On Binary Methods

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Giving types to binary methods causes significant problems for object-oriented language designers and programmers. This paper offers a comprehensive description of the problems arising from typing binary methods, and collects and contrasts diverse views and solutions. It summarizes the current debate on the problem of binary methods for a wide audience.

1 Introduction

Binary methods have caused great difficulty for designers of strongly typed object-oriented languages and for programmers using those languages. In this paper we study the sources of these problems and compare and contrast a variety of solutions.

The authors of this paper have differing views on what the most appropriate solutions are. We have attempted here to collect together the solutions that individuals among us advocate and to present a consensus on what can be fairly stated as strengths and weaknesses of each approach. This paper grew from presentations and discussions at the 2nd Workshop on Foundations of Object-Oriented Languages, which was sponsored by NSF and ESPRIT and held in Paris in June, 1994 [19].

Let us begin by fixing some basic terminology. A class is the code that defines the instance variables and methods of some objects. The objects that conform to this definition are called instances of the class. (The issues that we discuss also arise in delegation-based languages; for simplicity we concentrate on classes.) In this article we use new as a primitive that generates an instance of a class from the class name (and some initial values for its instance variables). An interface type, also called an object type or simply a type contains the names of the object’s methods, and the types of each method’s arguments and results. Due to subtyping, an object may have multiple interface types; what we mean...
method that has an argument of the same type: the object passed as argument and the receiving object itself. In general, a binary method could also include other arguments (including other arguments of the same type); by a standard abuse of terminology we still refer to these as binary methods. We provide examples in an object-oriented style later.

A subclass is code that extends a class or classes (called the superclasses of the subclass). Subclasses inherit definitions of instance variables and methods from their superclasses. A subclass may also override the definitions of methods it would otherwise inherit by redefining them. Because a subclass inherits code for methods, it also inherits interface type information for the methods that it does not override.

The most significant problem with binary methods lies in their typing in the presence of inheritance. The source of this problem is that the type of the argument of a binary method naturally should change in parallel to changes in the type of the object produced by the subclass. The difficulty is that these type changes may result in subclasses which may no longer produce subtypes. On the other hand if inheritance is limited to always produce subtypes then useful subclasses can not be directly defined, and work-arounds must be found.

A second problem is the asymmetry of a binary method: the method may have privileged access to only one of the two objects the method is invoked on. These two problems are described in more detail in Section 2.

Sections 3 and 4 concentrate on solutions to the problem of typing binary methods in the presence of inheritance. We consider the question from two sides: in

2 The Problem of Binary Methods

This section describes the problems caused by binary methods. The first subsection describes typing problems in the presence of inheritance, and the second describes problems with privileged access.
2.1 Typing Binary Methods in the Presence of Inheritance

In procedural or functional languages, the type of a binary function that takes two arguments of type \( \tau \) and returns a value of type \( \sigma \) is written \( \tau \times \tau \rightarrow \sigma \). In an object-oriented language, functions or procedures are typically replaced by methods belonging to a class corresponding to one of the arguments. Figure 1 shows a standard example of a class with a binary method.\(^1\) In this example, binary operations—when regarded as methods belonging to a class—are asymmetric: the receiver plays a role somewhat different than the parameter. This distinction is highlighted when we define a subclass of a class with a binary method.

Figure 2 defines a subclass \( \text{ColorPointClass} \) of \( \text{PointClass} \). In \( \text{ColorPointClass} \), the type of the parameter of \( \text{equal} \) is changed to \( \text{ColorPoint} \) to match the type of the receiver, allowing two \( \text{ColorPoint} \) objects to be compared by the \( \text{equal} \) method, which overrides the behavior of \( \text{equal} \) for points.

\[
\text{equal}(p: \text{ColorPoint}): \text{bool} \quad \text{return} \quad ( \text{cValue}=p.c) \land (\text{yValue}=p.y)
\]

end class

We generally write object types similarly to the type of the record of methods.\(^2\) Therefore instances of \( \text{PointClass} \) and \( \text{ColorPointClass} \) have the following object types:

\[
\text{Point} \equiv \text{OT} \langle \langle \text{xValue: real; yValue: real}; \text{equal: Point \rightarrow bool} \rangle
\]

\[
\text{ColorPoint} \equiv \text{OT} \langle \langle \text{xValue: real; yValue: real; cValue: string}; \text{equal: ColorPoint \rightarrow bool} \rangle
\]

\(^1\)A few notes on our notation. Methods are functions or procedures whose body occurs after the keyword \text{is}. We write parameterless functions and procedures by omitting the parentheses. We write the type of the parameterless functions as if they were variables of their return type. That is, we omit an implicit \text{unit} \rightarrow before the result type. Methods are selected by dot notation; thus \text{c}.\text{m} denotes the method of name \text{m} defined for the object \text{c}. Commented text is preceded by “-”.

\(^2\)We presume that instance variables are not accessible from outside of the object.

