Abstract—This note presents a new algorithm that is designed to identify the frequency, magnitude, phase and offset of a biased sinusoidal signal. The structure of the algorithm includes an orthogonal system generator based on a second-order generalized integrator. The proposed strategy has the advantages of a fast and accurate signal reconstruction capability and a good rejection to noise.

Index Terms—Sinusoidal parameters estimation, continuous least-squares, adaptive control.

I. INTRODUCTION

The estimation problem of the parameters set of a sinusoidal signal is a crucial task in a wide range of applications such as control theory [1], [2], signal processing [3], biomedical engineering [4], instrumentation and measurements, power systems [5], [6], [7], to name just a few. The problem is relevant in power systems area where grid-connected devices require an accurate and fast detection of the phase angle, amplitude and frequency of the utility voltage to assure the correct generation of the reference signals. Especially in this case, a relevant issue associated with grid-connected systems, is the presence of an offset in the measured grid voltage. This voltage offset is typically introduced by the measurements and data conversion processes and causes an error, for the estimated parameters, at the same frequency of the grid voltage. As a consequence, a filtering process can be performed at the expense of dynamic performance degradation [8]. Therefore, a very attractive challenge is the synthesis of an accurate method for the on line estimation of the entire parameters set of a biased sinusoidal signal. Several papers that address this problem can be found in the recent literature. Among them, a large part deals with the case of multi-sinusoidal estimation problem [9], [10], [11], [12], [13], [14]. Although the estimation of a single biased sinusoid is a sub-case of this general problem, however many of these methods fail in presence of a bias term since they require signals with strictly positive frequencies and they can not be adapted to deal with the problem here discussed. For these reasons many techniques are proposed to deal with a biased sinusoidal signal [8], [15], [16], [17]. This paper belongs to this category and it provides an estimation algorithm for the entire set of parameters.

In [15] a seventh-order global estimator is presented for the recovery of frequency, amplitude and offset. However it is not able to reconstruct the input signal as it does not estimate the initial phase angle. Moreover the presence of the quadratic term of the input signal may heavily affect the estimation in case of noisy data. In [16] a new approach to the problem of globally convergent frequency estimator is proposed where the estimator is represented by a fourth-order system. In [17] a scheme is proposed to estimate offset, amplitude and frequency. The main advantage of this method, with respect to similar schemes, is to be small and simple, showing good performances in the simulations [18]. In fact, a frequency parameter adaptive law is proposed that is robust with regard to unaccounted disturbances. However, as remarked in [18], the tuning procedure should be done carefully. Amplitude and bias estimators have been also presented but bad transients could be obtained due to possible divisions by zero. Therefore, these estimators should not be taken into account while not close to steady-state conditions.

The most commonly employed approach to estimate the entire set of parameters uses phase-locked-loop (PLL) topologies [19], [20], [21]. The structure of the main PLL topology is a feedback control system that automatically adjusts the phase of a locally generated signal to match the phase of the input signal. The main difference among single-phase PLL topologies consists in the orthogonal voltage system generation subsystem. In [19] an interesting PLL scheme is proposed to reconstruct an unbiased sinusoid by estimating in-phase and quadrature-phase amplitudes of the fundamental component of the input signal. In order to deal with the case of biased sinusoid signal, in [8] and references therein, an effective method is used to create orthogonal reference signals based on a second-order generalized integrator (OSG-SOGI) according to the information about the signal frequency that can be obtained by using a PLL system in a feedback closed-loop. The reliability of such a method has been tested in a wide range of practical scenarios but the dependence on initial conditions of the estimate may limit its use. Other applications of OSG-SOGI can be found in [22], [24].

Based on previous results in [25], an estimation scheme, relying on OSG-SOGI, is here presented. A continuous-time least-squares algorithm is used to estimate both bias and frequency. The remaining parameters are obtained via simple relationships between the OSG-SOGI output signals. The exponential convergence of this estimator is proved. The resulting estimator permits to reconstruct the unknown input signal with satisfactory accuracy. To properly test the proposed method, a comparison makes sense with techniques that directly deal with biased sinusoid. In the following the method presented in [17] and the PLL-based one presented in [8] are considered. In particular, as far as our knowledge is concerned, the method used in [8] is the most recently discussed one that is able to match the phase angle of a biased sinusoid. Unlike OSG-SOGI based approach, the method in [17] does not provide the estimation of the initial phase angle. Furthermore, the OSG-SOGI method is less sensitive to initial conditions choice than the method in [8] that requires an initial condition sufficiently close to the unknown frequency to be estimated.

