Explicit solution of the finite time $L_2$-norm polynomial approximation problem

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Abstract

The aim of this paper is to present a new approach to the finite time $L_2$-norm polynomial approximation problem. A new formulation of this problem leads to an equivalent linear system whose solution can be investigated analytically. Such a solution is then specialized for a polynomial expressed in terms of Laguerre and Bernstein basis.

1. Introduction

In many applications fields it is very important to model signals of interest by polynomials (see [1,2] and the references therein). This research field is widely investigated and a large amount of efficient approaches have been proposed. The problem to find a good polynomial approximation is relevant in mathematical functions evaluation when software or hardware implementation are required [3]. Further applications can be found in speech compression [4], raw image encoding [5], control theory [6,7], to name just a few. Roughly speaking, the first natural target for every polynomial approximation scheme is to obtain a polynomial which is sufficiently close to a given function. Obviously, the property of closeness depends on the a priori chosen criteria to measure the quality of the approximation. In this paper, a different approach is presented to solve the polynomial approximation problem in finite time $L_2$-norm. Statistical considerations show that $L_2$-norm is the most appropriate choice for data fitting when the errors in the data have a normal distribution [1]. The practical relevance of considering such a norm is pointed out in several papers [8–10].

Let $y(t)$ be the function to be approximated, then the $L_2$-norm polynomial approximation problem is formulated as follows.

Problem $I_{L_2}^{(m,T)}$. Find a polynomial $\hat{y}(t)$ of degree $m - 1$, as

$$\hat{y}(t) = \sum_{k=1}^{m} \hat{a}_k t^{k-1},$$

such that the coefficients vector $\hat{a}_m = \{\hat{a}_k\}_{k=1}^{m}$ is determined as

$$\hat{a}_m = \arg\min_{a_m} J_m(T)$$

with

$$J_m(T) = \int_0^T [y(t) - \hat{y}(t)]^2 dt,$$

where $T$ is the finite observation time.
The problem $L_{2}^{m,T}$ can be easily coped with orthogonal polynomials such as Legendre polynomials [1]. However, in a very large number of applications, it is necessary to consider expansions of a target function in different polynomial basis, for example Laguerre and Bernstein polynomials. The importance of considering Laguerre polynomial basis is pointed out by their practical relevance in a wide range of applications [11–15]. Bernstein polynomials have been largely investigated in several works [16–20]. Even if they do not provide orthogonality property, however they are used as polynomial root solver since it has proved that polynomials expressed in Bernstein form are not only stable but also better root conditioned than polynomials in power form [21,22]. In [23] the importance of choosing Bernstein polynomial basis is also documented.

The approximation problem is addressed by investigating an equivalent representation of the system to be solved. No assumption on the particular choice of the polynomial family basis is made and the solution of the problem $L_{2}^{m,T}$ involving the inversion of the Hilbert matrix is presented. Such a matrix is known to be highly ill-conditioned even if the analytical inversion formula in terms of product of binomial coefficients exists [24] and then the entries of the inverse matrix are all integer. This fact allows to explicit the solution of the problem $L_{2}^{m,T}$. The cases of Laguerre and Bernstein polynomial basis are considered as examples of applications. The rest of this paper is organized as follows: in Section 2 a new formulation of the problem is presented while its analytical solution is discussed in Section 3; in Section 4 the proposed solution is specialized for a polynomial expressed in terms of Laguerre and Bernstein basis; Section 5 is devoted to conclusions.

2. Preliminary results

In this section the classical optimal solution for the problem $L_{2}^{m,T}$ is derived together with a new formulation in terms of repeated integrals of the function to be approximated.

To minimize the index $J_{m}(T)$, the set of equations

$$
\frac{\partial J_{m}(T)}{\partial x_{m,i}} = 0 \quad \forall i = 1, \ldots, m
$$

has to be solved.

By considering Eq. (3), the partial derivatives in Eq. (4) can be rewritten as

$$
\int_{0}^{T} (y(t) - \hat{y}(t)) \frac{\partial \hat{y}(t)}{\partial x_{m,i}} dt = 0 \quad \forall i = 1, \ldots, m.
$$

Therefore the optimal solution for the problem $L_{2}^{m,T}$ is obtained by solving the following equation:

$$
\int_{0}^{T} y(t)t^{i-1} = \sum_{j=1}^{m} \hat{x}_{m,j} \frac{T^{2j-1}}{2j-1} \quad \forall i = 1, \ldots, m
$$

that can be rewritten in matrix form as

$$
\hat{x}_{m} = A_c^{-1} b_c
$$

where

$$
A_c(i,j) = \frac{T^{i+j-1}}{i+j-1}, \quad i, j = 1, \ldots, m,
$$

$$
b_c(i) = \int_{0}^{T} y(t)t^{i-1} dt, \quad i = 1, \ldots, m.
$$

In the following theorem an equivalent expression to Eq. (6) involving repeated integrals of the function to be approximated is presented.