The prefix \( \text{OT} \) is used to distinguish object types from record types. Note that both of these definitions happen to be recursive: the type being defined appears on the right-hand side of the \( \equiv \). It is not uncommon for the type being defined to appear as either an argument or result type in its methods.

Informally, a type \( \sigma \) is a \text{subtype} of \( \tau \), written \( \sigma :<: \tau \), if an expression of type \( \sigma \) can be used in any context that expects an expression of type \( \tau \) (cf. \([15, 16, 49])\). Associated with subtyping is the principle of \text{subsumption} (subtype polymorphism): if \( \sigma :<: \tau \) and a program fragment has type \( \sigma \), it also has type \( \tau \). A simple example of subtyping in object-oriented programming is that an object type is a subtype of the type with some methods removed, as any context that expects the object with fewer methods will not directly use the extra methods and thus no type errors will occur. In fact it is also possible to replace the type of any method by a subtype and still have the resulting object types in the subtype relation. Thus the general rule is \( \text{OT} \langle \langle m_1:S_1, \ldots, m_n:S_n,..., m_{nk}:S_{nk}\rangle \rangle :<: \text{OT} \langle \langle m_1:T_1, \ldots, m_n:T_n\rangle \rangle \) (with \( k \geq 0 \)) if and only if, for each \( i \in \{1..n\}, S_i :<: T_i \).

The rule for subtyping functions states that \( \sigma :<: \tau \rightarrow \sigma' \rightarrow \tau' \) if and only if \( \sigma' :<: \sigma \) and \( \tau :<: \tau' \) \([15]\). (This is sometimes called the “contravariant rule” because it is contravariant in the left argument of \( \rightarrow \).)

This rule is informally justified by the following. If \( f \) is expected to have type \( \sigma' \rightarrow \tau' \), but actually has type \( \sigma \rightarrow \tau \), then \( f \) can be passed an argument of type \( \sigma' \) when (by subsumption) \( \sigma' :<: \sigma \); furthermore, the result of such a call will have type \( \tau' \), which (by subsumption) can be considered to be of type \( \tau' \). Hence all functions of type \( \sigma \rightarrow \tau \) can be used as if they had type \( \sigma' \rightarrow \tau' \) without type error.

Subtype polymorphism is a useful feature of object-oriented programming: if subclasses correspond to subtypes, a subclass object can always be passed to a function or method expecting a superclass object, allowing

\begin{verbatim}
class ColorPointClass subclass of PointClass
  instance variables
    cValue : string
  methods
    -- x and y are inherited
    c: string is return(cValue)
    -- equal is overridden
    equal(p: ColorPoint): bool is
      return (cValue==p.c) & (yValue==p.y))
end class
\end{verbatim}

Figure 2: The class \( \text{ColorPointClass} \).
ere use of code. Unfortunately, subclasses do not always generate subtypes; this can happen if the types of methods need to change in subclasses to require more specialized behavior from their arguments. In particular, since the equal method in ColorPointClass checks its argument for color as well as position, the argument type needs to change, as the example indicates. Because of the contravariance of the subtyping relation on the domain of equal, ColorPoint is not a subtype of Point. For the subtype relation to hold, the type of equal in ColorPoint would have to be a subtype of the type in Point. Thus ColorPoint → bool must be a subtype of Point → bool. But, by the subtyping principle for functions, this requires Point to be a subtype of ColorPoint, exactly the opposite of what we are after and clearly untrue.

This loss of subtyping in this case is not due to any problem with the definition of subtyping for functions; the procedure breakit of Figure 3 illustrates how allowing this subtyping would be unsound. When breakit is applied to an actual parameter of type Point, there is no problem. However if the actual parameter is a ColorPoint, a run-time error will occur when p.equal(nuPt) is evaluated. Since the value of p will be a ColorPoint, the code for equal in ColorPointClass will be executed. When nuPt is sent the message c, it will fail because it has no corresponding method. Thus, in a sound type system, a call of breakit with an actual parameter of type ColorPoint must not type check.

```
procedure breakit(p: Point)
  var
    nuPt: Point
  begin
    nuPt := new PointClass(3.2, 4.5)
    if p.equal(nuPt) then
      ...
  end
```

Figure 3: The procedure breakit.

Most statically-typed object-oriented languages require subclasses to generate subtypes, even in the presence of binary methods. One type requirement that has been used to this end is that the types of methods may not be changed upon inheritance; this is done, for example, in C++ [52] Object Pascal [53], and Modula-3 [46]. In such languages, one cannot write ColorPointClass as in Figure 2, with the typing discussed above.

Eiffel does allow argument types to be specialized in a subclass’s methods; for example, it would allow ColorPointClass to be written as in Figure 2. We call such argument specialization covariant argument specialization, because it goes against the contravariant rule in argument positions. Eiffel in addition preserves the invariant that subclasses generate subtypes, but this

2.2 Privileged Access to Object Representations

A completely different kind of problem with binary operations on objects—whether they are methods or free-standing procedures—is that they must often be given privileged access to the instance variables of both of their arguments.