The paper is organized as follows: Section II presents OSG-SOGI scheme; in Section III the estimation method is discussed; Section IV contains simulation results. Finally Section V is devoted to conclusions.
II. OSG BASED ON SOGI

The closed-loop diagram representing the OSG-SOGI is depicted in Fig. 1. In order to generate two orthogonal signals, the reference signal $v(t)$ and the resonant frequency $\omega_c$ are needed in input. In the operating mode, if the resonant frequency is equal to the reference signal frequency, OSG-SOGI generates two sine waves ($v_1(t)$ and $v_2(t)$) that have the same magnitude of $v(t)$ and with a phase shift of $\pi/2$ each other. Moreover, $v_1(t)$ is in phase with the fundamental of the input signal. The input signal $v(t)$ is the biased sinusoid

$$ v(t) = A_0 + A_c \sin(\rho_c(t)) $$

where $\rho_c(t) = \omega_c t + \xi_c$ is the phase angle and $\{A_0, A_c, \omega_c, \xi_c\}$ are the unknown parameters to be estimated.

![Fig. 1. Block diagram of the Orthogonal Signals Generator based on SOGI.](image)

The OSG-SOGI system is governed by the following set of differential equations:

$$ \frac{dq_2(t)}{dt} = \omega_s v_1(t) $$

$$ \frac{dv_1(t)}{dt} = K_s \omega_s (v(t) - v_1(t)) - \omega_s q_2(t) $$

$$ v_2(t) = q_2(t) - K_s (v(t) - v_1(t)). $$

The gain $K_s$ affects the bandwidth of the OSG-SOGI. Small values of the frequency $\omega_s$ result in a slowdown of the dynamic response. In such a case, the steady-state condition for the OSG-SOGI method is obtained for sufficiently long observation times. On the other hand, high values of the resonant frequency permit to quickly reach the steady-state condition obtaining estimations for the unknown parameters even for small observation times. As it is evident, the orthogonal component $q_2(t)$ is directly affected by the presence of an offset, which does not appear in $v_1(t)$ and $v_2(t)$ because of the derivative actions on the input signal $v(t)$. In order to simplify the estimation algorithm analysis, the response of the OSG-SOGI is approximated as its sinusoidal steady-state response. Even if the input signal parameters may be time-varying, they will be assumed to vary slowly enough that the approximation is valid. Moreover, due to the structure of the OSG-SOGI poles $p_{1,2} = -K_s \pm \sqrt{K_s^2 - 4} \omega_s/2$, by choosing $K_s \in (0, 2)$, the transient response decays as $e^{-K_s \omega_s/2}$. It is effortless to show that the output signals converge exponentially fast to the following steady-state signals:

$$ v_{1,\infty}(t) = m_1 A_c \sin(\omega_c t + \xi_c + \xi) $$

$$ v_{2,\infty}(t) = -m_1 A_c \cos(\omega_c t + \xi_c + \xi) $$

$$ q_{2,\infty}(t) = A_0 K_s - m_2 A_c \cos(\omega_c t + \xi_c + \xi) $$

where

$$ m_1 = \frac{K_s \omega_c}{\sqrt{(\omega_s^2 - \omega_c^2)^2 + K_s^2 \omega_s^2 \omega_c^2}} $$

and

$$ m_2 = m_1 \frac{\omega_s}{\omega_c} $$

with

$$ \xi = \text{sgn} [\omega_s - \omega_c] \pi / 2 - \text{atan} \left( \frac{K_s \omega_c \omega_s}{\omega_s^2 - \omega_c^2} \right). $$

The sign function $\text{sgn}(\cdot)$ is defined as

$$ \text{sgn}(x) = \begin{cases} +1 & \text{iff } x \geq 0, \\ -1 & \text{iff } x < 0. \end{cases} $$

Remark 1: Generally, the OSG-SOGI structure requires an adaptive tuning with respect to its resonant frequency. This can be achieved by adjusting the resonant frequency of the SOGI on-line using, for example, the frequency provided by the feedback control loop of a PLL structure as proposed in [8] or the frequency tracking method based on modulating functions as proposed in [22], [23]. In the next section, it will be shown that the unknown parameters can be estimated through the measurements of the OSG-SOGI output signals without any adaption of the resonant frequency.

