On an integral representation of a class of Kapteyn (Fourier–Bessel) series: Kepler’s equation, radiation problems and Meissel’s expansion

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A B S T R A C T

In this paper, an integral representation of a class of Kapteyn series is proposed. Such a representation includes the most used series in practical applications. The approach uses the property of uniform convergence of the considered class and the integral representation of the Bessel functions. The usefulness of the proposed method is highlighted by providing an integral solution of Kepler’s equation and of some Kapteyn series arising in radiation problems. Moreover it allows us to generalize a result due to Meissel.

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1. Introduction

Kapteyn series [1] and related ones are frequently used in astronomical problems. It is well known that the solution of Kepler’s problem can be expressed in terms of Kapteyn series [2]. Kepler’s equation is probably the most famous transcendental equation in all branches of science and, although it has a simple form, for a long time an analytical solution was thought impossible to find. The relevance of Kapteyn series leads to a very large set of applications for which it is often helpful in obtaining explicit solutions: modern theory of electromagnetic radiation [3,4], Markov chains and queuing theory [5,6], atmospheric phenomena [7,8], spectral analysis of power conversion systems [9] and the analysis of ripple stability in Pulse-Width-modulated systems [10], just to name a few. It is well known that Kapteyn series often suffer from convergence problem. Many authors have proposed methods to improve convergence speed as in [4] where Wynn’s epsilon method [11] is used to accelerate these series convergence. In this paper a new approach is presented that leads to an integral representation of a class of Kapteyn series. Such a class includes Kepler’s problem and some Kapteyn series arising in radiation problems [4]. A generalization of a result due to Meissel (cited in [1]) is also given.

2. A class of Kapteyn series

Let us consider the following class of Kapteyn series

\[ \gamma(a, b, c, d, g, s, \nu, \phi) = \sum_{n=1}^{\infty} \frac{a^n}{(dn+b)^{s}} J_{gn+\nu} [(gn+\nu) \epsilon] \sin (2\pi nc + \phi) \]  

(1)

where \( J_n(z) \) is the Bessel function of the first kind defined as [1]

\[ J_n(z) = \frac{1}{\pi} \int_0^\pi \cos [n\theta - z \sin(\theta)] d\theta. \]  

(2)

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Taking into account the expression of Bessel function, the series in (1) can be rearranged as

\[ \gamma = \frac{1}{\pi} \sum_{n=1}^{\infty} \int_{0}^{\pi} \frac{a^n}{(dn+b)^r} \cos[(gn + \nu)(\theta - \epsilon \sin(\theta))] \sin(2\pi nc + \phi) \, d\theta \]

where the dependence of \( \gamma \) on parameters is omitted for ease of notation.

Eq. (3) can be rewritten as

\[ \gamma = \frac{1}{\pi} \int_{0}^{\pi} \sum_{n=1}^{\infty} \frac{a^n}{(dn+b)^r} \cos[(gn + \nu)(\theta - \epsilon \sin(\theta))] \sin(2\pi nc + \phi) \, d\theta \]

since the series on the right is uniformly convergent as proved in Lemma 1 (see Appendix). An integral expression for the series in Eq. (4) is presented in the following theorem.

**Theorem 1.** Let \( \gamma_c(\theta) \) be the function defined as

\[ \gamma_c(\theta) = \frac{a}{4d^3} \left\{ -e^{(\alpha_1(\theta) + \beta_1(\theta))} \Phi \left[ ae^{\alpha_1(\theta), s, \frac{b}{d} + 1} \right] + e^{-j\alpha_1(\theta) + \beta_1(\theta))} \Phi \left[ ae^{-j\alpha_1(\theta), s, \frac{b}{d} + 1} \right] 
\]

\[ \text{where} \]

\[ \alpha_1(\theta) = 2c\pi - g\theta + e\sin(\theta), \]

\[ \alpha_2(\theta) = 2c\pi + g\theta - e\sin(\theta), \]

\[ \beta_1(\theta) = \phi - \theta v + e v \sin(\theta), \]

\[ \beta_2(\theta) = \phi + \theta v - e v \sin(\theta), \]

\( \Phi \) is the Lerch transcendent function [12] and \( j \) is the imaginary unit.

