \).

Fix \( u \in W^\psi(\Omega) \) and let \( A, A' \in \mathcal{A} \) with \( A' \subset A \). We modify the function \( u \) in the following way: take \( A'' \in \mathcal{A}_0 \) such that \( A' \subset A'' \subset \Omega \), let \( \phi \) be a cut-off function between \( A' \) and \( A'' \) and set \( \tilde{u} := \phi u \). Since \( \tilde{u} \) has compact support, by (3.2) we obtain that \( \tilde{u} \in W^\psi(\Omega) \) and that \( \tilde{u} + v \in W^\psi(\Omega) \) for every \( v \in W^\psi(\Omega) \). Consider the functional \( G : L^1(\Omega) \times \mathcal{A} \to [0, +\infty) \) defined by
\begin{equation}
G(v, B) := \mathcal{F}(v + \tilde{u}, B).
\end{equation}
Then \( G \) satisfies all hypotheses of Proposition 4.9. Indeed, (i), (ii), (iv) and (v) are trivially satisfied, whereas for all \( v \in W^\psi(\Omega) \) and all \( B \in \mathcal{A} \) we have
\begin{align}
0 \leq G(v, B) &= \mathcal{F}(v + \tilde{u}, B) \\
&\leq \int_B (b(x) + \beta \psi(x, |D\tilde{u} + Dv|)) \, dx \\
&\leq \int_B (g(x) + \gamma \psi(x, |Dv|)) \, dx
\end{align}
where \( \gamma = 2^{n-1} c \beta \) and \( g(x) = b(x) + 2^{n-1} c \psi(x, |D\bar{u}(x)|) \in L^1_{\text{loc}}(\Omega) \), with \( c = c(A'') \) and \( q = q(A'') \) given by (3.2). Therefore from Steps 1-2 above it follows that there exists a Carathéodory function \( g : \Omega \times \mathbb{R}^{N \times n} \to [0, +\infty) \), satisfying (4.7) with \( \gamma \) and \( g(x) \) instead of \( \beta \) and \( b(x) \), such that

\[
(4.10) \quad G(v, B) \leq \int_B g(x, Dv(x)) \, dx \quad \forall v \in W^\psi(\Omega), \quad \forall B \in \mathcal{A},
\]

with equality for \( v \) piecewise affine in \( \Omega \). In addition, arguing as for (4.8), we can prove that for every \( B' \in \mathcal{A}_0 \) the functional

\[
(4.11) \quad v \mapsto \int_{B'} g(x, Dv(x)) \, dx
\]

is continuous in \( W^\psi_{\text{loc}}(\Omega) \). We now prove that

\[
(4.12) \quad \mathcal{F}(u, A') = \int_{A'} \varphi(x, Du(x)) \, dx;
\]

since \( \mathcal{F}(u, \cdot) \) is a measure, taking \( A' \not\subset A \) we will obtain (4.6) for all \( A \in \mathcal{A} \) and \( u \in W^\psi(\Omega) \). By Lemma 4.10 there exists a sequence \( \{u_j\} \) of functions in \( W^\psi(\Omega) \), piecewise affine in \( \Omega \), such that \( u_j \to \bar{u} \) in \( L^1(\Omega) \) and in \( W^\psi_{\text{loc}}(\Omega) \). Then, using the locality (i) of \( \mathcal{F} \), Step 2, Step 1, (4.10) and the continuity of the functionals (4.8) and (4.11), we obtain

\[
\begin{align*}
\int_{A'} g(x, 0) \, dx & = G(0, A') = \mathcal{F}(\bar{u}, A') = \mathcal{F}(u, A') \leq \int_{A'} \varphi(x, Du) \, dx \\
& = \lim_{j \to +\infty} \int_{A'} \varphi(x, D\bar{u}) \, dx = \lim_{j \to +\infty} \mathcal{F}(u_j, A') \\
& = \lim_{j \to +\infty} G(u_j - \bar{u}, A') \leq \lim_{j \to +\infty} \int_{A'} g(x, D(u_j - \bar{u})) \, dx \\
& = \int_{A'} g(x, 0) \, dx
\end{align*}
\]

and (4.12) is proved.

**Step 4:** \( \mathcal{F}(u, A) = \int_A \varphi(x, Du(x)) \, dx \) for \( u|A \in W^\psi_{\text{loc}}(A; \mathbb{R}^N) \) and \( A \in \mathcal{A} \).

If \( u \in L^1(\Omega), A \in \mathcal{A} \) and \( u|A \in W^\psi_{\text{loc}}(A) \), then for every \( A' \in \mathcal{A}_0, A' \subset \subset A \), we can find a function \( v \in W^\psi(\Omega) \) such that \( v|A' = u|A' \) (it suffices to take \( v = \phi u \), where \( \phi \in C^\infty_0(\Omega) \) is a cut-off function between \( A' \) and \( A'' \), with \( A' \subset \subset A'' \subset \subset A \)). Then, by the locality of \( \mathcal{F} \) and Step 3 we have

\[
\mathcal{F}(u, A') = \mathcal{F}(v, A') = \int_{A'} \varphi(x, Dv(x)) \, dx = \int_{A'} \varphi(x, Du(x)) \, dx
\]

and we obtain the assertion as \( A' \not\subset A \), by the measure property of \( \mathcal{F} \).

**Step 5:** quasi-convexity of \( \varphi \).

It is enough to prove that, for every \( A \in \mathcal{A}_0 \) with Lipschitz boundary, \( \varphi(x, \cdot) \) is quasi-convex on \( \mathbb{R}^{N \times n} \) for a.e. \( x \in A \). If \( A \in \mathcal{A}_0 \) is fixed and \( c = c(A), q = q(A) \), by (3.2) the restriction \( \varphi : A \times \mathbb{R}^{N \times n} \to [0, +\infty) \) is a Carathéodory integrand with

\[
0 \leq \varphi(x, z) \leq b(x) + \beta \psi(x, |z|) \leq b(x) + \beta (c|z|^q + 1)
\]
where \( b(x) \in L^1(A) \). In addition, by lower semicontinuity v) the functional (4.8) is sequentially weakly l.s.c. on \( W^{1,q}(A) \); hence by [10, Thm. 4.1.5] we obtain that \( f(x,\cdot) \) is quasi-convex in \( \mathbb{R}^{N \times n} \) for a.e. \( x \in A \). \( \Box \)

**Proof of Theorem 4.7.** We apply Proposition 4.9 to the relaxed functional \( \mathcal{F}(u, A) \). Indeed the locality property (i) is well known (see e.g. [16, Prop. 16.15]), the measure property (ii) is proved in Theorem 3.3, growth condition (iii) follows from (3.21), whereas (iv) and (v) are trivially satisfied. Therefore, Proposition 4.9 implies that (4.6) holds for all \( u \in W^{1,c}_\text{loc}(A) \) and \( A \in \mathcal{A} \), with \( \varphi \) quasi-convex in \( z \). Finally (3.21) and (3.8) yield the growth estimate (3.1) for \( \varphi \) and the integral representation (4.3) on all of \( L^1(\Omega) \). \( \Box \)

We are now able to state

**Theorem 4.11.** Under the hypotheses of Theorem 4.7, suppose that \( \psi \) enjoys the maximal property. Then, if \( f \) is a Carathéodory integrand, the function \( \varphi \) in (4.3) is equal to the quasi-convex envelope of \( z \mapsto f(x,z) \).

This is an easy consequence of Theorem 4.7 and of the following lower semicontinuity result, that we state in full generality.

**Theorem 4.12.** (Lower Semicontinuity) Under the hypotheses of Theorem 4.7, suppose that \( \psi \) enjoys the maximal property. Let \( \varphi : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to [0, +\infty) \) be a quasi-convex Carathéodory function satisfying

\[
0 \leq \varphi(x,u,z) \leq b(x) + C(\psi(x,|u|) + \psi(x,|z|))
\]

where \( C > 1 \) and \( b(x) \in L^1_{\text{loc}}(\Omega) \) with \( b(x) \geq 0 \). Then for every sequence \( \{u_k\} \subset W^{1,1}(\Omega; \mathbb{R}^N) \) with \( u_k \rightharpoonup u \) in \( L^1(\Omega; \mathbb{R}^N) \) and

\[
\sup_k \int_{\Omega} \psi(x,|Du_k|) \, dx < +\infty
\]

we have that \( u \in W^{1,c}_{\text{loc}}(\Omega; \mathbb{R}^N) \), \( \psi(x,|Du|) \in L^1(\Omega) \) and

\[
\int_{\Omega} \varphi(x,u(x), Du(x)) \, dx \leq \liminf_{k \to +\infty} \int_{\Omega} \varphi(x,u_k(x), Du_k(x)) \, dx.
\]

**Proof.** Our starting point is the classical lower semicontinuity proof of Acerbi & Fusco for quasi-convex integrals with \( p \)-growth. Hence we refer to the proof of [2, Thm II.4], where we will point out the differences. The main ingredients are the density property in \( W^\psi(\Omega; \mathbb{R}^N) \) and the fact that \( \psi \) enjoys the maximal property, see Definitions 4.5 and 4.4. Set now for every Borel set \( A \subset \Omega \)

\[
\mathcal{F}(u, A) := \int_A \varphi(x, u(x), Du(x)) \, dx.
\]

We divide the rest of the proof in four steps.

**Step 1:** \( u \in W^{\psi}_{\text{loc}}(\Omega; \mathbb{R}^N) \) and \( \int_{\Omega} \psi(x,|Du|) \, dx < +\infty \).

Setting \( \Omega_j := \{ x \in \Omega \mid |x| < j \text{ and } \text{dist}(x, \partial \Omega) > 1/j \} \in \mathcal{A}_0 \), if \( p = p(\Omega_j) > 1 \) is given by (3.2), passing to a subsequence we have that \( u_k \rightharpoonup u \) weakly in \( W^{1,p}(\Omega_j) \), and hence weakly in \( W^{1,1}(\Omega_j) \). Then we can apply Theorem 2.5 with \( A = \Omega_j \) and \( g(x, u, z) = \psi(x,|z|) \) to obtain

\[
\int_{\Omega_j} \psi(x,|Du|) \, dx \leq \liminf_{k \to +\infty} \int_{\Omega_j} \psi(x,|Du_k|) \, dx.
\]
for every \( j \). Hence (4.14) gives \( \psi(x, |Du|) \in L^1(\Omega) \) and finally (3.3) yields \( u \in W_{\text{loc}}^\psi(\Omega) \).

\textbf{Step 2: preliminary reductions.}

Since the supremum of lower semicontinuous functions is lower semicontinuous, we can restrict to prove (4.15) on a ball (or a hypercube) compactly contained in \( \Omega \). Hence, relabelling by \( \Omega \) such ball, that we shall take for the sake of simplicity as \( B_1 \), and possibly passing to a subsequence, which we relabel \( \{u_k\} \), we can suppose that the lower limit in (4.15) is a finite limit. Moreover, we can suppose that (3.2) holds on the whole of \( \Omega \). Then, setting \( z_k := u_k - u \), by (3.2) and Step 1 we have that (4.14) holds for \( \{z_k\} \). Hence, the Sobolev type property (Definition 3.2) yields that \( \{z_k\} \subset W^\psi(\Omega) \) and there exists \( M < +\infty \) such that:

\[
\sup_k \int_\Omega (\psi(x, |z_k|) + \psi(x, |Dz_k|)) \, dx < M. \tag{4.16}
\]

By applying the density property (Definition 4.5) to \( z_k \), since \( \varphi \) is a Carathéodory function satisfying (4.13), by the Dominated convergence theorem we can find for each \( k \) a sequence \( \{w_j\} \subset C_\infty(\Omega) \) such that \( w_j \to z_k \) in \( L^1(\Omega) \) as \( j \to +\infty \) and

\[
\lim_{j \to +\infty} F(u + w_j, A) = F(u + z_k, A) \quad \forall A \subset \subset \Omega.
\]

Again using the fact that the supremum of a family of lower semicontinuous integrals is semicontinuous, taking a smaller ball \( \Omega \), we can finally assume the sequence \( \{z_k\} \) to be in \( C_\infty(\Omega) \).