Theorem 1. An equivalent solution to the problem $L_{2}^{m,T}$ is

$$
\hat{x}_{m} = A_n^{-1} b_n,
$$

with

$$
A_n(i,j) = \frac{T^{i+j-1}}{i!\left(i+j-1\right)}, \quad i, j = 1, \ldots, m,
$$

$$
b_n(i) = \int_{0}^{T} y(t), \quad i = 1, \ldots, m.
$$
where it is denoted by
\[ \int (\phi(t). \]
the integral expression
\[ \int_0^T \cdots \int_{x_{k-1}}^x \phi(x_j) \, dx_j \cdots \, dx_t \]
with the definition
\[ \int (\phi(t) = \int_0^T \phi(x_1) \, dx_1. \]

**Proof.** The proof follows by considering a matrix \( M \in \mathbb{R}^{m \times m} \), whose the generic element is
\[ M(i,j) = (-1)^{i+j-1} (j-1)! \binom{i}{j-1} T^{i-j}, \quad i,j = 1, \ldots, m \]
and by showing that
\[ (c_1) M A_n = A_c, \]
\[ (c_2) M b_n = b_t. \]

The \((i,j)\)-element of the matrix \( A_c = MA_n \) can be expressed as
\[ A_c(i,j) = T^{i-j} \sum_{k=1}^i (-1)^{k-1} \binom{i-1}{k-1} \binom{k+j-1}{k}, \quad i,j = 1, \ldots, m. \] (8)

The summation in (8) can be arranged as
\[ (-1)^{i+j-1} \frac{(i-1)!(j-1)!}{(i+j-1)!} \sum_{k=0}^{i-1} (-1)^k \binom{i+j-1}{k} \]
and by using the equality [25]
\[ \sum_{k=0}^n (-1)^k \binom{r}{k} = (-1)^n \binom{r-1}{n} \]
with simple algebraic manipulations, the proof of \( (c_1) \) follows.

As far as condition \( (c_2) \) is concerned, it must be shown that
\[ \sum_{k=1}^i (-1)^{k-1} (k-1)! \binom{i-1}{k-1} T^{i-k} \int y(t) = \int_0^T y(t) t^{i-1} dt, \quad i = 1, \ldots, m. \]

By using the equality [26]
\[ \int y(t) t^q = \sum_{k=0}^s (-1)^k \binom{k+q-1}{k} \binom{s}{k} k! t^{i-k} \int y(t) \]
for \( q \geq 1 \) and \( s \geq 0 \), the condition \( (c_2) \) is easily proved. □

**Remark 1.** Using the well-known Cauchy formula
\[ \int u(t) = \frac{1}{(\alpha - 1)!} \int_0^t (t - \xi)^{\alpha-1} u(\xi) \, d\xi, \]
the repeated integrals reported in Eq. (7) can be rewritten in terms of a singular integral.

**3. An analytical solution for a generic polynomial basis**

Let \( P_r(t) \) be the generic polynomial of order \( r-1 \)
\[ P_r(t) = \sum_{k=1}^i 2r_k t^{k-1} \] (9)
and consider the polynomial basis
\[ \mathcal{B} = \{ P_1(t), P_2(t), \ldots, P_m(t) \}. \]

A polynomial expressed in such a basis is chosen as approximating function \( \hat{y}(t) \), i.e.,
\[ \hat{y}(t) = \sum_{r=1}^{m} \gamma_r P_r(t). \]

By considering Eq. (9), \( \hat{y}(t) \) is rearranged as
\[ \hat{y}(t) = \sum_{r=1}^{m} \sum_{k=1}^{m} \alpha_{rk} \gamma_r t^{k-1} \quad \alpha_{ij} = 0 \quad \forall j > i. \]

A relationship between the \( \hat{\alpha} \) coefficients and the \( \gamma \) ones can be established considering the generic expression of \( \hat{y}(t) \) in terms of coefficients \( \hat{\alpha}_{mk}, k = 1, \ldots, m \)
\[ \hat{\alpha}_{mk} = \sum_{r=1}^{m} \alpha_{rk} \gamma_r, \]

which can be expressed in matrix notation as
\[ \hat{\alpha}_m = A \gamma \]

with
\[ A_{ij}(i,j) = \alpha_{ij}, \quad i, j = 1, \ldots, m. \]

By the results stated in Theorem 1, the optimal \( \gamma^* \) coefficients in terms of \( L_2 \)-norm are computed as
\[ \gamma^* = (A_n \cdot A_y)^{-1} b_n. \quad (10) \]

A factorization for the matrix product \( A_n \cdot A_y \) is proposed as
\[ A_n \cdot A_y = D_1 \cdot M_1 \cdot D_2 \cdot M_2, \]
where
\[ D_1 = \text{diag}(T^i)_{i=1, \ldots, m}, \]
\[ D_2 = \text{diag}(T^{i-1})_{i=1, \ldots, m}, \]
\[ M_1(i,j) = \frac{1}{i! \left( i + j - 1 \right)}, \quad i, j = 1, \ldots, m \quad (11) \]

and
\[ M_2(i,j) = \alpha_{ij}, \quad i, j = 1, \ldots, m. \quad (12) \]

The reported factorization leads to the following expression for the optimal \( \gamma^* \) coefficients
\[ \gamma^* = M_2^{-1} D_2^{-1} M_1^{-1} D_1^{-1} b_n. \]

Note that the optimal solution depends on the inverse of the matrices \( M_1 \) and \( M_2 \). While \( M_1 \) has a fixed structure depending only on the dimension of the problem, the elements of the matrix \( M_2 \) are related to the coefficients of the particular polynomial basis. In the next proposition the analytic inversion of \( M_1 \) is presented. As far as \( M_2 \) is concerned, its inverse is investigated once the polynomial basis is chosen. In the next sections the cases of Laguerre and Bernstein basis are discussed.