The series in (4) is equivalent to

\[ \gamma = \frac{1}{\pi} \int_{0}^{\pi} \gamma_c(\theta) \, d\theta \]

for \( \text{Re} \left[ \frac{b}{d} + 1 \right] > 0 \) and one of the following conditions satisfied:

- \( |a| \leq 1, \text{Re}[\text{ae}^{\alpha_i(\theta)}]|_{i=1,2} \neq 1, \text{Re}[s] > 0, \)
- \( \text{ae}^{\alpha_1(\theta)} = 1, \text{Re}[s] > 1 \) or \( \text{ae}^{\alpha_2(\theta)} = 1, \text{Re}[s] > 1. \)

**Proof.** By applying the well-known Euler’s formula and by standard algebra it is possible to rewrite Eq. (4) as

\[ \gamma = \frac{1}{4\pi d^3} \int_{0}^{\pi} \sum_{n=1}^{\infty} \frac{a^n}{(n + \frac{b}{d})^{r+1}} \left\{ e^{-j[n\alpha_1(\theta) + \beta_1(\theta)]} - e^{j[n\alpha_1(\theta) + \beta_1(\theta)]} + e^{-j[n\alpha_2(\theta) + \beta_2(\theta)]} - e^{j[n\alpha_2(\theta) + \beta_2(\theta)]} \right\} \, d\theta. \]

Rearranging the terms in Eq. (8), the proof follows by the definition of the Lerch transcendent function [12]

\[ \Phi(z, t, h) = \sum_{n=0}^{\infty} \frac{z^n}{(n + h)^r} \]

and the fact that

\[ \sum_{n=1}^{\infty} \frac{z^n}{(n + h)^r} = z\Phi(z, t, h + 1). \]

Hypotheses on parameters \( s, b, d, a \) and \( \epsilon \) are related to the following conditions that guarantee \( \Phi(z, t, h) \) to be analytic, i.e. \( \text{Re}[h] > 0 \) and either \( |z| \leq 1, z \neq 1, \text{Re}[t] > 0 \) or \( z = 1, \text{Re}[t] > 1 \) [12].

**Remark 1.** Note that if \( s = 1 \) and there is a value of \( \theta \in (0, \pi) \) such as \( \text{ae}^{\alpha_1(\theta)} = 1 \) or \( \text{ae}^{\alpha_2(\theta)} = 1 \) then the function \( \gamma_c(\theta) \) has a singular point in \( \theta \).
3. Kepler’s problem

In this section the most famous transcendental equation

\[ E - \epsilon \sin(E) = M \]  \hspace{1cm} (11)

is analyzed. Without loss of generality, Kepler’s problem is to solve (11) for \( E \in (0, \pi) \), given \( M \in (0, \pi) \) and \( \epsilon \in (0, 1) \) [2]. An exact analytical solution for (11) is extremely important since it assumes a key role in a wide range of applications in many fields. The classical solution in terms of Kapteyn series [1,13] is

\[ E = M + \sum_{n=1}^{\infty} \frac{2}{n} \gamma(n \epsilon \sin(nM)) \] \hspace{1cm} (12)

which can be obtained by Eq. (1) as \( \gamma \left( 1, 0, \frac{M}{2\epsilon}, 1, 1, \epsilon, 0, 0 \right) \). By using the results stated in Theorem 1 and the Jonquière function \( Li_s(x) \), commonly called the polylogarithm function, that is a particular case of the Lerch function, i.e. \( Li_s(z) = z \Phi(z, s, 1) \), an integral representation of Kepler’s solution can be expressed as

\[ E = M + \frac{1}{2\pi} \int_{0}^{\pi} \left\{ \ln[1 - e^{\frac{1}{2}x(\theta + M)]} - \ln[1 - e^{-\frac{1}{2}x(\theta - M)]} + \ln[1 - e^{-\frac{1}{2}x(\theta + M)]} \right\} \right\} d\theta \] \hspace{1cm} (13)

where the relationship between polylogarithm function and ordinary natural logarithm, namely \( Li_s(x) = -\ln(1-x) \), is considered. The logarithm of a complex number can be expressed in terms of more elementary functions by the equation \( \ln(z) = \ln|z| + j \arg(z) \) so that Eq. (13) becomes

\[ E = M + \frac{1}{\pi} \int_{0}^{\pi} \left\{ \tan \left( \frac{\pi - M + \theta - \epsilon \sin(\theta)}{2} \right) \right\} d\theta, \] \hspace{1cm} (14)