\textbf{Step 3: The case \( \text{supp } z_k \subset \Omega \).}

First we extend each \( z_k \) to the whole \( \mathbb{R}^n \) by letting \( z_k \equiv 0 \) outside \( \Omega \). We define (according to [2])

\[
(M^* z_k)(x) := (Mz_k)(x) + \sum_{i=1}^n (MD_i z_k)(x). \tag{4.17}
\]

We observe that if the support of \( u \) is contained in \( \Omega \), then \( (Mv)(x) \) as defined in Definition 4.4, coincides with the standard maximal function as employed in [2].

By (3.2), (4.16), (4.17) and the fact that \( \psi \) enjoys the maximal property (Definition 4.4) we have

\[
\sup_k \int_\Omega \psi(x, (M^* z_k^{(i)})(x)) \, dx < +\infty \quad \forall i = 1, \ldots, N,
\]

where \( z_k = (z_k^{(1)}, \ldots, z_k^{(N)}) \), hence we can apply the Biting lemma [2, Lemma I.7] to obtain for each \( \epsilon > 0 \) a (not relabelled) subsequence \( \{z_k\} \), a set \( A_\epsilon \subset \Omega \), with \( |A_\epsilon| < \epsilon \), and a real number \( \delta > 0 \) such that

\[
\sup_k \int_B \psi(x, (M^* z_k^{(i)})(x)) \, dx < \epsilon \quad \forall i = 1, \ldots, N, \tag{4.18}
\]

for every Borel set \( B \subset \Omega \setminus A_\epsilon \) with \( |B| < \delta \). Also, let \( \eta : \mathbb{R}^+ \to \mathbb{R}^+ \) be a continuous increasing function, with \( \eta(0) = 0 \), such that for every measurable set \( B \subset \Omega \)

\[
\int_B [b(x) + C(\psi(x, |u(x)|) + \psi(x, |Du(x)|))] \, dx < \eta(|B|). \tag{4.19}
\]
From this point on we shall closely follow the proof of Theorem II.4 from [2]. Once defined \( \lambda, H^\lambda_{i,k}, H^\lambda_k, g^i_k, v^i, g_k, v \) and \( v \) as in [2, Thm II.4], by (4.19), (4.18) and growth condition (4.13), since \( |(\Omega \setminus A_i) \setminus H^\lambda_{i,k}| < \delta \), if \( q \) and \( c \) are given by (3.2) with \( A = \Omega \), we obtain

\[
F(u + g_k, (\Omega \setminus A_i) \setminus H^\lambda_k) \\
\leq c \cdot 2^{q-1} \{ \eta(N\epsilon) + c(n, \Omega) \int_{(\Omega \setminus A_i) \setminus H^\lambda_k} \psi(x, \lambda) \, dx \} \\
\leq c \cdot 2^{q-1} \{ \eta(N\epsilon) + c(n, \Omega) \sum_{i=1}^{N} \int_{(\Omega \setminus A_i) \setminus H^\lambda_{i,k}} \psi(x, (M^* z_k^i)(x)) \, dx \} \\
\leq c \cdot 2^{q-1} \{ \eta(N\epsilon) + N \cdot c(n, \Omega) \epsilon \} = o_\epsilon,
\]

where \( o_\epsilon \to 0 \) when \( \epsilon \to 0 \); this last estimate replaces the one at the top of p. 131 in [2]. The rest of the proof in this case follows [2, Thm. II.4].

**Step 4: the general case \( \{ z_k \} \subset C^\infty(\Omega) \).**

In the following we adopt the notation of Lemma 2.6. We fix \( 0 < s < t < 1 \) and take \( \epsilon \in (0, 1) \); according to Lemma 2.6 (applied to \( f_k := \psi(x, |Dz_k|) \)) we select \( N = N(\epsilon, M) \) and \( M \) is from (4.16) (recall that we already reduced to the case \( \Omega \equiv B_1 \)). Therefore we find a thin layer \( A_h \) and a not relabelled subsequence \( \{ z_k \} \) such that

\[
(4.20) \quad \sup_{k} \int_{A_h} \psi(x, |Dz_k|) + \psi(x, |Du|) \, dx \leq \epsilon.
\]

Now we take a cut-off function \( \eta \) between \( B_{s_h} \) and \( B_{s_h+1} \) such that \( \| D\eta \| \leq 2N/(t-s) \) and define \( \tilde{z}_k := \eta z_k \). Since \( D\tilde{z}_k = D\eta \otimes z_k + \eta Dz_k \) and by (3.2)

\[
\int_{B_1} \psi(x, |D\tilde{z}_k|) \, dx \leq c \int_{B_1} (\psi(x, |Dz_k|) + \psi(x, |z_k|)) \, dx
\]

for some absolute constant \( c > 0 \) possibly depending on \( B_1 \) and \( \eta \), by (4.16) we obtain that (4.14) holds for \( \{ \tilde{z}_k \} \). Therefore, by Step 3 and condition \( \text{supp} \tilde{z}_k \subset B_{s_h+1} \),

\[
(4.21) \quad \int_{B_1} f(x, Du) \, dx \leq \liminf_{k \to +\infty} \int_{B_{s_h+1}} f(x, Du + D\tilde{z}_k) \, dx.
\]

As a consequence, again by (3.2)

\[
\int_{B_{s_h+1}} f(x, Du + D\tilde{z}_k) \, dx = \int_{B_{s_h}} f(x, Du + Dz_k) \, dx + \int_{A_h} f(x, Du + D\tilde{z}_k) \, dx
\]

\[
\leq \int_{B_1} f(x, Du + Dz_k) \, dx + c \int_{A_h} (\psi(x, |Du|) + \psi(x, |Dz_k|)) \, dx
\]

\[
+ c(\| D\eta \|) \int_{B_1} \psi(x, |z_k|) \, dx.
\]

Therefore, using (4.20), and combining with (4.21), since by the Rellich’s type property \( \int_{B_1} \psi(x, |z_k|) \, dx \to 0 \) as \( k \to +\infty \), we obtain

\[
\int_{B_1} f(x, Du) \, dx \leq \lim_{k \to +\infty} \int_{B_1} f(x, Du + Dz_k) \, dx + o_\epsilon,
\]
with \( o_\epsilon \to 0^+ \) as \( \epsilon \to 0^+ \). Finally the full statement follows by first letting \( \epsilon \to 0 \) and then \( s \to 1^- \).

**5. Continuity estimates for the maximal function.** Throughout this section we shall always assume that \( \Omega \) is a bounded open set. We will prove that in case \( \psi(x, t) \) is equal to \( r^{\alpha(x)A(t)} \) or to \( t^p + a(x) t^q \), under suitable hypotheses in both cases \( \psi \) enjoys the maximal property, as described in Definition 4.4.

**Proposition 5.1.** Let \( 0 \leq a(x) \leq L \) be such that \( a(x) \in C^{0, \alpha}(\Omega) \), where \( 0 < \alpha \leq 1 \) and (4.5) holds. Then for every function \( f \in L^1(\Omega) \) with

\[
\int_{\Omega} (|f(x)|^p + a(x) |f(x)|^q) \, dx < +\infty
\]

we have

\[
(5.1) \quad \int_{\Omega} (|\langle M f \rangle(x)|^p + a(x) |\langle M f \rangle(x)|^q) \, dx \leq \tilde{C} \left( \int_{\Omega} (|f(x)|^p + a(x) |f(x)|^q) \, dx + 1 \right)^{q/p}
\]

where \( \tilde{C} \) is a positive constant depending on \( n, \Omega, p, q, L, |a|_{0, \alpha} \).

**Proof.** Let us first prove that for every \( x \in \Omega \) and \( r > 0 \)

\[
(5.2) \quad a(x) |\langle M_r f \rangle(x)|^q \leq |\langle M_r (a(\cdot)^{1/q} f) \rangle(x)|^q + C |\langle M_r f \rangle(x)|^p \cdot \|f\|_{L^p(\Omega)}^{q-p}
\]

Denoting \( B = B_r(x) \), for simplicity, and

\[
a_r(x) := \inf\{a(y) \mid y \in \Omega, |y - x| < r\},
\]

trivially if \( |B \cap \Omega| > 0 \) then

\[
(5.3) \quad a_r(x) |\langle M_r f \rangle(x)|^q \leq \left| \frac{1}{|B|} \int_{B \cap \Omega} a(y)^{1/q} |f(y)| \, dy \right|^q = |\langle M_r (a(\cdot)^{1/q} f) \rangle(x)|^q.
\]

Moreover

\[
(5.4) \quad (a(x) - a_r(x)) |\langle M_r f \rangle(x)|^q \leq (a(x) - a_r(x)) |\langle M_r f \rangle(x)|^{q-p} \cdot |\langle M_r f \rangle(x)|^p
\]

whereas by (4.5) we have \( \alpha/(q-p) \geq n/p \) and hence, for \( 0 < r \leq 1 \), we estimate

\[
r^{\alpha/(q-p)} \leq r^{n/p}. \quad \text{Then, by Hölder inequality, since} \ f \in L^p(\Omega),
\]

\[
(5.5) \quad (a(x) - a_r(x)) |\langle M_r f \rangle(x)|^{q-p} \leq |a|_{0, \alpha} \cdot r^{n/p} \cdot |B_r|^{-1/p} \|f\|_{L^p(\Omega)}^{q-p}
\]

Also, (5.5) trivially holds if \( r > 1 \) since \( a(x) \) is bounded. Now, (5.3), (5.4) and (5.5) yield (5.2) so that, taking the supremum on \( r \) and passing to the integrals on \( \Omega \), since \( f \in L^p(\Omega) \) and \( a(\cdot)^{1/q} f \in L^q(\Omega) \), by the standard Hardy-Littlewood maximal theorem [45, Thm. 1, Sec. I.1] we obtain:

\[
\int_{\Omega} (|\langle M f \rangle(x)|^p + a(x) |\langle M f \rangle(x)|^q) \, dx \leq C \left( \int_{\Omega} |f(x)|^p \, dx \right)^{q/p} + C \int_{\Omega} a(x) |f(x)|^q \, dx
\]

and finally (5.1). \( \square \)
The next proposition extends a result due to Diening [19].

**Proposition 5.2.** Let $A : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous function with $1 < s_1 \leq A(t) \leq s_2 < +\infty$ and such that $t \rightarrow t^{\gamma A(t)}$ is non-decreasing and convex in $\mathbb{R}^+$ for every $p > 1$. Moreover, let $p : \Omega \rightarrow (1, +\infty)$ be a uniformly continuous function such that $\inf_{\Omega} p(x) > 1$ and for some $C_0 > 1$

\begin{equation}
(5.6) \quad |p(x) - p(y)| \leq \frac{C_0}{\log |x - y|} \quad \forall x, y \in \Omega, \quad 0 < |x - y| < \frac{1}{2}.
\end{equation}

Set $1 < p := s_1 \inf_{\Omega} p(x) \leq s_2 \sup_{\Omega} p(x) =: q < +\infty$. Then, for every function $f \in L^1(\Omega)$ such that $\int_{\Omega} |f(x)|^{p(x)A(|f(x)|)} \, dx < +\infty$ we have

\begin{equation}
(5.7) \quad \int_{\Omega} |(M f)(x)|^{p(x)A(|M f(x)|)} \, dx \leq \bar{C} \left( \int_{\Omega} |(f(x)|^{p(x)A(|f(x)|)} \, dx + 1 \right)^{q/p}
\end{equation}

where $\bar{C}$ is a positive constant depending on $n, \Omega, p, q, s_1, s_2$.