**Proposition 1.** The inverse of matrix \( M_1 \) can be expressed as
\[ M_1^{-1} = H^{-1} Z, \quad (13) \]
where \( H \) is the Hilbert matrix and
\[ Z(i,j) = (-1)^{i+j-1}(i-1)! \binom{i-1}{j-1} \quad i, j = 1, \ldots, m. \quad (14) \]
\textbf{Proof.} By considering Eq. (13), it is easy to obtain that
\[ H = Z M_1. \]  
(15)  
The proof follows by showing that the matrix \( H = Z M_1 \) is equal to the Hilbert matrix. By using Eqs. (11) and (14), the \((i,j)\) element of \( H \) is
\[ \hat{H}(i,j) = \sum_{k=1}^{m} \frac{(-1)^{k+1}}{k} \binom{v+j-1}{j-i} \binom{t-1}{k-1} i,j = 1,\ldots,m. \]  
(16)  
The summation in Eq. (16) is equal to the one in Eq. (8) from which the proof follows.  

\section*{4. Particular solution for laguerre and bernstein basis}  
\subsection*{4.1. Laguerre basis}  
In this subsection a polynomial basis is considered that is composed of \( m \) Laguerre polynomials defined as [27]
\[ L_j^v(t) = \sum_{i=1}^{j} (-1)^{i+1} \binom{v+j-1}{j-i} \frac{t^{i-1}}{(i-1)!} , \quad j \geq 1. \]  
(17)  
Taking into account Eq. (17), a simple comparison term by term between Eqs. (12) and (17) leads to the following expression for the elements \( M_2(i,j), i,j = 1,\ldots,m, \)
\[ M_2(i,j) = \frac{(-1)^{i+1}}{(i-1)!} \binom{v+j-1}{j-i} , \quad i,j = 1,\ldots,m. \]  
(18)  
Thanks to the particular structure assumed by the matrix \( M_2 \), the explicit expression of its inverse is obtained as proved in the next theorem.

\textbf{Theorem 2.} The inverse matrix of matrix \( M_2 \) can be expressed as
\[ M_2^{-1}(i,j) = (-1)^{i+1} (j-1)! \binom{v+j-1}{j-i} , \quad i,j = 1,\ldots,m. \]  
(19)  
\textbf{Proof.} Let us define \( \tilde{M} = M_2^{-1} \). It is sufficient to prove that
\[ \tilde{M}(i,j) = \delta_{ij} \quad i,j = 1,\ldots,m, \]
where \( \delta_{ij} \) represents the Kronecker’s delta. Taking into account Eqs. (18) and (19), the \((i,j)\)– element of the matrix \( \tilde{M} \) is equal to
\[ (-1)^{i} \frac{v-j-1}{(i-1)!} \sum_{k=1}^{j} (-1)^{k} \binom{v+k-1}{k-i} \binom{v+j-1}{j-k} . \]  
(20)  
The summation in Eq. (20) can be expressed in terms of Gamma function [25] such that the following is immediately obtained
\[ \tilde{M}(i,j) = \frac{1}{T(1+i+j)} \delta_{ij} \quad i,j = 1,\ldots,m. \]  
(21)  
Therefore
\[ \tilde{M}(i,j) = \delta_{ij} \quad i,j = 1,\ldots,m \]
and the proof follows.  

\subsection*{4.2. Bernstein basis}  
In this subsection, the solution of the problem \( T^{m,T} \) is investigated where Bernstein polynomial basis is considered. For every natural number \( m \), the Bernstein polynomials of degree \( m-1 \) on \([0,1]\) are defined by
\[ B^m_k(t) = \binom{m-1}{k-1} (1-t)^{m-k} , \quad k = 1,\ldots,m. \]
By specializing the coefficients $a_{ik}$ for $i, j = 1, \ldots, m$, to the Bernstein basis case, the matrix $M_2$ is now expressed as

$$M_2(i,j) = (-1)^{i+j} \binom{m-1}{i-1} \binom{m-j}{j-1}, \quad i, j = 1, \ldots, m$$

and by applying the results stated in [23] it is straightforward to obtain an explicit expression for its inverse as

$$M_2^{-1}(i,j) = \binom{i-1}{j-1} \binom{m-1}{m-i}, \quad i, j = 1, \ldots, m.$$  

5. Conclusions

In this paper, a novel approach to finite time $L_2$-norm polynomial approximation has been presented. The discussed formulation allows to obtain an explicit expression for the optimal solution in terms of repeated integrals of the function to be approximated. The cases of Laguerre and Bernstein polynomial basis for the approximation problem have been investigated.

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References