Note that \( \frac{\pi - M + \theta - \epsilon \sin(\theta)}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2}) \), therefore Eq. (14) can be rewritten as

\[ E = \frac{\epsilon}{\pi} + \frac{1}{4}(2M + \pi) + \frac{1}{\pi} \int_{0}^{\pi} \left\{ \tan \left( \frac{\pi - M + \theta - \epsilon \sin(\theta)}{2} \right) \right\} d\theta. \] \hspace{1cm} (15)

Since the term \( \frac{\pi - M + \theta - \epsilon \sin(\theta)}{2} \) belongs to the interval \( (0, \pi) \) then it is not possible to use the well-known relationship between the inverse and the direct trigonometric function [15]

\[ \arctan[\tan(x)] = x - n\pi, \quad n\pi - \frac{\pi}{2} < x < n\pi + \frac{\pi}{2}, \]

because the range of \( x \) changes with \( \theta \).

Remark 2. Taking into account the statement of the Remark 1, it is easy to show that both the integrand in Eqs. (14) and (15) have a discontinuity point in \( (0, \pi) \) which coincides with the solution. Moreover the integrand in Eq. (14) is a piecewise constant function with a change of sign in the two sections. These facts allow, among other things, to solve Kepler’s problem via a bisection algorithm on the integrand in Eq. (14).

Remark 3. It is important to note that Eq. (14) also covers the parabolic case \( (\epsilon = 1) \) of Kepler’s problem, unlike other solutions presented in literature [2].

4. Other applications

4.1. Radiation problems

In [4] the following series arising in radiation problems are considered

\[ F^+ = \sum_{n=1}^{\infty} \frac{\xi_n^2(nb)}{n} \] \hspace{1cm} (16)

\[ F^- = \sum_{n=1}^{\infty} (-1)^n \frac{\xi_n^2(nb)}{n}. \]

In the aforementioned paper, integral forms for both summations:

\[ F^+ = -\frac{1}{\pi^2} \int_{0}^{\pi} d\phi \int_{0}^{\pi} d\theta \ln \left[ \frac{\sin^2(\theta - b \cos \phi \sin \theta) \sin^2(\theta + b \cos \phi \sin \theta)}{\sin^4 \theta} \right] \] \hspace{1cm} (17)
and
\[
F^- = -\frac{1}{\pi^2} \int_0^{\pi} d\phi \int_0^{\pi} d\theta \ln \left[ \frac{\sin^2(\theta - b \cos \phi \cos \theta) \sin^2(\theta + b \cos \phi \cos \theta)}{\sin^4 \theta} \right]
\] (18)
are provided. Taking into account the equality \[4\]
\[
g^2_{2n}(nb) = \frac{2}{\pi} \int_0^{\pi} d\psi f_{2n}[2nb \cos(\psi)],
\] (19)
the proposed approach, by considering \(\gamma(1, 0, 0, 1, b \cos(\psi), 0, 0)\) for \(F^+\) and \(\gamma(-1, 0, 0, 1, b \cos(\psi), 0, 0)\) for \(F^-\), leads to the following integral forms
\[
F^+ = -\frac{2}{\pi} \int_0^{\pi} d\psi \int_0^{\pi} d\theta \ln[2 \sin(\theta - b \cos \psi \sin \theta)]
\] (20)
and
\[
F^- = -\frac{2}{\pi} \int_0^{\pi} d\psi \int_0^{\pi} d\theta \ln[2 \cos(\theta - b \cos \psi \sin \theta)].
\] (21)