**Remark 5.3.** With a slightly different proof it is possible to obtain the following inequality:

\begin{equation}
\int_{\Omega} A(|(M f)(x)| \cdot |(M f)(x)|^{p(x)} \, dx \leq \bar{C} \left( \int_{\Omega} A(|f(x)|) \cdot |f(x)|^{p(x)} \, dx + 1 \right)^{q/p}
\end{equation}

in the case $t \rightarrow t^\alpha A(t)$ is convex for any $\alpha > 1$ and $t^{s_1} \leq A(t) \leq L(1 + t^{s_2})$ where, this time, $s_2 \geq s_1 > 1$, $q := \sup p(x) + s_2$ and $p := \inf p(x) + s_1$.

**Proof.** First of all we can assume $f$ is defined on the whole $\mathbb{R}^n$ by letting $f \equiv 0$ outside $\Omega$ so that

\begin{equation}
(M_r f)(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| \, dy,
\end{equation}

in any case; this will allow us to apply Jensen inequality in the second inequality from (5.10). Let us first prove that for every $x \in \Omega$ and $r > 0$

\begin{equation}
(5.8) \quad |(M_r f)(x)|^{p(x)A(|M_r f(x)|)} \leq \gamma [(M_r f)(\cdot)]^{p(x)A(|f(\cdot)|)}(x) + 1,
\end{equation}

where

\begin{equation}
(5.9) \quad \gamma := c \left( \int_{\Omega} |f(x)| \, dx + 1 \right)^{q/p}
\end{equation}

and $c$ depends on $n, p$ and $q$. Set, for every ball $B \subset \mathbb{R}^n$ with $|B \cap \Omega| > 0$,

\begin{equation}
p_B^- := \inf_{x \in B \cap \Omega} p(x), \quad p_B^+ := \sup_{x \in B \cap \Omega} p(x)
\end{equation}

and we have

\begin{equation}
|B|^{p_B^- - p_B^+} \leq c(n, p, q).
\end{equation}

Indeed the previous inequality is trivial when $r > 1/4$ and it is a consequence of (5.6) in the other case. Since $z \mapsto |z|^{p_B^- A(|z|)}$ is convex, and $f(\cdot)^{p_B^- A(|f(\cdot)|)} \in L^1(B)$, by
Jensen inequality, denoting $B = B_r(x)$ for simplicity, we have

$$\begin{align*}
|M(f(x))|^q(x) &= |(M(f(r))(x))^{p(x)A(|(M(r)f(x)|x)|)} \\
&= |(M(f(r))(x))^{p(x)-p(\gamma)}A(|(M(r)f(x)|x)|) \\
&\times |(M(r)f(x))^{p(\gamma)}A(|(M(r)f(x)|x)|) \\
&\leq \gamma |B|^{p(\gamma) - p(x)}x \gamma (M(r)f(x))^{p(\gamma)}A(|(M(r)f(x)|x)|) \\
&\leq c \gamma M(r)[f(\cdot)^p A(|(f(\cdot)|x)|] \\
&\leq c \gamma M(r)[f(\cdot)^p A(|(f(\cdot)|x)|] + c.
\end{align*}$$

(5.10)

Setting now $q(x) := p(x) s_1/p > 1$, since $\inf p(x) > 1$ then

$$p/s_1 > 1 \quad \text{and} \quad \int_{\Omega} |f(x)|^{q(x)A(|(f(x)|x)| x} ^{p/s_1} dx < +\infty$$

we can use the boundedness of the maximal operator as follows

$$\int_{\Omega} |M[f(\cdot)]^{q(x)A(|(f(\cdot)|x)| x} ^{p/s_1} dx \leq c \int_{\Omega} |f(x)|^{q(x)A(|(f(x)|x)| x} ^{p/s_1} dx \leq c \int_{\Omega} |f(x)|^{p(x)A(|(f(x)|x)| x} dx$$

(5.11)

where the above constant $c$ depends on $n$, $\Omega$ and $p/s_1$. Applying (5.8) (which is a pointwise inequality) with $q(x)$ instead of $p(x)$, we finally obtain, using also (5.9), (5.11) and Hölder inequality

$$\begin{align*}
\int_{\Omega} |(Mf(x))|^{p(x)A(|(Mf(x)|x)| x} dx &= \int_{\Omega} |M[f(\cdot)]^{q(x)A(|(f(\cdot)|x)| x} |^{p(x)} dx \\
&\leq c \left( \int_{\Omega} |f(x)|^{p(x)} dx + 1 \right)^{(q/p) - 1} \int_{\Omega} |[(M[f(\cdot)]^{q(x)A(|(f(\cdot)|x)| x}) + 1]^{p/s_1} dx \\
&\leq c \left( \int_{\Omega} |f(x)|^{p(x)} dx + 1 \right)^{(q/p) - 1} \int_{\Omega} |[(M[f(\cdot)]^{p(x)A(|(f(\cdot)|x)| x}) + 1] dx.
\end{align*}$$

Therefore (5.7) immediately follows as $p \leq \inf p(x) A(|(f(x)|x)|$. ☐

**Remark 5.4.** It is interesting to note that conditions (4.5) and (5.6) are sharp in order to guarantee the validity of the maximal property, that is (5.1) and (5.7), respectively. This is again a consequence of the counterexamples in Sec. 7 and Sec. 8 below. Indeed suppose (4.5) and (5.6) fail to hold but (5.1) and (5.7) are satisfied; then by Proposition 4.6 also the density property holds true and in turn Theorem 4.7 would imply that $F(u, A)$ is an absolutely continuous Radon measure as soon as $Du \in L^2(A; \mathbb{R}^N)$, in the case $F \equiv F_2$ and $F \equiv F_1$ respectively (see (1.6)). This is in contrast to the counterexamples presented in Sec. 7 and Sec. 8, where it is shown that, in general, the failure of (4.5) and (5.6) causes the rising of a singular Borel measure in the Relaxation procedure (see in particular Theorem 7.4 for (4.5) and Theorem 8.3 for (5.6)). This observation, together with the forthcoming examples in Sec. 7 Sec. 8, clarifies the unifying role of the continuity assumptions of the type (4.5) and (5.6).
In this section we want to outline how to apply the previous results to general classes of functionals, including many model examples available in the literature to which standard relaxation techniques do not apply. If \( \{\psi_i\}_i \) is a finite collection of functions satisfying (i) and (ii) from Sec. 3 together with the maximal property (and hence satisfying also the density property by Proposition 4.6), then the new function defined by

\[
\bar{\psi}(x,t) := \sum_i a_i(x) \psi_i(x,t) \quad L^{-1} \leq a_i(x) \leq L < +\infty
\]

also enjoys the same properties. Using this simple observation it immediately follows that the maximal estimates of the previous section allow to use the model examples introduced there as building blocks to construct new functionals to which our theory applies. The main point we would like to stress here is that the model functionals presented in Sec. 5 describe the way the presence of the variable \( x \) in the energy density \( f \) modifies the growth with respect to the gradient variable \( z \). Using the previous observation, Theorem 4.12 may be applied, via the maximal estimates of Section 5 and Proposition 4.6, in the cases when

\[
\begin{align*}
\psi_1(x,|z|) &:= a(x)|z|^{p(x)} \log(1 + |z|), \quad L^{-1} \leq a(x) \leq L \\
\psi_2(x,|z|) &:= A(|z|)^{p(x)}, \quad |z| \leq A(|z|) \leq L(1 + |z|) \\
\psi_3(x,|z|) &:= (e + |z|^2)^{p(x)}(\theta_1 + \theta_2 \sin \log \log(1 + |z|^2)), \quad \text{for suitable } \theta_1, \theta_2 \\
\psi_4(x,|z|) &:= f_p(x,|z|) + a(x)f_q(x,|z|), \quad |z|^s \leq f_s(x,|z|) \leq L(1 + |z|^r) \quad s = p, q \quad 0 \leq a(x) \leq L .
\end{align*}
\]

In turn, any finite combination of \( \psi_i \) works and so on. Let us observe that energies related to \( \psi_1 \) appear in the context of Prandtl-Eyring fluids (see [27]), while \( \psi_2 \) is related to electrorheological fluids (see [44] and [4]). The function \( \psi_3 \) has been introduced in the setting of functionals with non-standard growth conditions in [29] while \( |z|^{p(x)} \) and \( \psi_4 \) have been introduced, in the context of Homogenization theory, by Zhikov [46]. Finally we want to briefly mention that the results of the previous sections could be extended to the case of the so called anisotropic functionals, i.e. functionals in which each direction is penalized with a different exponent. Functionals of this type come up when studying reinforced materials. In this case (3.1) is replaced by

\[
L^{-1} \sum_{i=1}^n a_i(x)|D_i u|^{p_i(x)} \leq f(x,Du) \leq L \left( 1 + \sum_{i=1}^n a_i(x)|D_i u|^{p_i(x)} \right) \quad 1 \leq L < +\infty ,
\]

where, in the models for reinforced materials, the exponents are constants: \( p_i(x) \equiv p_i \equiv \text{constant} \).

7. **A sharp example with energy concentration.** Let \( \Omega = B_1 \), the unit ball of \( \mathbb{R}^n \), and \( f(x,z) := |z|^p + a(x)|z|^q \), see Example 3.4, where \( a(x) \) is a suitable bounded non-negative function in \( C^{0,\alpha}(B_1) \) for some \( 0 < \alpha < 1 \).

In this section we will first show (Theorem 7.4) that energy concentration does occur in the process of relaxation in the case (4.5) is violated; more precisely when

\[
1 < p < n < n + \alpha < q < p^* \quad (7.1)
\]

where, as usual, \( p^* := np/(n-p) \). Secondly, if in particular

\[
q > n \left( 1 + \alpha \right) \quad \text{and} \quad n \frac{1 + \alpha}{2 + \alpha} < p < n , \quad (7.2)
\]
we are then able to give a complete representation of the relaxed functional (Theorem 7.6). We emphasize here that it is a significant feature of our analysis that the examples proposed in this section and in the next one already work in the scalar case $N = 1$, to which we specialize henceforth. For every $0 < \alpha \leq 1$ we define

$$
(7.3) \quad a(x) := \max \left\{ \left( x_n^2 - \sum_{i=1}^{n-1} x_i^2 \right), 0 \right\}^{\alpha} |x|^{-\alpha}, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n,
$$

so that $a(x) \in C^{0,\alpha}(\mathbb{R}^n)$ and $a(x) > 0$ in the open cone

$$
C^+ := \left\{ x \in \mathbb{R}^n \mid x_n^2 - \sum_{i=1}^{n-1} x_i^2 > 0 \right\}.
$$

By (7.1) the assumptions of Theorem 3.3 are satisfied (see Example 3.4). Then, in this section we denote by $F(u, A)$ and $\mathcal{F}(u, A)$ the functionals given by (2.1) and (2.2), respectively, with $\Omega = B_1$ and, when not differently specified, $f(x, z) := |z|^p + a(x) |z|^q$, where $a(x)$ is given by (7.3) and (7.1) holds, so that $\mathcal{F}(u, A)$ satisfies the measure property.