The equality methods of points and colored points are examples of the simpler case where this need does not arise—the necessary attributes of the argument are already publicly available through existing methods. In order to write the equal method for the point class, we only needed to compare the receiver’s instance variables xValue and yValue to the values returned by the x and y methods of the argument p. There is no need to access p’s instance variables directly. Indeed, p might not even have instance variables named xValue and yValue; there is no need to know anything at all about its internal representation. The situation is similar for the equal of the colored point class.

On the other hand, suppose we want to write a class definition for simple integer set objects with the following interface type:

```
IntSet ≡ OT ⟨⟨add: int → unit;
  member: int → bool;
  union: IntSet → IntSet;
  superSetOf: IntSet → bool⟩⟩
```

We can easily choose a representation for integer sets, (say, as lists of integers) and implement the add and member methods as in Figure 4. But when we come to implementing the union and superSetOf methods, we get stuck: given the interface type we have chosen for sets, there is no way to find out what elements a given set contains.

The obvious thing to do is to extend the public interface of sets with an enumerate method that (for example) returns a list of the elements of the set. But suppose
class IntSetClass
  instance variables
    elts: IntList
  methods
    add(i: int): unit is elts := elts.cons(i)
    member(i: int): bool is return(elts.memq(i))
    union(s: IntSet): IntSet is ???
    superSetOf(s: IntSet): bool is ???
end class

Figure 4: The class IntSetClass, for which writing superSetOf and union is problematic.

we want to use a more efficient internal representation for sets, storing the elements in a bit string. We would certainly expect not only the `add` and `member` methods to be efficient, but `superSetOf` and `union` as well. But, to achieve good performance, `union` needs to work directly with the bit string representations of the two sets, so the `enumerate` method has to be replaced by an `asBitString` method that returns the underlying representation. Doing so is unsatisfactory, because it makes representation details visible to users.

In order to handle binary operations like `equal` as methods, we need the `type` of a parameter of a method to be the same as the type of the receiving object; for methods like `superSetOf` and `union` we need an additional mechanism for constraining the implementation of the parameter to be the same as the receiver's. Indeed, such a mechanism is required whether or not we want to consider `union` a proper method of set objects: an external procedure for computing the union of two set objects will also need to gain privileged access to the internal representations of both of its arguments.

## 3 Avoiding Binary Methods

Sometimes the simplest solution to a problem is to ignore it. In this section we explore the position that binary operations like `equal`, `union`, and `+` should not be regarded as `methods` of either of their argument objects, thus sidestepping the thorny typing issues raised so far.

There are some theoretical benefits to taking this step. For example, aside from binary methods, the types of methods are always positive, in the sense that the object type itself appears only in result positions. In this case, the classic encoding of object types as recursive records [15, 25] may be replaced by an encoding where objects are modeled by existential types [48, 33].

It may also be argued that keeping binary operations separate from their arguments avoids conceptual confusion. Turning a symmetric operation like `+` into a method gives one of its arguments an artificially special status, requiring programmers to think in terms of

| function eqPoint(p1,p2: Point): bool is |
| return( (p1.x == p2.x) && (p1.y == p2.y)); |

| function eqColorPoint(cp1,cp2: ColorPoint):bool is |
| return( (cp1.x == cp2.x) && (cp1.y == cp2.y) & (cp1.c == cp2.c)) |
Ordinarily, one advantage of using methods instead of functions is dynamic dispatch: each class can choose its own code to execute in response to a given message. Therefore, moving from binary methods to binary functions may seem a step backwards. The programmer must now know when to apply methods such as eqColorPoint, instead of relying on the objects themselves “knowing” which equality is appropriate. (To be fair, it is worth noting that it is difficult to achieve dynamic dispatch for binary methods such as equal, without adding additional methods to the classes PointClass and ColorPointClass, as in Section 4.3.)

The loss of dynamic dispatch when functions are used instead of binary methods is a serious problem. This problem manifests itself by causing code duplication which would not be needed if methods were used. When methods are used, it often occurs that the body of one method, m, invokes another method, n, on the receiving object itself. If n is overridden in a subclass, then invocations of m on objects generated by the subclass will correctly call the new version of n. However, if n happens to be a binary method, then replacing it with two (or more) binary functions results in a loss of dynamic dispatch. The loss of dynamic dispatch means that either the correct version of n will not be called from the inherited method in the subclass, or extra code must be written in the subclass (for method m) to call the proper version of n.