III. ESTIMATION METHOD

In the nominal case, i.e. when the input signal is not affected by noise, Eqs. (6), (7) lead to the following linear relationship:

$$ y(t) = \phi^T(t) \theta^* $$

where $\phi(t)$ is the regressive vector,

$$ \phi(t) = \begin{bmatrix} q_{2,\infty}(t) \\ -K_s \end{bmatrix} $$

$\theta^*$ is the vector of unknown parameters

$$ \theta^* = \begin{bmatrix} \omega_c^2 \\ A_0 \omega_c^2 \\ 0 \end{bmatrix} $$

and

$$ y(t) = \omega_s^2 v_{2,\infty}(t). $$

To obtain the unknown vector $\theta^*$, an estimate $\hat{\theta}(t)$ is computed by minimizing the cost function

$$ J(\hat{\theta}) = \frac{1}{2} \int_0^T e^{-\lambda(t-\tau)} \phi^T(\tau) \hat{\theta}(\tau) - y(\tau))^2 d\tau + \frac{1}{2} e^{-\lambda T} (\hat{\theta}(T) - \hat{\theta}_0)^T Q_0 (\hat{\theta}(T) - \hat{\theta}_0) $$

where $Q_0 = \lambda^0 > 0$ and $\lambda \geq 0$ are design constants, $\hat{\theta}_0 = \theta(0)$ is the initial parameters estimate and the linear term in the integrand function provides a zero-convergent estimation error in the nominal case, as explained in the following. The cost function (17) includes the forgetting factor $\lambda$ to possibly discount the past data and a penalty on the initial error between the estimate $\hat{\theta}_0$ and $\theta^*$ [26].
Setting $\partial J/\partial \hat{\theta} = 0$ and assuming $\hat{\theta}(\tau) = \text{constant}$ in $[0, t]$, the estimate that minimizes Eq. (17) is

$$\hat{\theta}(t) = P(t) \left[ e^{-\lambda t}Q_0 \hat{\theta}_0 + \int_0^t e^{-\lambda(t-\tau)} \tau \phi(\tau) y(\tau) d\tau \right]$$  \hspace{1cm} (18)

with

$$P(t) = \left( e^{-\lambda t}Q_0 + \int_0^t e^{-\lambda(t-\tau)} \tau \phi(\tau) \phi^T(\tau) d\tau \right)^{-1}.$$  \hspace{1cm} (19)

To achieve computational efficiency, it is desirable to compute $P(t)$ recursively. This amounts to replace the above equation by the differential equation

$$\frac{dP^{-1}(t)}{dt} = -\lambda P^{-1}(t) + t \phi(t) \phi^T(t).$$  \hspace{1cm} (20)

By using the identity

$$\frac{d}{dt} \left[ P(t)P^{-1}(t) \right] = \frac{dP(t)}{dt} P^{-1}(t) + P(t) \frac{dP^{-1}(t)}{dt} = 0$$  \hspace{1cm} (21)

we obtain

$$\frac{dP(t)}{dt} = \lambda P(t) - tP(t)\phi(t)\phi^T(t)P(t)$$  \hspace{1cm} (22)

with $P(0) = Q_0^{-1}$.

Differentiating Eq. (18) and using Eq. (22), we find that the parameters update satisfies

$$\frac{d\hat{\theta}(t)}{dt} = -tP(t)\phi(t) \left( \phi^T(t) \hat{\theta}(t) - y(t) \right).$$  \hspace{1cm} (23)

Therefore, Eqs. (2)-(4) and the two matrix differential equations (Eqs. (22),(23)) define the overall 7-th order estimation algorithm. In fact, as it can be noted, the algorithm consists of two differential equations (Eqs. (2),(3)), two equations related to vector $\hat{\theta}(t)$ (Eq. (23)) and the remaining three equations to update the components of the symmetric matrix $P(t)$.