**Remark 7.1.** For every $u \in L^1(B_1)$ and $A \in \mathcal{A}$, it is possible to find a sequence $\{u_k\} \subseteq L^1(B_1)$ with $u_k \rightarrow u$ in $L^1(B_1)$ and $u_k|_A \in W^{1,q}(A)$ for every $k \in \mathbb{N}$. Moreover, since

$$
(7.4) \quad \mathcal{F}(u, A) = \inf \left\{ \liminf_{k \rightarrow +\infty} \int_A (|Du_k|^p + a(x) |Du_k|^q) \, dx \mid \{u_k\} \subset W^{1,q}_{\text{loc}}(A), \quad u_k \rightarrow u \text{ in } L^1(A) \right\},
$$

by Remark 2.1 for every $A \in \mathcal{A}$ and $u \in L^1(B_1)$

$$
\mathcal{F}(u, A) = \inf \left\{ \liminf_{k \rightarrow +\infty} \int_A (|Du_k|^p + a(x) |Du_k|^q) \, dx \mid \{u_k\} \subset W^{1,q}_{\text{loc}}(A), \quad u_k \rightarrow u \text{ in } L^1(A) \right\}.
$$

Let us now introduce some notation. If $n = 2$ we are going to use the following polar coordinates

$$
x_1 = \rho \sin \phi, \quad x_2 = \rho \cos \phi, \quad \rho \geq 0, \quad 0 \leq \phi \leq 2\pi.
$$

If $n \geq 3$ we use the spherical coordinate transformation $x = F(\rho, \phi, \Theta)$, $\Theta := (\theta_1, \ldots, \theta_{n-2})$, where $(\rho, \Theta) \in I(\phi, \Theta) := [0, \pi] \times \left( \prod_{i=1}^{n-3} [0, \pi] \right) \times [0, 2\pi]$ and

$$
x_1 = \rho \sin \phi \cdot \prod_{j=1}^{n-2} \cos \theta_j, \quad x_{n-1} = \rho \sin \phi \cdot \sin \theta_1, \quad x_n = \rho \cos \phi,
$$

$$
x_i = \rho \sin \phi \cdot \prod_{j=1}^{n-1-i} \cos \theta_j \cdot \sin \theta_{n-i}, \quad i = 2, \ldots, n-2.
$$

Moreover, for any function $u$ on $\mathbb{R}^n$, in the sequel we will always denote

$$
\tilde{u}(\rho, \phi, \Theta) := u(F(\rho, \phi, \Theta))
$$

the corresponding function written in spherical coordinates. For example, if $a(x)$ is given by (7.3) we have

$$
\tilde{a}(\rho, \phi, \Theta) = (\rho (\cos(2\phi))^+)^{\alpha}, \quad C^+ = \{ x = F(\rho, \phi, \Theta) \mid \cos(2\phi) > 0 \}.$$
where \( y^+ \) denotes the positive part of real number \( y \), i.e., \( y^+ := \max\{y, 0\} \). Finally,
\[
\partial C^+ := \{ x \in \mathbb{R}^n \mid x_n^2 = \sum_{i=1}^{n-1} x_i^2 \} = \{ x = F(\rho, \phi, \Theta) \mid \cos(2\phi) = 0 \}
\]
is the boundary of \( C^+ \), for every \( 0 < \beta < \pi/4 \)
\[
C^\beta := \{ x = F(\rho, \phi, \Theta) \mid \cos(2\phi) > c_\beta \}, \quad c_\beta := \cos\left(\frac{\pi}{2} - 2\beta\right)
\]
is the subset of \( C^+ \) given by a cone of smaller angle and, for \( 0 < r < 1 \),
\[
C^+ := C^+ \cap B_r, \quad C^\beta := C^\beta \cap B_r
\]
is the intersection with the open ball \( B_r \) of radius \( r \); moreover we introduce the following “half cones”:
\[
+C^\beta := \{ x \in F(\rho, \phi, \Theta) \in C^\beta \mid 0 \leq \phi < \pi/4 - \beta \},
-\beta := \{ x \in F(\rho, \phi, \Theta) \in C^\beta \mid 3\pi/4 + \beta < \phi < \pi \},
\]
being the upper and the lower part, respectively, of \( C^\beta \); accordingly we define
\[
+C^+ := \{ x \equiv F(\rho, \phi, \Theta) \equiv C^+ \mid 0 \leq \phi < \pi/4 \}
-\beta^+ := \{ x \equiv F(\rho, \phi, \Theta) \equiv C^+ \mid 3\pi/4 < \phi \leq \pi \}.
\]

The following result will allow us to consider the traces in the origin of a function \( u \in L^1(C^\beta_r) \), see (7.8), in the case (7.1) holds.

**Lemma 7.2.** Let \( u \in L^1(B_r) \), \( 0 < r < 1 \), be such that \( \int_{B_r} a(x)|Du|^q \, dx < +\infty \), where \( a(x) \) is given by (7.3) and \( q > n + \alpha \). Then for every \( n < s < q \), with \( q/s > (n + \alpha)/n \), we have \( \int_{C^\beta_r} |Du|^s \, dx \) for every \( 0 < \beta < \pi/4 \). In particular \( u \in C^{0,1-n/s}(+\overline{C^\beta_r}) \) and \( u \in C^{0,1-n/s}(-\overline{C^\beta_r}) \), i.e. there exists a constant \( c \), only depending on \( \int_{B_r} a(x)|Du|^q \, dx \) such that

\[
(7.5) \quad |u(x_1) - u(x_2)| \leq c|x_1 - x_2|^{1-n/s} \quad \forall x_1, x_2 \in +\overline{C^\beta_r}
\]
\[
|u(y_1) - u(y_2)| \leq c|y_1 - y_2|^{1-n/s} \quad \forall y_1, y_2 \in -\overline{C^\beta_r}.
\]

**Proof.** First note that \( a(x)^{s/(s-q)} \in L^1(C^\beta_r) \). In fact, by the area formula [23, 3.2.3] we have
\[
\int_{C^\beta_r} a(x)^{s/(s-q)} \, dx \\
\leq c(n) \int_{[0, \pi/4 - \beta] \cup [3\pi/4 + \beta, \pi]} \frac{(\sin \phi)^{n-2}}{(\cos(2\phi))^{n-s/(q-s)}} \, d\phi \int_{C^\beta_r} |x|^{n/s} \, dx \\
\leq c(n) c_\beta^{n/(s-q)} \int_0^{\rho} \rho^{n-1+\alpha s/(s-q)} \, d\rho
\]
which is finite since \( n + \alpha s/(s-q) > 0 \) if \( q/s > (n + \alpha)/n \). Then by Hölder inequality we have
\[
(7.6) \quad \int_{C^\beta_r} |Du|^s \, dx \leq \left( \int_{C^\beta_r} a(x)|Du|^q \, dx \right)^{s/q} \cdot \left( \int_{C^\beta_r} a(x)^{s/(s-q)} \, dx \right)^{(q-s)/q} < +\infty.
\]
The assertions concerning Hölder continuity follow via Sobolev embedding theorem and Morrey’s theorem, since $s > n$.

With a stronger assumption on the exponent $q$, that is replacing (7.1) by (7.2), we can similarly prove the following.

**Lemma 7.3.** Under the hypotheses of Lemma 7.2, suppose in particular that $q > n(1 + \alpha)$. Then for every $n < s < q/(1 + \alpha)$, with $q/s > 1 + \alpha$, we have $\int_{C^+_r} |Du|^s \, dx < +\infty$ and hence $u \in C^{0,1-n/s}(\overline{C}^+_r)$ and $u \in C^{0,1-n/s}(\overline{C}^-_r)$ with analogous estimates to (7.5) with $\beta = 0$.

**Proof.** Now we have $a(x)^{s/(s-q)} \in L^1(C^+_r)$. In fact, for $n \geq 3$ ($n = 2$ is similar)

$$\int_{C^+_r} a(x)^{s/(s-q)} \, dx = c(n) \int_{[0,\pi/4] \cup [3\pi/4,\pi]} \frac{(\sin \phi)^{n-2}}{(\cos(2\phi))^{\alpha s/(q-s)}} \, d\phi \int_0^\rho \rho^{n-1+\alpha s/(q-s)} \, d\rho,$$

which is finite since $1/(\cos(2\phi))^{\alpha s/(q-s)}(0,2\pi)$ as $q/s > 1 + \alpha$. Then (7.6) holds again, with $C^+_r$ instead of $C^+_r$. The rest follows as for Lemma 7.2.

**Traces at $0_{\mathbb{R}^n}$.** Let now $u \in L^q(B_1)$ be such that $a(x)|Du|^q \in L^1_{\text{loc}}(A)$ for some open set $A \subset \mathbb{R}^n$. Since $B_r \subset A$ for $r$ sufficiently small, if $q > n + \alpha$ by Lemma 7.2 we can therefore define for every $0 < \beta < \pi/4$

$$\lambda_1 := \lim_{\rho \to 0^+} \bar{u}(\rho, \phi, \Theta), \quad \lambda_2 := \lim_{\rho \to 0^+} \bar{u}(\rho, \phi, \Theta)$$

($\phi \in [0,\pi/4-\beta] \cup [3\pi/4+\beta,2\pi]$) and $\phi \in [3\pi/4+\beta,5\pi/4-\beta]$, respectively, if $n = 2$, where the finite limits exist uniformly in $\Theta$ since $u$ is Hölder continuous up to the closure of both $+C^\beta_r$ and $-C^\beta_r$. Moreover, if $q > n(1 + \alpha)$ by Lemma 7.3 we obtain (7.8) with $\beta = 0$, i.e., the traces in the origin do exist in both the upper and lower half cones of $C^+$.

We will now prove the following result, which actually shows that (4.5) is a sharp condition to prevent energy concentration in the process of relaxation.

**Theorem 7.4.** Let $F(u,A)$ and $\overline{F}(u,A)$ be given by (2.1) and (2.2) with $\Omega = B_1$ and $f(x,z) := |z|^p + a(x)|z|^q$, where $a(x)$ is given by (7.3) and (7.1) holds. Let $0_{\mathbb{R}^n} \in A \subset \mathbb{A}$ and $|Du|^p + a(z) |Du|^q \in L^1_{\text{loc}}(A)$, so that (7.8) holds. Then, if $\lambda_1 \neq \lambda_2$, we have $\overline{F}(u,A) = +\infty$, hence an infinite singular measure is concentrated in the origin.