Figure 5 is an example that illustrates the problems caused by loss of dynamic dispatch. (Although we have no hard data on how common such examples are, this example is a combination of standard idioms.) LinkClass is a simple class of linked list objects, and DoubleLinkClass is a subclass that uses double links (a more complete implementation would include methods such as reverse, map, and length). The type MyType given to variables next and link in the example represents the type of objects of the current class. That is, it means Link in the class LinkClass, but means DoubleLink in the class DoubleLinkClass and in the instance variables and methods it inherits from LinkClass. MyType will be discussed in more detail in Section 4.1 below; also cf. [51, 12, 13, 29]. The objects now have only one interesting method, append, which is inherited by DoubleLinkClass. This method uses setNext, a binary method, to set the pointer next, and setNext is overridden in DoubleLinkClass to also properly maintain the prev link to the previous object.

In a hypothetical function encoding, the setNext method would be replaced by functions setNextLink and setNextDoubleLink that lie outside the class definition (ignoring for now questions of privileged access). However, since append invokes setNext, it would have to be re-written as two almost identical functions, one invoking setNextLink and the other invoking setNextDoubleLink, causing unnecessary code duplication. An in-place reverse method of its arguments is another method for which inheritance would suffer under this encoding. Thus dispatch can be statically resolved, but only at the cost of code duplication if this scheme is used.

### 3.2 Making Both Arguments into One Object

Even in a “purist” object-oriented language where every operation is treated as a message sent to some object, we may place binary operations outside of the objects on which they operate by turning the two argument objects into a single pair object and invoking the method on the pair. To see how this would work, imagine that the types Point and ColorPoint do not have any binary methods. For example, they could be:

```
Point = OT ⟨ ⟨ x : real; y : real ⟩ ⟩
ColorPoint = OT ⟨ ⟨ x : real; y : real; c : string ⟩ ⟩
```

With this definition, ColorPoint would be a subtype of Point.

Now define two new classes, PointPairClass and ColorPointPairClass, each with a method named equal, as shown in Figure 6. Note that the types of the objects generated by these classes are the same, OT ⟨ ⟨ equal : bool ⟩ ⟩, since they have the same public interface.

```java
class PointPairClass
  instance variables
    p1: Point
    p2: Point
  methods
    equal: bool is
      return (p1.x == p2.x) && (p1.y == p2.y)
  end class

class ColorPointPairClass
  instance variables
    p1: ColorPoint
    p2: ColorPoint
  methods
    equal: bool is
      return (p1.x == p2.x) && (p1.y == p2.y)
      && (p1.c == p2.c)
  end class
```

Figure 6: The classes PointPairClass and ColorPointPairClass.

So the former binary methods are now unary methods of these new classes. What would originally have been written as:

```
aCpt.equal(anotherCpt)
```
to compare two colored points, will now be written with these new classes as:

\[(\text{new PointPairClass}(\text{aCPt, anotherCPT}).\text{equal})\]

If the types of aCpt and anotherCpt are both ColorPoint, then one might wish instead to compare them as colored points, in which case one would write:

\[(\text{new ColorPointPairClass}(\text{aCPt, anotherCPT}).\text{equal})\]

There is a benefit in making these pair objects: it clarifies the perspective desired for the equality comparison. When one creates a PointPairClass object, it is clear what behavior is expected from its \text{equal} method; this expectation is borne out even when the two points that make up the PointPairClass object are actually ColorPoint objects.

Because the classes generate objects of the same (interface) type, one can have a variable \text{myPointPair} that denotes objects generated by either PointPairClass or ColorPointPairClass. In this case a message send such as \text{"myPointPair.equal"} results in the invocation of the \text{equal} method defined in whichever class was used to generate the object. Thus, sending the \text{equal} message to a pair object gets the view with which the pair was created. This should be contrasted with the function call \text{"eqPoint(aCPt, anotherCPT)"}, which always compares its two arguments as points. It can also be contrasted with a message-send of the form \text{"aCPt.equal(anotherCpt)"}, which always uses the \text{equal} code of ColorPointClass.

This approach has problems similar to the function approach that was discussed previously—the LinkClass example of Figure 5 would require code duplication for inherited methods such as \text{append}.

4 Embracing Binary Methods

Having examined what happens if binary methods are avoided, in this section we consider the typing mechanisms that must come into play if one chooses not to avoid them.

Two important solutions have been proposed to the typing problems posed by binary methods. One solution, first proposed by the Abel project at HP labs [25], develops a method that partially solves the Point/ColorPoint problem by relaxing the requirement that subclasses generate subtypes. As they put it, “Inheritance is not subtyping.” They did not, however, propose a concrete mechanism for realizing their ideas in an object-oriented language. In Section 4.1, we show one way this may be done using the concept of matching [12, 13].