4.2. Meissel’s expansion

It is known \[1\] that the series
\[
\sum_{n=1}^{\infty} \frac{J_{2n}(2\pi e)}{n^{2m}},
\] (22)
with \(e \in (0, 1)\) is a polynomial in \(e\) and Meissel gave its values in the case \(m = 1, 2, 3, 4, 5\) \[16\]. Considering \(\gamma(1, 0, 0, 1, 2m, e, 0, 0)\), the proposed approach generalizes Meissel’s results providing an integral representation of the considered polynomial for a generic \(m\)
\[
P_m(e) = \frac{(-1)^{m+1}(2\pi)^{2m-1}}{(2m)!} \int_0^{\pi} B_{2m} \left( \frac{\theta - e \sin \theta}{\pi} \right) d\theta
\] (23)
with \(B_n(x)\) the Bernoulli polynomial that is related to the polylogarithm function by the following equation \[12\]
\[
\text{Li}_n(e^{2\pi i x}) + (-1)^n \text{Li}_n(e^{-2\pi i x}) = -\frac{(2\pi i)^n}{n!} B_n(x)
\] (24)
which holds for \(n = 0, 1, \ldots\) and for \(0 \leq \text{Re}[x] < 1\) if \(\text{Im}[x] \geq 0\), and for \(0 < \text{Re}[x] \leq 1\) if \(\text{Im}[x] < 0\).

Appendix

**Lemma 1.** The series
\[
\tilde{\gamma} = \sum_{n=1}^{\infty} \frac{a^n}{(n+b)^s} \cos \left[ (gn + \nu)(\theta - e \sin(\theta)) \right] \sin (2\pi nc + \phi)
\] (A.1)
is uniform convergent for \(s\) integer, \(s \geq 1\), \(|a| \leq 1\), \(b \geq 0\) and
\[
2\delta \leq 2\pi c - g\theta + eg \sin(\theta) \leq 2\pi - 2\delta,
\] (A.2)
\[
2\delta \leq 2\pi c + g\theta - eg \sin(\theta) \leq 2\pi - 2\delta
\] (A.3)
where \(\delta\) is an arbitrary small positive constant.

**Proof.** The proof follows by using the Hardy’s test \[17\]. The series
\[
\sum_{n=1}^{\infty} a_n(z)f_n(z)
\] (A.4)
converges uniformly if \(\sum_{n=1}^{p} a_n(z) \leq k\) where \(a_n(z)\) is real, \(k\) is finite and independent of \(p\) and \(z\), and if \(f_n(z) \geq f_{n+1}(z)\) and \(f_n(z) \to 0\) uniformly as \(n \to \infty\). Eq. (A.1) can be rewritten by algebraic manipulations as
\[
\tilde{\gamma} = \sigma_1 + \sigma_2
\] (A.5)
with

\[ \sigma_i = \sum_{n=1}^{\infty} \frac{a^n}{(n+b)^2} \sin[n\alpha_i + \beta_i] \quad i = 1, 2 \]  

(A.6)

where

\[ \alpha_1 = 2c\pi - g\theta + \epsilon g \sin(\theta) \quad \beta_1 = \phi - \theta v + \epsilon v \sin(\theta), \]  

(A.7)

and

\[ \alpha_2 = 2c\pi + g\theta - \epsilon g \sin(\theta) \quad \beta_2 = \phi + \theta v - \epsilon v \sin(\theta). \]  

(A.8)

By considering \( f_n(z) = \frac{a^n}{(n+b)^2} \), the conditions on \( f_n(z) \) function hold if \( |a| \leq 1 \) and \( b \geq 0 \) with \( s \) integer, \( s \geq 1 \). Taking into account the following equality \[ 15 \]

\[ \sum_{\xi=1}^{p} \sin(x + \xi y) = \sin\left(x + \frac{p+1}{2} y\right) \sin\left(\frac{p y}{2}\right) \cosec\left(\frac{y}{2}\right) \]  

(A.9)

it is straightforward to obtain

\[ \left| \sum_{n=1}^{\infty} \sin[n\alpha_i + \beta_i] \right| \leq \left| \cosec\left(\frac{\alpha_i}{2}\right) \right| \leq \left| \cosec(\delta) \right|, \quad i = 1, 2 \]  

(A.10)

from which the proof follows. □

References