**Example 7.5.** In particular, for $n \geq 3$, if $u_0 : B_1 \to \mathbb{R}$ is given in spherical coordinates by

$$\bar{u}_0(\rho, \phi, \Theta) := \begin{cases} 1 & \text{if } 0 \leq \phi \leq \pi/4 \\ \sin(2\phi) & \text{if } \pi/4 \leq \phi \leq 3\pi/4 \\ -1 & \text{if } 3\pi/4 \leq \phi \leq \pi, \end{cases}$$

and similarly for $n = 2$, then since $\lambda_1 = 1$ and $\lambda_2 = -1$ there is energy concentration in the origin, i.e.,

$$\overline{F}(u_0,A) = +\infty \quad \forall A \in \mathbb{A} \quad \text{such that } 0_{\mathbb{R}^n} \in A.$$

**Proof of Theorem 7.4.** We argue by contradiction supposing that $\overline{F}(u,A) < +\infty$; then we pick a radius $r > 0$ such that $B_r \subset A$ and a sequence $\{u_k\} \subset C^1(B_r)$ such
that \( u_k \to u \) in \( L^1(B_r) \) and a.e. and
\[
\lim_{k \to +\infty} \int_{B_r} (|Du_k|^p + a(x) |Du_k|^q) \, dx = \mathcal{F}(u, B_r) < +\infty.
\]

By the Hölder estimates in Lemma 7.2 we obtain that the sequence \( \{u_k\} \subset C^1(B_r) \) is equi-uniformly continuous on both \( +\overline{C}_r \) and \( -\overline{C}_r \); since each \( u_k \) is continuous this yields that the sequence \( \{u_k\} \subset C^1(B_r) \) is equi-uniformly continuous on the whole \( \overline{C}_r \). Then by Ascoli-Arzelà theorem, up to a not relabelled subsequence, \( u_k \to u \) uniformly on \( \overline{C}_r \) that, in turn, yields to the continuity of \( u \) at 0 in \( \mathbb{R}^n \). This is a contradiction since \( \lambda_1 \neq \lambda_2 \) makes the function \( u \) discontinuous at 0 in \( \mathbb{R}^n \). \( \Box \)

With a bit more of effort, if (7.2) holds we are able to prove the following complete representation result.

**Theorem 7.6.** Let \( F(u, A) \) and \( \overline{F}(u, A) \) be given by (2.1) and (2.2) with \( \Omega = B_1 \) and \( f(x, z) \) be a Carathéodory function such that
\[
(7.11) \quad c_1 (|z|^p + a(x) |z|^q) \leq f(x, z) \leq c_2 (|z|^p + a(x) |z|^q + 1)
\]
for a.e. \( x \in \Omega \) and all \( z \in \mathbb{R}^n \), where \( c_2 > c_1 > 0 \). If \( a(x) \) is given by (7.3) and (7.2) holds, then we have
\[
(7.12) \quad \mathcal{F}(u, A) = \begin{cases} \int_A (Cf)(x, Du) \, dx + \mu(u, A) & \text{if } |Du|^p + a(\cdot) |Du|^q \in L^1(A) \\ +\infty & \text{elsewhere on } L^1(\Omega) \end{cases}
\]
where \( Cf \) denotes the usual convexification of \( f \) and \( \mu(u, \cdot) \) is an infinite singular measure concentrated in the origin. More precisely, we have
\[
(7.13) \quad \mu(u, A) = \begin{cases} 0 & \text{if } \mathbb{R}^n \notin A \\ \chi_{A}^{\lambda_1} & \text{if } \mathbb{R}^n \in A \end{cases}
\]
where \( \lambda_1 \) and \( \lambda_2 \) are defined by (7.8) and
\[
\chi_{A}^{\lambda_1} := \begin{cases} 0 & \text{if } \lambda_1 = \lambda_2 \\ +\infty & \text{if } \lambda_1 \neq \lambda_2. \end{cases}
\]

**Proof.** We will first give the proof in the case
\[
f(x, z) := |z|^p + a(x) |z|^q.
\]
The first part of the statement is trivial. In fact, following Lemma 3.5, Theorem 2.5 yields that if \( \mathcal{F}(u, A) < +\infty \) for some \( A \in \mathcal{A} \), then \( |Du|^p + a(\cdot) |Du|^q \in L^1(A) \). Moreover, we note that for every \( A \in \mathcal{A} \) with \( A \subset \subset (B_1 \setminus \partial C^+) \) we have that \( L^{-1} \leq a(x) \leq L \) on \( A^+ := A \cap C^+ \), for some positive constant \( L \) depending on \( A \), whereas \( a(x) = 0 \) on \( A \setminus A^+ \). Hence, by convexity of \( z \to |z|^p + L |z|^q \) and by the dominated convergence theorem, if \( \mathcal{F}(u, A) < +\infty \) we can easily find a sequence of smooth maps \( \{u_k\} \in C^1(A) \) with \( u_k \to u \) in \( L^1(A) \) and
\[
\lim_{k \to +\infty} \int_A (|Du_k(x)|^p + a(x) |Du_k(x)|^q) \, dx = \int_A (|Du(x)|^p + a(x) |Du(x)|^q) \, dx.
\]
Then, by (7.4) and by inner regularity of \( \mathcal{F}(u, \cdot) \), for every \( A \in \mathcal{A} \) with \( A \cap \partial C^+ = \emptyset \) we have that
\[
(7.14) \quad \mathcal{F}(u, A) = \int_A (|Du(x)|^p + a(x) |Du(x)|^q) \, dx
\]
if $u \in L^1(B_1)$ is such that $|Du|^p + a(|x|) |Du|^q \in L^1(A)$. As a consequence, we infer that the absolute continuous part of the measure $\mathcal{F}(u, \cdot)$ is the integral given in (7.12), and that its singular part $\mu(u, \cdot)$ is concentrated in the $(n-1)$-dimensional cone $\partial C^+$. We now show that there is no energy concentration on open sets which do not contain the origin.

**Proposition 7.7.** Under the hypotheses of Theorem 7.6, if $A \in \mathcal{A}$, $0 \in A$ and $|Du|^p + a(|x|) |Du|^q \in L^1(A)$, then $\mu(u, A) = 0$ in (7.12) and hence

$$\mathcal{F}(u, A) = \int_A (|Du|^p + a(x) |Du|^q) \, dx.$$

**Proof.** We adapt the approximation and reflection arguments of Lemma 3.4 and Lemma 3.5 in [22]. Indeed following this paper it is possible to show that for every $A' \in \mathcal{A}_0$, with $A' \subset A$, there exists a sequence of functions $\{u_k\} \subset W^{1,q}(A')$ such that $u_k \to u$ in $L^1(A')$ and

$$\lim_{k \to +\infty} \int_{A'} (|Du_k|^p + a(x) |Du_k|^q) \, dx = \int_{A'} (|Du|^p + a(x) |Du|^q) \, dx.$$

Then, by (7.4), this yields

$$\mathcal{F}(u, A') \leq \int_{A'} (|Du|^p + a(x) |Du|^q) \, dx,$$

and hence, by inner regularity, letting $A' \to A$ one obtains the assertion by the fact that $\mu(u, A) \geq 0$. We explicitly remark that in [22] the proof is given for the case the function $a(x)$ is replaced (in polar coordinates) by

$$\bar{a}(|\rho|, \theta) = \rho^\alpha \cos(2\phi)^+.$$

Then the proof is achieved taking advantage of the fact that the function $\cos(2\phi)$ satisfy the so called Muckenhoupt condition $A_q$; this gives the possibility to build an approximation procedure based on a reflection argument (where the Muckenhoupt property enters). It is easy to see that the same argumentation works here for the function $(\cos(2\phi))^\alpha$, which comes from the study of our case. \qed

Now let $0 \in A \in \mathcal{A}$ and $|Du|^p + a(|x|) |Du|^q \in L^1_{\text{loc}}(A)$. By Theorem 7.4 it follows that $\mu(u, A) = +\infty$ if $\lambda_1 \neq \lambda_2$ in (7.8). To conclude with (7.13), it then remains to show that $\mu(u, A) = 0$ if $\lambda_1 = \lambda_2$. To this aim, by (7.4) it suffices to prove the following

**Proposition 7.8.** Let $0 \in A \in \mathcal{A}$ and $u \in L^1(B_1)$ be such that $|Du|^p + a(|x|) |Du|^q \in L^1_{\text{loc}}(A)$, with $\lambda_1 = \lambda_2$ in (7.8). Then for each $\epsilon > 0$ there exists a sequence $\{w_k\} \subset W^{1,q}(A)$ such that $w_k \to u$ in $L^1(A)$ and

$$\liminf_{k \to +\infty} \int_A (|Dw_k|^p + a(x) |Dw_k|^q) \, dx \leq \int_A (|Du|^p + a(x) |Du|^q) \, dx + \epsilon. \quad (7.15)$$

**Proof.** Observe that we may and do assume that the right-hand side of (7.15) is finite. We will denote by $\nu$ the outward unit normal to $\partial B_R$ and by $\tau := (\tau_1, \ldots, \tau_{n-1})$ an orthonormal basis to the tangent $(n-1)$-space to $\partial B_R$. Then, setting $D_\tau u := (D_1 u, \ldots, D_{n-1} u)$, we have that $|Du|^2 = |D_\nu u|^2 + |D_\tau u|^2$. Also, if $u \in W^{1,p}(B_1)$ and $0 < R < 1$, we will denote by $T_R u := T[\partial B_R]u$ the usual trace operator: that is $T_R u \in W^{1-\frac{1}{p}, p}(\partial B_R)$ is the trace of $u$ on $\partial B_R$. 


Fix now $0 < \delta < \text{dist}(0_{\mathbb{R}^n}, \partial A)$ and let $r \in (0, \delta/2)$. Then, by Remark 7.1 and Proposition 7.7, we select a sequence $\{u_k\} \subset W^{1,q}(A \setminus B_r)$ such that $u_k \to u$ in $L^1(A \setminus B_r)$ and

$$\lim_{k \to +\infty} \int_{A \setminus B_r} (|Du_k|^p + a(x)|Du_k|^q) \, dx$$

(7.16)

$$= F(u, A \setminus B_r)$$

$$= \int_{A \setminus B_r} (|Du|^p + a(x)|Du|^q) \, dx < +\infty.$$  

Up to passing to a not relabelled subsequence, by uniform convexity, (7.16) yields

$$\lim_{k \to +\infty} \int_{A \setminus B_r} (|Du_k - Du|^p + a(x)|Du_k - Du|^q) \, dx = 0.$$  

In particular, by an estimate similar to (7.6), with $\beta = 0$, which is allowed since now (7.2) is in force (see also (7.7)), we have

$$\lim_{k \to +\infty} \int_{B_{2r} \setminus B_r \cap C^+} |Du_k - Du|^s \, dx = 0$$

for some $s > n$. As a consequence, by Sobolev, Morrey and Rellich’s theorems, passing again to a not relabelled subsequence, we can select $R \in (r, 2r)$ such that $T_{Ru} \in W^{1,p}(\partial B_R) \cap W^{1,q}(\partial B_R \cap C^+)$, $T_{Ru_k} \in W^{1,q}(\partial B_R)$ for every $k$,

$$\int_{\partial B_R} (|D_y u_k|^p + a(x)|D_y u_k|^q) \, d\mathcal{H}^{n-1}$$

(7.17)

$$\leq \int_{\partial B_R} (|D_y u|^p + a(x)|D_y u|^q) \, d\mathcal{H}^{n-1} + \frac{\epsilon}{3} R,$$

(7.18)

$$\int_{\partial B_R} |u_k - \lambda|^p \, d\mathcal{H}^{n-1} \leq \int_{\partial B_R} |u - \lambda|^p \, d\mathcal{H}^{n-1} + \frac{\epsilon}{3} R^{q-1},$$

(7.19)

$$\int_{\partial B_R \cap C^+} |u_k - \lambda|^q \, d\mathcal{H}^{n-1} \leq \int_{\partial B_R \cap C^+} |u - \lambda|^q \, d\mathcal{H}^{n-1} + \frac{\epsilon}{3} R^{q-\alpha-1},$$

where $\lambda := \lambda_1 = \lambda_2$ is given by (7.8). Define now

$$v_k(x) := \begin{cases} u_k(x) & \text{if } x \in A \setminus B_R \\ \frac{|x|}{R} \left( u_k \left( \frac{R \frac{x}{|x|}}{x} \right) \right) - \lambda + \lambda & \text{if } x \in B_R. \end{cases}$$