The other important solution was presented in two papers at the 1991 OOPSLA conference [2, 31]. These papers deal with the static type-checking of languages with \text{multi-methods} (also called \text{generic functions} or \text{overloaded functions}). Multi-methods as in CLOS allow, as we show in Section 4.2.1, the Point/ColorPoint example to be typed preserving the subtyping of the two classes. But this is obtained at the expense of the encapsulation of the methods, since the generic functions, like the functions in Section 3.1, are separated from objects (objects encapsulate only data). In Sec-

---

```plaintext
class LinkClass
  instance variables
    value: integer
    next: MyType
  methods
    getValue: integer is return(value)
    getNext: MyType is return(next)
    setValue(n: integer): unit is value := n
    setNext(link: MyType): unit is next := link
    append(link: MyType): unit is
      if next == nil then self.setNext(link) else next.append(link)
end class

class DoubleLinkClass subclass of LinkClass
  instance variables -- value and next are inherited
    prev: MyType
  methods -- getValue, getNext, setValue, and append are inherited; setNext is overridden
    getPrev: MyType is return(prev)
    setNext(link: MyType): unit is next := link; link.setPrev(self)
    setPrev(link: MyType): unit is prev := link
end class
```

Figure 5: The classes LinkClass and DoubleLinkClass.
In Section 4.1 we show how to reconcile multi-methods with objects encapsulating data and code [17, 45]. Closely related to the solutions of Section 4.2 is Ingalls’ solution to the multiple dispatch problem [34]. He presented his solution in an untyped framework, but it can be adapted to a typed language, as Section 4.3 shows. Lastly, we show in Section 4.4 how a general principle of giving more “precise” types to binary methods produces more flexible typings across a range of approaches, even in the case where binary operations are not treated as methods.

4.1 Matching

This section describes how a relation called “matching,” which is weaker than subtyping, can replace subtyping in many situations [12]. In particular, we will see below that this generalization of subtyping provides us with the ability to handle binary methods smoothly.

4.1.1 Generalizing subtyping to matching

As seen in Section 2.1, languages that insist that subclasses generate subtypes often compensate for the resulting type problems by restricting the programmer’s ability to change the types of parameters of inherited methods. This effectively eliminates the use of binary methods in these cases. If one feels that binary methods are important, then an obvious solution is to give up the identification of subclasses with subtypes. An important advantage of this decision, not discussed further here, is to separate the notion of interface (type) from that of implementation (class). In the remainder of this section we assume such a separation has been made, and thus that the notions of subtyping and matching (defined below) depend only on the interfaces of objects, not the classes generating them.

Most object-oriented languages provide a name for the receiver of a message (e.g., self or this), which can be used inside method bodies. Similarly, we use MyType as a keyword that denotes the type of the receiver [51]. It may be used in the definition of methods whose parameters or return types should be the same as that of the receiver. One can think of the object type in the following as simply another way of writing the type Point given in section 2.1, and it could also be the type of objects of a polar implementation of points.

\[
\text{Point} \equiv OT \langle \langle x: \text{real}; y: \text{real}; equal: \text{MyType} \rightarrow \text{bool} \rangle \rangle
\]

One advantage of MyType is that it makes it easier for human readers to compare types like Point and PolarPoint. A more important advantage is that it works well with inheritance of methods, because its meaning changes in the subclass. For example, when MyType is used in the definition of ColorPointClass, all occurrences of MyType in the methods automatically represent ColorPoint rather than Point.

\[
\text{ColorPoint} \equiv OT \langle \langle x: \text{real}; y: \text{real}; colouring; equal: \text{MyType} \rightarrow \text{bool} \rangle \rangle
\]

As before we can show the type ColorPoint is not a subtype of Point. However, there is a relationship between the types ColorPoint and Point, which is clearly apparent when looking at their types written using MyType. One can see that the only difference is the addition of a new method, c, to ColorPoint.

We say one object type matches another if the first has at least the methods of the second and the corresponding method types are the same, considering MyType in one to be “the same” as MyType in the other. We use \(<\#\) to denote this relationship. In symbols,

\[
OT \langle m_1: \tau_1; \ldots; m_n: \tau_n \rangle \ <\# \ OT \langle m_1: \tau_1; \ldots; m_k: \tau_k \rangle
\]

holds iff \(k \leq n\). (In fact, a more general definition is possible in which the types of corresponding methods of the first are all subtypes of the corresponding types of the second [12]. This means that the corresponding result types are subtypes—vary in a covariant way—while the corresponding parameter types are supertypes—vary in a contravariant way. However, this more general relation will not be needed here.)

Because the meaning of MyType changes in subclasses, the meanings of the types of methods in subclasses need not be the same as those of the corresponding methods in the superclass. However, type-safe rules for defining subclasses can ensure that the types of the objects from the subclass always match the types of the objects generated from the superclass. In order to obtain type safety, it is necessary to type check the methods of a class under the assumption that MyType only matches the types of objects being defined by the class. This ensures that these methods will continue to be type-safe when inherited in subclasses [12]. While some routines will not type check with this assumption, even though they would have passed under the stronger assumption that MyType is exactly the type of objects generated by this class, in our (admittedly not comprehensive) experience, very few routines fail. Matching tells you what messages can be sent to an object, and what their types will be. However, if \(S \ <\# T\), it does not allow the use of a parameter of type \(S\) where one of type \(T\) is expected. Nor does it allow assignment of expressions of type \(S\) to variables of type \(T\).