### A. Convergence analysis

Let $\epsilon(t) = \hat{\theta}(t) - \theta^*$ be the estimation error and consider the term $P^{-1}(t)\epsilon(t)$. From Eqs. (20) and (23), one can obtain

$$\frac{d}{dt} \left[ P^{-1}(t)\epsilon(t) \right] = -\lambda P^{-1}(t)\epsilon(t)$$  \hspace{1cm} (24)

from which it is straightforward to derive the following expression for the estimation error:

$$\epsilon(t) = e^{-\lambda t} P(t)Q_0 \epsilon(0).$$  \hspace{1cm} (25)

To investigate the convergence properties of the proposed estimator, a preliminary result is here introduced. The proof follows the one presented in [26] based on the concept of persistence of excitation.

**Proposition 1:** The norm of the matrix $P^{-1}(t)$ is unbounded as $t \to \infty$.

**Proof:** The inequality

$$P^{-1}(t) \geq e^{-\lambda \delta} \int_{t-\delta}^t \tau \phi(\tau)\phi^T(\tau) d\tau$$  \hspace{1cm} (26)

holds with $\delta \in [0, \xi]$ because of the non-negative definition of the integrand function.

Solving the integral in Eq. (26) with $\delta = \frac{2\pi}{\omega_c}$, one has

$$P^{-1}(t) \geq e^{-\lambda \frac{2\pi}{\omega_c}} \left( t - \frac{\pi}{\omega_c} \right) M_1 + \frac{A_c m_2 \pi}{\omega_c} sin(\rho(t)) M_2$$  \hspace{1cm} (27)

where

$$M_1 = \frac{2\pi}{\omega_c} \left[ A_0^2 K_0^2 + \frac{1}{2} m_2^2 A_c^2 - A_0 K_0^2 \right],$$  \hspace{1cm} (28)

$$M_2 = \left[ -4A_0 K_s - m_2 A_c \cos(\rho(t)) \right] \frac{2K_s}{K_s}$$  \hspace{1cm} (29)

and $\rho(t) = \omega_c t + \epsilon t + \xi$.

Note that $M_1$ is a constant symmetric positive definite matrix and each element of $M_2$ is bounded. Therefore the Frobenius norm of $P^{-1}(t)$ tends to infinity as $O(t)$.

As an immediate consequence of previous results, the norm of the absolute error, i.e.

$$||\epsilon(t)|| \leq \frac{e^{-\lambda t}}{||P^{-1}(t)||} ||Q_0 \epsilon(0)||$$  \hspace{1cm} (30)

tends to zero exponentially.

Given the estimate of $A_0$ and $\omega_c$, the amplitude $A_c$ is estimated as

$$\hat{A}_c(t) = \sqrt{\frac{\omega_c^2 (\omega_c^2 + \Omega_d^2) - m_2^2 \zeta(\tau)^2}{m_1^2}}$$  \hspace{1cm} (31)

where

$$\zeta(\tau) = \Omega_{2\omega_c} - K_s \hat{A}_0(\tau).$$  \hspace{1cm} (32)

Moreover the phase angle of the sinusoidal signal can be estimated as

$$\hat{\phi}(\tau) = \arg \eta(\tau) - \xi$$  \hspace{1cm} (33)

with

$$\eta(\tau) = -\frac{\hat{\omega}_c}{\omega_s} \zeta(\tau) + \frac{\hat{\omega}_c}{\omega_s} v_{1\omega}(\tau)$$  \hspace{1cm} (34)

where $j$ is the imaginary unit.

If the signal $v(t)$ is corrupted by additional bounded disturbance, namely $w(t)$, the noise affects both the regressors and the signal $y(t)$. However the noise present in the output signals is bounded due to the filtering characteristics of the OSG-SOGI system. In this case the steady-state error estimation is expressed as

$$\epsilon(t) = \hat{P}(t)\epsilon_n$$  \hspace{1cm} (35)

where $\epsilon_n$ depends on the noise $w(t)$ as

$$\epsilon_n = \int_0^t e^{-\lambda(t-\tau)} \tau \phi(\tau) w(\tau) d\tau$$  \hspace{1cm} (36)

and the bar notation stands for noisy variables.