(7.20)

Trivially $\{v_k\} \subset L^q(A)$ and $v_k \to u$ in $L^1(A \setminus B_R)$ whereas, since for a.e. $x \in B_R$ \n
$$|Du_k(x)|^2 = R^{-2} \left| u_k \left( \frac{R \frac{x}{|x|}}{x} \right) - \lambda \right|^2 + |D_y u_k \left( \frac{R \frac{x}{|x|}}{x} \right)|^2,$$

we infer

$$\int_{B_R} |Du_k|^q \, dx \leq c(q) \int_{\partial B_R} (R^{1-q} \cdot |u_k - \lambda|^q + R \cdot |D_y u_k|^q) \, d\mathcal{H}^{n-1}$$
and hence \( \{v_k\} \subset W^{1,q}(A) \). We now show that, using the forerunning information, for any \( r \in (0, \delta/2) \) we can find \( R \in (r/2, r) \) such that

\[
\liminf_{k \to +\infty} \int_{\partial B_R} (|Dv_k|^p + a(x)|Dv_k|^q) dx \\
\leq \int_{\partial B_R} (|Du|^p + a(x)|Du|^q) dx + O(R) + \epsilon,
\]

(7.21)

where \( O(R) \to 0^+ \) as \( R \to 0^+ \). To this aim, since \(|a(x)| \leq R^\alpha \) for \( x \in B_R \), we first estimate

\[
\int_{B_R} (|Dv_k|^p + a(x)|Dv_k|^q) dx \leq c(p,q) \left\{ R^{1-p} \int_{\partial B_R} |u_k - \lambda|^p d\mathcal{H}^{n-1} \\
+ R^{1+\alpha-q} \int_{\partial B_R \cap C^+} |u_k - \lambda|^q d\mathcal{H}^{n-1} \\
+ R \int_{\partial B_R} (|D\tau u_k|^p + a(x)|D\tau u_k|^q) d\mathcal{H}^{n-1} \right\}.
\]

(7.22)

We now make use of the following embedding result (see [42, Lemma 5.8] for a proof).

**Lemma 7.9.** If \( u \in W^{1,p}(B_\delta) \) with \( 1 \leq p < n \), \( B_\delta \subset \mathbb{R}^n \) being the \( n \)-ball of radius \( \delta \), and \( \lambda \in \mathbb{R} \), then for a.e. \( 0 < R < \delta \) we have

\[
R^{1-p} \int_{\partial B_R} |u - \lambda|^p d\mathcal{H}^{n-1} \\
\leq c(n,p) \left\{ \int_{B_R} |Du|^p dx + \left( \int_{B_R} |u - \lambda|^p dx \right)^{p/p^*} \right\}
\]

(7.23)

where \( p^* := np/(n-p) \) is the Sobolev conjugate of \( p \).

Now, condition \( B_\delta \subset \subset A \) yields that \( u(\cdot) - \lambda \in W^{1,p}(B_\delta) \). Then, by (7.18), (7.23), Sobolev embedding theorem and absolute continuity we obtain

\[
R^{1-p} \int_{\partial B_R} |u_k - \lambda|^p d\mathcal{H}^{n-1} \leq O(R) + \frac{\epsilon}{3}.
\]

(7.24)

Recall now that since \( a(\cdot)|Du|^q \in \mathcal{L}^1(A) \), by Lemma 7.3 and Morrey’s theorem [5, Thm. 5.4], since both \( \pm C^+_R \) have Lipschitz boundaries, we have

\[
|u(x) - u(y)| \leq c_0 \|Du\|_{\mathcal{L}^q(C^+_R)} |x - y|^{1-n/s} \quad \forall x, y \in \pm C^+_R,
\]

where \( c > 0 \) is an absolute constant and \( s > n \). In particular, by (7.8), with \( \lambda = \lambda_1 = \lambda_2 \), for every \( x \in \partial B_R \cap C^+ \) we then infer

\[
|u(x) - \lambda| \leq c_0 \|Du\|_{\mathcal{L}^q(C^+_R)} R^{1-n/s}.
\]

(7.25)

Now, since by (7.7) (with \( r = 1 \))

\[
c_2 := \|a(\cdot)^{-1}\|_{\mathcal{L}^{q/(s-q)}(C^+_R)} = \left( \int_{C^+_R} a(x)^{s/(s-q)} dx \right)^{(q-s)/s} < +\infty
\]

by homogeneity of \( a(x) \) we compute

\[
\|a(\cdot)^{-1}\|_{\mathcal{L}^{q/(s-q)}(C^+_R)} = c_2 R^{(n(q-s)-\alpha s)/s}.
\]
Moreover, by (7.6) (with \( \beta = 0 \)) and (7.26) we estimate
\[
\|Du\|^{q}_{L^{q}(C_{R}^{+})} \leq \|a(\cdot)\|_{L^{1}(\partial B_{R})} \cdot \|a(\cdot)^{-1}\|_{L^{1}(\partial B_{R})} c_{2} R^{n(\gamma - \alpha)}/s.
\]
As a consequence, by (7.25) we have
\[
|u(x) - \lambda|^{q} \leq c_{0}^{q} R^{n-\eta q}/s \|a(\cdot)\|_{L^{1}(C_{R}^{+})} \cdot c_{2} R^{n(\gamma - \alpha)}/s
\]
for every \( x \in \partial B_{R} \cap C^{+} \) and hence
\[
R^{1+\gamma-q} \int_{\partial B_{R} \cap C^{+}} |u - \lambda|^{q} \, dH^{n-1} \\
\leq c(n) R^{n+\gamma-q} c_{0}^{q} c_{2} R^{n-\eta q}/s \|a(\cdot)\|_{L^{1}(C_{R}^{+})} \cdot c_{2} R^{n(\gamma - \alpha)}/s
\]
\[
= C \|a(\cdot)\|_{L^{1}(C_{R}^{+})}.
\]

Then, by absolute continuity and (7.19) we obtain
\[
R^{1+\alpha-q} \int_{\partial B_{R} \cap C^{+}} |u_{k} - \lambda|^{q} \, dH^{n-1} \leq C \|a(\cdot)\|_{L^{1}(C_{R}^{+})} + \frac{\epsilon}{3} \\
\leq O(R) + \frac{\epsilon}{3}.
\]

Finally, since \( |Du|^{p} + a(\cdot) |Du|^{q} \in L^{1}(B_{\delta}) \), setting
\[
f(\rho) := \int_{\partial B_{\rho}} (|D_{\tau} u|^{p} + a(x) |D_{\tau} u|^{q}) \, dH^{n-1}, \quad 0 < \rho < \delta,
\]
by the coarea formula one has \( f(\rho) \in L^{1}(0, \delta) \). Therefore, since \( f(\rho) \geq 0, \) we have \( \lim_{\rho \to 0^{+}} \rho \cdot f(\rho) = 0 \). As a consequence, without loss of generality we can choose \( R \) so that \( R \cdot f(R) = O(R) \) and hence, by (7.17),
\[
R \int_{\partial B_{R}} (|D_{\tau} u_{k}|^{p} + a(x) |D_{\tau} u_{k}|^{q}) \, dH^{n-1} \leq O(R) + \frac{\epsilon}{3}.
\]

Then, by (7.24), (7.27) and (7.28) the right-hand side of (7.22) is smaller than \( O(R) + \epsilon \) and finally, by lower semicontinuity and (7.16), we obtain (7.21).

We finally make use of a diagonal argument, as follows. We first select \( r_{j} \) \( \leq 0 \) and \( R_{j} \in (r_{j}, 2r_{j}) \) as above; then for any fixed \( j \) via (7.20) we define \( \{ u^{(j)}_{k} \} \subset W^{1,q}(A \setminus B_{r_{j}}) \) so that \( u^{(j)}_{k} \to u \) in \( L^{1}(A \setminus B_{r_{j}}) \) and (7.16) holds with \( r = r_{j} \); we then construct \( \{ u^{(j)} \} \subset W^{1,q}(A) \) such that \( u^{(j)} \to u \) in \( L^{1}(A \setminus B_{R_{j}}) \) and (7.21) holds with \( R = R_{j} \). Finally we set \( w_{k} := u^{(k)}_{k} \), so that \( \{ w_{k} \} \subset W^{1,q}(A) \), \( w_{k} \to u \) in \( L^{1}(A) \) and by (7.21)
\[
\lim_{k \to +\infty} \int_{A} (|Dw_{k}|^{p} + a(x) |Dw_{k}|^{q}) \, dx \\
\leq \lim_{k \to +\infty} \left\{ \int_{A \setminus B_{R_{k}}} (|Du|^{p} + a(x) |Du|^{q}) \, dx + O(R_{k}) + \epsilon \right\}
\]
so that (7.15) holds, as required.

**End of the Proof of Theorem 7.6.** In order to prove Theorem 7.6 for general integrands \( f \), since we have shown that the density property (Definition 4.5) holds out of the origin, arguing as in Proposition 4.9, and keeping into account that \( Qf \equiv Cf \) in the scalar case \( N = 1 \), we obtain (7.12) where the singular measure \( \mu(u, \cdot) \) is concentrated in the origin. Finally, (7.13) follows from growth condition (7.11).
8. Another sharp example with energy concentration. In this section we describe another counterexample, probably involving the finest analysis of the paper: we show that if \( \Omega = B_1 \), the unit ball of \( \mathbb{R}^2 \), and \( f(x,z) := |z|^{p(x)} \), where \( p : \Omega \to (1, +\infty) \) is a suitable continuous exponent, energy concentration does occur in the process of relaxation in case (4.4) is violated: more precisely, following Zhikov [46] we set

\[
(8.1) \quad p(x) := 2 + \frac{x_1 x_2}{|x|} \left( \log \frac{2}{|x|} \right)^{-t}, \quad x = (x_1, x_2) \in B_1
\]

where \( 0 < t < 1 \) is fixed. In this case the assumptions of Theorem 3.3 (see Example 3.4) hold, while the ones of Theorems 4.11 and 4.12 are not satisfied (see Example 4.8). Zhikov considered in [46] the homogeneous extension \( \Pi(x) := \varphi(x/|x|) \) on \( B_1 \) of the function \( \varphi \) defined in standard polar coordinates \( x = (\cos \theta, \sin \theta) \) on \( \partial B_1 \) by

\[
(8.2) \quad \varphi(\theta) := \begin{cases} 
1 & \text{if } -\alpha \leq \theta \leq \pi/2 + \alpha \\
2(\theta + \alpha - \pi)/(4\alpha - \pi) & \text{if } \pi/2 + \alpha \leq \theta \leq \pi - \alpha \\
0 & \text{if } \pi - \alpha \leq \theta \leq 3\pi/2 + \alpha \\
2(\pi - \theta - 3\pi/2)/(\pi - 4\alpha) & \text{if } 3\pi/2 - \alpha \leq \theta \leq 2\pi - \alpha 
\end{cases}
\]

where \( 0 < \alpha \ll \pi/4 \) is a fixed small angle. Of course \( |Du|^{p(x)} \in L^1(\Omega) \), but he showed that the Dirichlet problem for the \( p(x) \)-energy with boundary condition \( \Pi \) on \( \partial B_1 \) is not regular, i.e., the infimum over \( W^{1,p(x)} \)-maps is strictly less that the infimum over smooth maps. For future purposes, we remark that the key point in Zhikov’s argument is the summability near 0 of the function \( \rho \mapsto \rho^{-1+c(\log(2\rho^{-1}))^{-t}} \) for any \( 0 < t < 1 \) and \( c > 0 \), since for some \( c_1, c_2 \equiv c_1, c_2(c) > 0 \) we have

\[
(8.3) \quad \int_0^1 \rho^{-1+c(\log(2\rho^{-1}))^{-t}} d\rho \leq c_1 \int_0^1 \rho^{-1+c(\log(2\rho^{-1}))^{-t}} d\rho
\]