As stated earlier, ColorPoint is not a subtype of Point. ColorPoint and Point provide an example of two types which match, but are not subtypes. Basically, if a class has a binary method, that is, a method with a parameter of type MyType, subclasses of that class that add new methods will not generate subtypes. On the other hand if a method’s return type is MyType, this will not stand in the way of subtyping. Both of these follow easily from the subtyping rule for recursive types in [3], and the fact that MyType can be seen as an abbreviation for a recursive definition of types.

What if we want to use a ColorPoint as an actual parameter in a procedure or function that originally ex-
We can introduce a language feature to support a form of bounded polymorphism using matching. With this feature, functions can be specified to take type parameters whose values are restricted to “match” another type. Of course, unrestricted type parameters can also be provided, but in a large number of situations some sort of restriction is necessary.

As an example, suppose we wish to write a routine to sort an array whose elements are drawn from some ordered set. In an object-oriented language, the requirement that the elements be ordered can be modeled by demanding that they support (at least) \textit{less} than and \textit{equal} methods. Define:

\[
\text{Comparable} \equiv \text{OT} \ (\langle \text{less} \text{.} \text{than} : \text{MyType} \rightarrow \text{bool}, \\
\quad \text{equal} : \text{MyType} \rightarrow \text{bool} \rangle)
\]

With this definition, the header of our polymorphic sort routine is as follows, where the notation \textit{T} \#\textit{Comparable}\textit{)} means that the type parameter \textit{T} must match the type \textit{Comparable}:

\[
\text{procedure sort}(\text{T} \#\text{Comparable}; \ a: \text{Array of T});
\]

And the function then has type\(^3\):

\[
\text{sort : All(}\text{T} \#\text{Comparable}) \ (\text{Array of T}) \rightarrow \text{unit}.
\]

If \text{PhoneEntry} is an object type supporting at least methods \textit{less} than and \textit{equal} of type \text{MyType} \rightarrow \text{bool},

and if \text{pArray} is an array of elements of type \text{PhoneEntry}, then \text{sort(PhoneEntry, pArray)} is a legal call of \text{sort}.

It is worth noting here that the type \text{Comparable} has no useful proper subtypes because of the appearance of \text{MyType} as the type of a parameter in its methods. Thus, if the bounds on type parameters were only expressed in terms of subtyping, it would be impossible to apply the \text{sort} routine to any interesting arguments.

The use of bounded matching is equivalent to the use of F-bounded polymorphism suggested in [14]. It is also very similar in effect to the restrictions on type parameters expressible in CLU and Ada (as well as the type classes of Haskell). For example, in Ada one would write the \text{sort} routine as:

\[
\text{generic}
\]

\[
\quad \text{type t is private;}
\]

\[
\quad \text{with function } "<"\ (x, y: t) \text{ return BOOLEAN is } < >;
\]

\[
\quad \text{with function } "="\ (x, y: t) \text{ return BOOLEAN is } = >;
\]

\[
\quad \text{procedure sort} \ (A: \text{in out array } (<>) \text{ of t}) \text{ is } ...
\]

This is similar to the \text{sort} procedure written with bounded matching. Object-oriented languages containing similar constructs are Emerald [10], School [50], and Theta [40].

Returning to our example with \text{Point}, if \text{f(p: Point)} is a function accepting an argument of type \text{Point} then it can often be rewritten in the form \text{f(T="# Point; p:T)} so that it accepts a type parameter matching \text{Point} and an object of that type. If this type checks, then it will be possible to apply it to the type \text{ColorPoint} as well as an object of type \text{ColorPoint}. Of course this rewriting will not succeed in all cases—\text{breakit} being a prime example. The reason this transformation will not succeed in \text{breakit} is that the formal parameter \text{nuPt} will be of some type \text{T="# Point}, while \text{p} will always be of type \text{Point}. Thus we cannot guarantee that the type of the argument to \text{equal} in the body of \text{breakit} will be the same as the type of \text{p}, and the type check must fail.

What can actually be done with the information that one type matches another? The matching relation guarantees that certain messages may be sent to an object. If \text{T="# Comparable} then objects of type \text{T} can be sent messages \text{less}\_\text{than} and \text{equal} (and their parameters must also be of type \text{T}). It turns out that for most situations this is all that is needed in order to ensure that the object is usable. The stronger information that a type is actually a subtype of another is generally not needed.

In particular, bounded matching can be viewed as an explicit, weakened (and hence more generally applicable) form of subtyping. If subsumption were necessary to type a function call, the code could be rewritten so the function constrains the type parameter, like \text{Comparable} above, and function invocations explicitly pass the “smaller” type as argument. Simple subtyping is handled by the case where the type constraint contains no occurrences of \text{MyType}. The disadvantage of this encoding of subtyping is that all subtypings must be explicitly given in the program.