By similar arguments, the norm of $P^{-1}(t)$ tends to infinity as $O(t)$. Moreover

$$||\epsilon_n|| \leq \frac{\lambda M + e^{-\lambda t} - 1}{\lambda^2}.$$  \hspace{1cm} (37)

Therefore $||\epsilon_n||$ tends to infinity in $O(t)$ time and, as a consequence, the boundedness of the estimation error norm can be guaranteed.
Remark 2: Note that in Eq. (31) the signal term \( \frac{\dot{\omega}_c(t)}{\omega} \zeta(t) \) where \( \zeta(t) \) is the auxiliary signal defined in Eq. (32), could be replaced with \( \frac{\dot{\omega}_c(t)}{\omega} v_{2n}(t) \). However, the signal \( v_{2n}(t) \) is more corrupted by the input noise because its dependence on the second-order derivative of the signal \( v(t) \).

IV. SIMULATIONS

In this section some examples are given to illustrate the behavior of the proposed method, namely FF. Input signals, sampled with a period of \( T_s = 3 \times 10^{-4} \) s, are affected by a zero mean Gaussian noise with a signal-to-noise (SNR) ratio equal to 10. The SNR is measured in decibels as the logarithm of the average power of the reference signal samples and the noise ones over the experiment time as

\[
SNR \triangleq 10 \log_{10} \frac{\sum_{k=0}^{n-1} y(kT_s)^2}{\sum_{k=0}^{n-1} w(kT_s)^2}
\]  

(38)

where \( n \) is the number of samples.

Example 1. The aim of this example is to highlight the capability of the proposed method to identify all the parameters of a biased sinusoid and then to on-line reconstruct the input signal. The biased sinusoid \( v(t) = 1 + 5 \sin (3t + \frac{\pi}{4}) \) is considered as OSG-SOGI input signal. The parameters of FF are chosen as \( K_s = 1, \lambda = 10, Q_0 = \frac{1}{2} I, \theta_0 = [\omega_s^2, \omega_s^2]^T \) and \( \omega_s \) is chosen equal to twice the unknown angular frequency initial value. Fig. 2 depicts the noisy input signal, the nominal and the reconstructed one. As it can be noted FF seems to show a fast and accurate signal reconstruction capability and a good rejection to noise.

Fig. 2. Example 1. Signal reconstruction.

Example 2. In this example, a signal with two frequency steps is used to compare the proposed method with the OSG-SOGI based PLL (CTA) in [8] and the linear adaptive scheme (ABKNS) in [17]. Let us assume an input signal

\[
v(t) = 2 + 2\sin(\omega_c t)\]

(39)

with

\[
\omega_c = \begin{cases} 
4, & 0 \leq t < 30, \\
8, & 30 \leq t < 60, \\
2, & 60 \leq t < 90.
\end{cases}
\]

(40)

All the methods are initialized with the same initial condition \( \dot{\omega}_c(0) = 3.2 \). The parameters of FF are chosen as \( K_s = 1, \omega_s = \dot{\omega}_c(0), \lambda = 1, Q_0 = \frac{1}{2} I, \theta_0 = [\omega_s^2, \omega_s^2]^T \). CTA method is tuned with \( k = 1, \omega = \dot{\omega}_c(0) \) and the PI regulator gains \( K_p = 1.51 \) and \( K_I = 0.89 \). As far as ABKNS method is concerned, the version with a correction term is considered as proposed in [17]. This modification permits to reduce estimation error in case of noisy signal and it ensures boundedness of the closed system signals. The chosen parameters are \( k = 10, \alpha = 1 \) and \( \theta_0 = 5 \). The results are depicted in Fig. 3.

Fig. 3. Example 2. Frequency estimation.

All the methods deal with the first frequency sweep in a satisfactory manner but CTA can not perform the tracking of the final frequency that is 75% less than the intermediate frequency. As it can be noted, the proposed method performs a better filtering action on the estimated frequency with respect to other methods. Note however that the filtering characteristics of CTA and ABKNS can be improved by an accurate choice of the gain parameters at the expense of longer transients. This means that, in the case of heavily noisy signals, a trade-off between filtering actions and transient responses is required making it difficult to correctly tune the free parameters.

V. CONCLUSIONS

In this paper, an estimator for the on-line recovering of the frequency, amplitude, phase and offset of a biased sinusoidal signal has been proposed. The discussed method is based on a continuous least squares approach by considering a cost function with a forgetting factor and a regressors vector that is weighted by a linear term. Such regressors permit to have an exponentially zero-convergent estimation error when no noises are present and a bounded estimation error in the case of noisy input signals. Simulations have been conducted in order to illustrate the reconstruction capability of the method and its behavior dealing with abrupt changes in the input signal frequency.

REFERENCES