Moreover, for future convenience, we take also note that

\[
(8.4) \quad \int_{\log 2}^{+\infty} x^t e^{-c_2 x^{1-t}} dx < +\infty \quad \forall t \in [0,1], \ c_2 > 0.
\]

In this section we denote by \( F(u,A) \) and \( \overline{F}(u,A) \) the functionals given by (2.1) and (2.2), respectively, where \( \Omega = B_1 \) and \( f : B_1 \times \mathbb{R}^2 \to [0, +\infty) \) is a Carathéodory function satisfying (8.7), where \( p(x) \) is given by (8.1), so that \( \overline{F}(u,A) \) satisfies the measure property (see Example 3.4). Since \( p(x) \) satisfies (4.4) out of the origin (it is actually Lipschitz continuous far from the origin), we infer that energy concentration can only occur in \( x = 0_{\mathbb{R}^2} \). More precisely, by Theorems 4.7 and 4.11 (see Example 4.8) we immediately obtain the following

**Proposition 8.1.** Let \( u \in L^1(B_1) \) be such that \( |Du|^{p(x)} \in L^1_{loc}(A) \) for some open set \( A \subset B_1 \) with \( 0_{\mathbb{R}^2} \notin A \), where \( p(x) \) is given by (8.1). Then

\[
(8.5) \quad \overline{F}(u,A) = \int_A |Du(x)|^{p(x)} \, dx.
\]

Now we define, for every \( 0 < \beta < \pi/4 \), the open cones

\[
C^\beta \equiv +C^\beta := \{ x = \rho e^{i\theta} \mid \beta < \theta < \pi/2 - \beta \}, \\
-C^\beta := \{ -x \mid x \in C^\beta \}, \quad \pm C^\beta := \pm C^\beta \cap B_r.
\]
Since inside $\pm C^\beta$ we have $p(x) > 2$ and $p(x) \to 2^+$ very rapidly as $x \to 0_{R^2}$, we are able to define the traces in the origin of a function with finite energy in the cones $\pm C_r^\beta$, see (8.6). Of course we do not have at our disposal a standard estimate of the type in Lemma 7.2, since in our case $p(x) \to 2$ (the borderline case of Sobolev embedding) as $x \to 0$; anyway we are able to prove the following

**Theorem 8.2.** (Trace theorem) Let $u \in L^1(B_1)$ be such that $|Du|^{p(x)} \in L^1_{\text{loc}}(A)$ for some open set $A \subset B_1$ with $0_{R^2} \in A$, where $p(x)$ is given by (8.1). Then for every $0 < \beta < \pi/4$ the following finite limits exist

$$
\begin{align*}
\lambda_1 := \lim_{x \to 0_{R^2}} \lim_{x \in C_r} u(x) & \quad \text{and} \quad \lambda_2 := \lim_{x \to 0_{R^2}} \lim_{x \in -C_r} u(x),
\end{align*}
$$

(8.6)

In particular, if $r > 0$ is such that $B_r \subset \subset A$, we have that $u$ is a continuous function up to the boundary of both the cones $\pm C_r^\beta$.

Thanks to Theorem 8.2, as in Theorem 7.4 we show that there is energy concentration in the origin if the traces in (8.6) take different values, for example when

$$u(x) \equiv u(x) := \varphi(x/|x|),$$

with $\varphi$ given by (8.2).

**Theorem 8.3.** Let $F(u, A)$ and $\overline{F}(u, A)$ be given by (2.1) and (2.2), with $\Omega = B_1$ and $f(x, z)$ being a Carathéodory function such that

$$c_1 |z|^{p(x)} \leq f(x, z) \leq c_2 (|z|^{p(x)} + 1),$$

(8.7)

where $p(x)$ is given by (8.1) and $c_2 > c_1 > 0$. Moreover, let $u \in L^1(B_1)$ be such that $|Du|^{p(x)} \in L^1_{\text{loc}}(A)$ for some open set $A \subset B_1$ with $0_{R^2} \in A$. Then, if $\lambda_1 \neq \lambda_2$ in (8.6), it follows that $\overline{F}(u, A) = +\infty$.

Remark 8.4. In contrast to the previous section, this time we do not give the complete representation of the relaxed functional (that is, an analog of Theorem 7.6), confining ourselves to emphasize the main concentration phenomenon in the origin. This for the sake of brevity: indeed severe technical complications intervene in the upper bound estimate for the energy in the case $\lambda_1 = \lambda_2$. Anyway it should be possible to obtain the same complete representation of the type in Theorem 7.6 also in this case.

**Proof of Theorem 8.2.** It is not restrictive to suppose

$$A = B_1 \quad \text{and} \quad \int_{B_1} |Du|^{p(x)} \, dx < +\infty.$$

Moreover, we will show existence of the first limit in (8.6), the second one being treated the same way. We remind the reader that in the following $c > 1$ goes on denoting a constant possibly varying from line to line; we shall emphasize the relevant connections.

**Step 1: dyadic type sequences.**

We consider a sequence $\{y_k\} \subset \overline{C}_r$ of the type

$$y_k := r_k (\cos \theta_k, \sin \theta_k),$$

(8.8)

where $\theta_k \in [\beta, \pi/2 - \beta]$ and $r_k \to 0^+$ is a decreasing sequence such that

$$L^{-1}/2^k \leq r_k \leq L/2^k$$

(8.9)
with $L \in [1, +\infty)$. We have

$$
|y_k - y_{k+1}| \leq |y_k| \cdot |\theta_k - \theta_{k+1}| + |r_k - r_{k+1}| \leq (\pi + 1) r_k.
$$

By applying Morrey’s theorem to the closure of the smooth set

$$
S_k := C^\beta \cap B^{L_{2-k}}_{L+1-2(k+1)},
$$

where $B^R_r := B_r \setminus B_r$, we have

$$
|u(y_k) - u(y_{k+1})| \leq c b_k |y_k - y_{k+1}|^{1-2/p_k},
$$

where

$$
p_k = 2 + c_\beta \left( \log(L^{2+2}) \right)^{-1}, \quad c_\beta := \frac{\sin(2\beta)}{2} > 0, \quad b_k := \frac{3}{p_k - 2},
$$

and $c$ is an absolute constant depending on $L$ and $\int_{S_k} (|Du|^p + 1) \, dx$, and hence on $\int_{B_L} (|Du|^p \, dx$. Note that we used the fact that $u \in W^{1,p}(S_k)$ since $p(x) \geq p_k$ whenever $x \in S_k$. For the validity of the previous inequality, and in particular the determination of the constant $b_k$, see Remark 8.5. Then for $k$ large so that $(\pi + 1) r_k < 1$, by (8.9) and (8.10) we have

$$
|u(y_k) - u(y_{k+1})| \leq c b_k |y_k - y_{k+1}|^{1-2/p_k} \leq c b_k (2^{-k})^{(c_\beta/\beta)(\log(L^{2+2}))^{-t}} \leq c b_k e^{\epsilon(\log(L^{2+2}))^{-t} \log(2^{-k})} \leq c (\log(L^{2+2}))^t e^{-\epsilon(\log(L^{2+2}))^{1-t}}
$$

where $c$ and $\hat{c}$ are positive constants depending on $\beta$, $L$ and $\int_{B_L} (|Du|^p + 1) \, dx$. Therefore

$$
\sum_{k=1}^{+\infty} |u(y_k) - u(y_{k+1})| \leq c \sum_{k=1}^{+\infty} (k + 2)^t e^{-\epsilon(\log(2)^{1-t}) (k+2)^{1-t}} < +\infty,
$$

the last series being convergent by (8.4). Observe that the constant $c$ depends on $L$ and $\int_{B_L} (|Du|^p + 1) \, dx$ and $\hat{c}$ depends on $\beta$, $L$; moreover $\hat{c} \to 0$ as $\beta \to 0$ or when $L \to +\infty$ whereas, by the definition of $b_k$, it follows that $c \to +\infty$ as $\beta \to 0$. Therefore we have that the sequence $u(y_k)$ converges to a certain limit value $l < +\infty$.

**Step 2: comparing dyadic type sequences.**

Now take $\{y^1_k\}$ and $\{y^2_k\}$ two sequences as in (8.8) satisfying (8.9) with different constants $L_1$, $L_2$ and define $L = \max\{L_1, L_2\}$. Arguing as in the previous step there exist $l_1, l_2$ such that $u(y^1_k) \to l_1$ and $u(y^2_k) \to l_2$. As in (8.10) we also deduce that

$$
|y^1_k - y^2_k| \leq c_3(L)/2^k.
$$

With the same notation as in (8.12) (with everything adapted to the new value of the constant $L$), we find

$$
|u(y^1_k) - u(y^2_k)| \leq c \frac{|y^1_k - y^2_k|^{1-2/p_k}}{p_k - 2} \leq c (k + 2)^t e^{(-\epsilon \log 2)^{-1-t} (k+2)^{1-t}} \to 0.
$$
where \( \tilde{c} \equiv \tilde{c}(\beta, L) > 0 \) and \( c \) depends both on \( L \) and \( \int_{B_1}(|Du|^p(x) + 1) \, dx \), as in previous step. Therefore we infer \( l_1 = l_2 \).

**Step 3: conclusion.**

It suffices to show that if \( \{x_k\} \subset C_\beta \setminus \{0_{\mathbb{R}^2}\} \) converges to \( 0_{\mathbb{R}^2} \) then \( u(x_k) \to l \) where \( l \) is defined as the limit of any sequence of Step 1. Note that by Step 2 we have that \( l \) does not depend on such a choice. Therefore we pick \( \lambda_1 := l \) in (8.6). In turn it suffices to show that from \( \{x_k\} \) it is possible to select a subsequence \( \{z_k\} \) such that \( u(z_k) \to l \). To this aim we let \( x_k := \tilde{\rho}_k(\cos \varphi_k, \sin \varphi_k) \) and we pass to a subsequence \( z_k := \rho_k(\cos \varphi_k, \sin \varphi_k) \) such that \( \rho_{k+1} \leq 4^{-1}\rho_k \). Next we consider the new sequence \( \{y_k\} \), built as follows. First we define \( \{\tilde{r}_k\} \) as the decreasing rearrangement of the set \( \{\rho_k \mid k \in \mathbb{N}\} \cup \{2^{-k} \mid k \in \mathbb{N}\} \equiv A \cup B \). Then, from this sequence we build yet another sequence by dropping certain terms: we delete \( \tilde{r}_h \) if and only if \( \tilde{r}_h \in B \setminus A \) and moreover \( \tilde{r}_{h+1} \in A \). (Roughly speaking, after rearranging the pieces of the original sequence \( A \) with the ones from the dyadic type sequence \( B \), we delete all the terms from \( B \setminus A \) which come immediately before a term of the sequence \( A \).) Observe that since we have chosen \( \{\rho_k\} \) such that \( \rho_{k+1} \leq 2^{-k} \), then between any two terms of the type \( 2^{-k} \) and \( 2^{-k-1} \) it falls at most one term of \( A \setminus B \); moreover, in the new sequence all the terms of the original one \( A \) do appear and, in the case they go to zero faster than \( 2^{-k} \), they are interpolated by the dyadic numbers). Relabelling, we finally get \( \{r_k\} \); then we set \( y_k := r_k(\cos \theta_k, \sin \theta_k) \) where \( \theta_k = \varphi_j \) if \( r_k \in A \) and \( r_k = \rho_j \) for some \( j \in \mathbb{N} \), while \( \theta_k = \pi/4 \) otherwise. The sequence \( \{y_k\} \) is now of the type in (8.8), for a suitable constant \( L \). Therefore by Step 1 and Step 2 it follows that \( u(y_k) \to l \) and so does \( \{u(z_k)\} \), being a subsequence of \( \{u(y_k)\} \). The proof is now complete.