In general the use of bounded matching requires one to “plan ahead,” by identifying the type parameter to be matched against. This was illustrated in the \text{sort} example: the type \text{Comparable} needs to be discovered by the programmer, and every use of \text{sort} requires that an explicit type parameter be passed. This is in contrast to subtyping, which is implicit and, as mentioned above, does not require any explicit type instantiation to be given in the program. (It is an interesting open problem to show how to automatically infer declarations that use matching.) If we were to decide to eliminate subtyping altogether in favor of matching, then all object subtypings would have to be recast as bounded matchings. Moreover, since we have, thus far, only defined matching for object types, we would not be able to capture the use of subtyping on other types without extending the definition of matching.

Another difficulty with relying only on matching is that it is not type-safe to perform an assignment to

\(^3\text{The notation } \text{All(S }\# T)E(S) \text{ is the universally polymorphic type that can be instantiated to } E(S) \text{ for all } S \text{ such that } S \# T.\)
and \(\tau < \sigma\) achieved by circular subtype assertions.

Details can be found in the paper cited above.

While understanding objects types as fixed points in this way is intuitively appealing, the ability to unfold recursive types does not interact well with the definition of matching as essentially a relation on these operators (rather than the fixed points themselves).

For example, look at the relationship between the following types:

\[
\text{EPoint} \equiv \text{OT} \langle x: \text{real}; y: \text{real}; \text{equal}: \text{EPoint} \to \text{bool} \rangle
\]

\[
\text{Point} \equiv \text{OT} \langle x: \text{real}; y: \text{real}; \text{equal}: \text{MyType} \to \text{bool} \rangle
\]

\[
\text{ColorPoint} \equiv \text{OT} \langle x: \text{real}; y: \text{real}; c: \text{string}; \text{equal}: \text{MyType} \to \text{bool} \rangle
\]

Understanding object types as recursive records, the first two would seem to be the same type. However, while ColorPoint matches Point according to our definition of matching, ColorPoint does not match EPoint. Thus these two seemingly identical types must be treated as being distinct. It is worth noting that there is some justification to treating the two types as distinct, as the equal method of a class which generates objects of type Point is type checked under weaker assumptions on the parameter (i.e., it has type MyType, which is only assumed to match Point) than the corresponding method of EPoint, in which the parameter has type EPoint.

This anomaly suggests that an encoding of object types in terms of something a bit weaker than fixed points might be necessary. It is an interesting open problem to find such an encoding of object types (including MyType) and a corresponding semantic definition of matching.

### 4.2 Multi-methods

A different solution whereby binary methods can be embraced is to use multi-methods. Contrary to matching, this solution does not introduce a new relation on types, since with multi-methods one can have both type safety and subtyping relations such as ColorPoint \(\subset\) Point.

A multi-method is a collection of method bodies associated with one message name. The selection of which method body to execute depends on the classes of one
of more of the parameters of the method (rather than just on the class of the receiver as in ordinary object-oriented languages).

In this survey we distinguish two different kinds of multi-methods: the ones used by the language CLOS [27], and the encapsulated multi-methods of [17, 45]. A unified analysis of both kinds of multi-methods is given in [17]. We now describe each kind in turn.

### 4.2.1 Multi-methods à la CLOS

Intuitively the idea is to consider (multi-)methods (in CLOS jargon, *generic functions*) as global functions that are dynamically bound to different method bodies according to the classes of the actual arguments. An object does not encapsulate its methods, just the data (its instance variables). There no longer exists the notion of a privileged receiver for a method (the one that encapsulates it, usually denoted by `self` or `this`) since a multi-method is applied to several arguments that equally participate in the selection of the body. In this case we talk of “multiple dispatching” languages, in antithesis to “single dispatching” ones where a privileged receiver is used. A class of objects is then characterized just by the internal variables of its instances.

For example, in a typed multi-method-based language, the classes given in Figures 1 and 2 would be defined as in Figure 7.

In order to simplify the exposition in this section, we identify classes and types, in the sense that the name of a class (for which we no longer use the suffix `Class`) is used as the type of its instances; therefore in this section (and in this section only) `p: Point` will also mean “`p` is an instance of class `Point`.” Thus, when discussing multi-methods à la CLOS, we write class names where types would otherwise appear. This allows one to consider multi-methods as overloaded functions, whose actual code is dynamically selected according to the type (i.e., the class) of the arguments they are applied to.