**Remark 8.5.** Here we briefly justify the validity of the inequality (8.11). It is well known that if we let

\[
R := (0,1) \times (\beta, \pi/2 - \beta)
\]

then for any function \( v \in W^{1,s}(R) \), with \( s > 2 \), Morrey’s imbedding inequality takes the form

\[
|v(x) - v(y)| \leq \frac{c}{1 - 2/s} |x - y|^{1 - 2/s} \left( \int_R |Dv(z)|^s \, dz \right)^{1/s},
\]

where \( c \) is an absolute constant. This can be inferred from [5, page 110]; similar inequalities are valid for general parallelepipeds in higher dimensions. Then we infer (8.11) from the previous inequality, letting of course \( s := p_k \), via a simple change of variable argument and the use of polar coordinates. The details follow. Using the non-singular map

\[
\phi_k : (\rho, \theta) \in R \to g_k(\rho)(\cos \theta, \sin \theta) \in S_k,
\]

\[
g_k(\rho) := (L2^{-k} - L^{-1}2^{-(k+1)})\rho + L^{-1}2^{-(k+1)},
\]

it turns out that

\[
\Phi_k \equiv \phi_k^{-1} : (x_1, x_2) \in S_k \to (f_k(|x|), \arctan(x_2/x_1)) \in R,
\]

where

\[
f_k(\rho) := (L2^{-k} - L^{-1}2^{-(k+1)})^{-1}(t - L^{-1}2^{-(k+1)}),
\]

and the following relations hold

\[
||D\phi_k||_\infty \leq c2^{-k}, \quad ||D\Phi_k||_\infty \leq c2^k
\]
\[
\det D\phi_k = g_k(\rho)(L2^{-k} - L^{-1}2^{-(k+1)}) > 0, \quad |\det D\phi_k|^{-1} \leq c4^k,
\]

where \( c \equiv c(L) \) denotes an absolute constant independent of \( k \in \mathbb{N} \). Finally, if \( u \in W^{1,s}(S_k) \) then \( v = u \circ \phi_k \in W^{1,s}(R) \); therefore, using (8.13), the change of variable formula and the previous relations we get, with \( x, y \in S_k \) and \( \phi_k(x) = x \) and \( \phi_k(y) = y \),

\[
|u(x) - u(y)| = |v(\tilde{x}) - v(\tilde{y})| \\ \leq \frac{CS}{s-2} |\tilde{x} - \tilde{y}|^{1-2/s} ||Dv||_{L^s(R)} \\ \leq \frac{CS}{s-2} ||D\Phi_k||^{1-2/s} ||D\phi_k||_\infty ||\det D\phi_k||^{-1/|s|} |x - y|^{1-2/s} \\ \times \left( \int_R |Du(\phi_k(\rho, \theta))| |s| \det D\phi_k \, d\rho \, d\theta \right)^{1/s} \\ \leq \frac{CS}{s-2} |x - y|^{1-2/s} \left( \int_{S_k} |Du(x)|^s \, dx \right)^{1/s},
\]

and (8.11) follows by taking \( s := p_k \), observing that \( \max p(x) \leq 3 \); also, from the previous estimates, the constant \( c \) clearly depends on \( L \) and blows up when so does \( L \), anyway, \( c \) is independent of \( k \in \mathbb{N} \).

**Proof of Theorem 8.3.** As for the proof of Theorem 7.6 we shall restrict, without loss of generality, to the case \( f(x, z) := |z|^{p(x)} \). Arguing as for Theorem 7.4, it then suffices to prove the following

**Proposition 8.6.** Let \( \{u_j\} \subset C^1(B_r) \) be such that \( u_j \rightharpoonup u \) in \( L^1(B_r) \) and a.e. in \( B_r \) and

\[
\sup_j \int_{C_r^\beta} |Du_j|^{p(x)} \, dx + \sup_j \int_{-C_r^\beta} |Du_j|^{p(x)} \, dx < +\infty.
\]

Then, possibly passing to subsequences, \( u_j \rightharpoonup u \) uniformly on the closure of \( C_r^\beta \cup -C_r^\beta \). Therefore \( \lambda_1 = \lambda_2 \) in (8.6).

**Proof.** Due to a.e. convergence \( u_j \rightharpoonup u \), by Ascoli-Arzelà theorem it suffices to show that \( \{u_j\} \) is equi-uniformly continuous on the closure of \( C_r^\beta \) and of \( -C_r^\beta \). We make use of the following

**Lemma 8.7.** Let \( v \in C^1(B_r) \) be such that \( \int_{C_r^\beta} |Dv|^{p(x)} \, dx < +\infty \); there exist a non-decreasing non-negative function \( g : \mathbb{R}^+ \to \mathbb{R}^+ \), depending on \( p(x) \) and \( \beta \), with \( g(R) \to 0^+ \) as \( R \to 0^+ \) and constants \( \tilde{c}_1 \), \( \tilde{c}_2 \) depending only on \( \beta \), both independent of the function \( v \), such that

\[
|v(x) - v(y)| \leq \tilde{c}_1 \min\{B(R)d_\alpha(\log(2/R))^{-t}, g(S)\}
\]

for every \( x, y \in \overline{C_r^\beta} \setminus \{0\} \). Here \( d := |x - y| \leq 1, S := \max\{|x|, |y|\}, R := \min\{|x|, |y|\} \) and \( B(R) \) is a function only depending on \( R \) and such that it is bounded on every interval of the type \([R_0, 1], R_0 > 0\). The same result holds replacing \( \overline{C_r^\beta} \) by \( -\overline{C_r^\beta} \).

**Proof.** We only treat the case of \( \overline{C_r^\beta} \), the proof for \( -\overline{C_r^\beta} \) being similar. By applying Morrey’s theorem to the set \( C_{R/2}^\beta \setminus \overline{C_r^\beta} \), we infer

\[
|v(x) - v(y)| \leq c B(R)|x - y|^{c_\beta(\log(4/R))^{-t}}
\]

where \( c_\beta := \sin(2\beta)/2 > 0 \), \( B(R) \) only depends on \( R \) and the constant \( c \) depends on \( \int_{C_r^\beta} |Du|^{p(x)} \, dx \); observe that \( B(R) \) is given by Morrey’s imbedding inequality.
and it turns out that $B(R) \to +\infty$ when $R \to 0$: this gives the first estimate for (8.14). In order to get the second one we argue as follows: if $k \in \mathbb{N}$ is such that $2^{-k} \le |x| < 2^{-k+1}$, and $x_i := 2^{-i}(\cos(\pi/4), \sin(\pi/4))$, arguing as in Theorem 8.2 (compare Remark 8.5), and setting $p(\rho) := 2 + c_\beta (\log(2/\rho))^{-t}$, we also infer

$$|v(x) - v(0_{B^2})| \le |v(x) - v(x_k)| + \sum_{i=k}^{+\infty} |v(x_i) - v(x_{i+1})|$$

$$\le c \frac{|x - x_k|^{1-2/p(2^{-k})}}{p(2^{-k}) - 2} + c \sum_{i=k}^{+\infty} |x_i - x_{i+1}|^{1-2/p(2^{-i+1})}$$

$$\le c \sum_{i=k}^{+\infty} \log(2^{i+1})^t \frac{(c_\beta/3)\log(2^{i+1})^{-t}}{p(2^{-i+1}) - 2}$$

$$\le c \sum_{i=k}^{+\infty} (i + 1)^t e^{-c_\beta \log(2)^{2^{-i+1}}}$$

$$=: c A_k .$$

Observe that, as for Theorem 8.2, the constant $c > 0$ depends only on $\int_{B^2} |Dv|^p dx$ and $\beta$, while $c_\beta > 0$ only depends on $\beta$, moreover $c \to +\infty$ when $\beta \to 0$ while $c_\beta \to 0$ when $\beta \to 0$; this follows by Remark 8.5 replacing $S_k$ by $\tilde{S}_k := C^2 \cap B^{2-k+1}_2$ (thereby taking $L = 1$); as before, it has been used that $u \in W^{1,p}(2^{-k}) (\tilde{S}_k)$.

In the same way, if $2^{-h} \le |y| < 2^{-h+1}$ for some $h \in \mathbb{N}$, then

$$|v(y) - v(0_{B^2})| \le A_h$$

and by triangle inequality

$$|v(x) - v(y)| \le 2 \max\{A_k, A_h\} = 2 A_{\min\{k,h\}} .$$

Then, since $A_k \to 0$ as $k \to +\infty$, see (8.4), we conclude setting

$$g(S) := \begin{cases} A_k & \text{if } 2^{-k} \le S < 2^{-k+1} \\ A_1 & \text{if } S \ge 1/2 ; \end{cases}$$

clearly, $g(S) \to 0$ if and only if $S \to 0$. \qed

End of Proofs of Proposition 8.6 and Theorem 8.3. Let $\{u_j\} \subset C^1(B_r)$ be the sequence as in the statement; as explained at the beginning of the proof of Proposition 8.6, it suffices to prove that $\{u_j\}$ is equi-uniformly continuous on the closure of $C^2$; in turn this is equivalent to prove the equi-uniform continuity on $C_r^2 \setminus \{0\}$. We have to prove that for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that whenever $x, y \in C_r^2 \setminus \{0\}$ satisfy $|x - y| \le \delta$ then $|u_j(x) - u_j(y)| \le \epsilon$, for every $j \in \mathbb{N}$. We argue by contradiction; if it were not so, there would exist $\epsilon_0 > 0$ such that for any positive integer $h$ there exist $j(h) \in \mathbb{N}$, $x_h, y_h \in C_r^2 \setminus \{0\}$ and a function $v_h := u_{j(h)}$ such that

$$d_h := |x_h - y_h| \le 1/h$$

but

$$|v_h(x_h) - v_h(y_h)| > \epsilon_0 .$$

By the estimate (8.14), if we set $S_h := \max\{|x_h|, |y_h|\} > 0$, then (8.16) implies that $g(S_h) \ge \epsilon_0/c_1 > 0$ (where $c_1$ is independent of $h \in \mathbb{N}$, as observed in Lemma 8.7)
and so there exists $S_0 > 0$ such that $S_h \geq S_0$ for every $h \in \mathbb{N}$. In turn, if we let $R_h := \min\{|x_h|, |y_h|\} > 0$, by (8.15) we get that $R_h \geq R_0 := S_0/2 > 0$ for every index $h > 2/S_0$. Hence, by (8.14) we get

$$B(R_h)d_h \tilde{c}_2(\log(2/R_h))^{-1} < \tilde{B}d_h \tilde{c}_2(\log(2/R_0))^{-1} \to 0,$$

and applying again (8.14) and (8.16) yields

$$0 < \epsilon_0 \leq \limsup_{h \to +\infty} |v_h(x_h) - v_h(y_h)| \leq \lim_{h \to +\infty} B(R_h)d_h \tilde{c}_2(\log(2/R_h))^{-1} = 0$$

which is impossible. Therefore $\{u_j\}$ is equi-uniformly continuous and the proofs are complete.

REFERENCES