The definitions of the methods in Figure 7 are completely disconnected from those of classes. There are two distinct definitions for `equal`, one for arguments of types `Point x Point` and the other for arguments of type `ColorPoint x ColorPoint`. We say that the message `equal` denotes a multi-method (or a generic function, or an overloaded function) formed by two branches (or method bodies). The type of a multi-method is the set of the types of its branches; thus `equal` has type:

```
{Point x Point -> bool, ColorPoint x ColorPoint -> bool}
```

When `equal` is applied to a pair of arguments, the system executes the branch defined for those parameters whose type “best matches” the type of the arguments. For example if `equal` is applied to two arguments in which at least one of them is of type `Point` and the other is a subtype of it, then the first definition of `equal` is executed; if both arguments have as type a subtype of `ColorPoint` then the second definition is selected. More generally, when a multi-method of type

```
{S₁ → T₁, S₂ → T₂, ..., Sₙ → Tₙ}
```

is applied to an argument of type `S`, the system executes the body defined for the parameter of type `S_j = \text{min}_{i=1..n}\{S_i \mid S <: S_i\}`. This selection is performed at run-time. In this way one obtains dynamic dispatch. Note that in this paradigm binary methods are really binary, since the implicit argument given by the receiver of the message is, in this case, explicit.

In [18] it is proved that to have a sound type system it suffices that every multi-method of type `{S₁ → T₁, S₂ → T₂, ..., Sₙ → Tₙ}` satisfies the following condition.

```
∀i, j ∈ [1..n] \text{ if } S_i <: S_j \text{ then } T_i <: T_j
```

(This is similar to the monotonicity condition of [49, 42].) Note that all the multi-methods defined in Figure 7 (and in particular `equal`) satisfy this condition. Therefore `ColorPoint <: Point` does not cause type insecurities.

Intuitively, the idea underlying the multi-method approach is that binary methods may be applied to arguments of different types and that, in general, it is not possible to choose the code to execute according to the type of just one argument. To determine which method body must be executed one needs to know the types of all the arguments of the method. In single dispatching the branch selection is based only on one argument—the receiver; therefore combining subtyping and binary methods with heterogeneous arguments is not type-safe.

In contrast, using a multi-method we can refine the selection by considering all the arguments. Thus it need never happen that the argument of a method has a supertype of the type of the corresponding parameter (as in the case of `breakit`). It is important to stress that this constitutes an approach completely different from matching, where the arguments of a binary method are statically forced to have the same type.

Note also that multi-methods allow one to specialize `equal` in a different way for each possible combination.

---

5If we were to distinguish between types and classes (i.e. between interfaces and implementations: cf 4.1.1), then a new notation would be needed to specify both a class and a type parameter for multi-methods. One possibility is to use the notation of Cecil [20, 22], which does separate these concepts.

6Some further conditions are required to assure that a best matching branch always exists for the selection (see [2], [22], and [18]).
object-oriented languages with single dispatching and to apply the ideas of multi-methods in more traditional pursuing that idea, in the next subsection, we show how to control instance variable access [22]. Instead of such problems may be to add a separate module system to control instance variable access [22]. Instead of pursuing that idea, in the next subsection, we show how to apply the ideas of multi-methods in more traditional object-oriented languages with single dispatching and classes.

```plaintext
class Point
    includes
    xValue: real
    yValue: real
end class

class ColorPoint subclass of Point
    includes -- xValue and yValue are inherited
cValue: string
end class

method x(p: Point):real is return(p.xValue)
method y(p: Point):real is return(p.yValue)
method c(p: ColorPoint):real is return(p.cValue)

method equal(p:Point, q:Point):bool is return((x(p)==x(q)) && (y(p)==y(q)) )
method equal(p:ColorPoint, q:ColorPoint): bool is return((c(p)==c(q)) && (x(p)==x(q)) && (y(p)==y(q)) )
```

Figure 7: Point and ColorPoint classes defined using multi-methods à la CLOS.

of arguments. It suffices to add the branches for the remaining cases:

```
method equal(p: Point, q: ColorPoint): bool is . . .
method equal(p: ColorPoint, q: Point): bool is . . .
```

As we have seen, CLOS’s multi-methods induce an object-oriented style of programming that is rather different from the one of traditional single dispatching object-oriented languages. Most of the languages that use multi-methods are untyped (e.g. CLOS [27], Dylan [7], which use classes instead of types to drive the selection of multi-methods). The only strongly-typed languages in our ken that use multi-methods are Cecil [22], and Polyglot [2].

The lack of encapsulation in multi-methods is both an advantage and a drawback. The drawback is methodological: an object (or a class of objects) is no longer associated with a fixed set of methods that have privileged access to its internal representation. The usual rule is that any method with a formal parameter of a given class can access the instance variables of the actual parameter object. The advantage is that this solves the privileged access problem described in Section 2.2, because a binary method can gain privileged access to both its arguments. However, because such methods can be defined anywhere in the program, one cannot restrict direct access to instance variables to a small area of the program text. One way to fix such problems may be to add a separate module system to control instance variable access [22]. Instead of pursuing that idea, in the next subsection, we show how to apply the ideas of multi-methods in more traditional object-oriented languages with single dispatching and classes.

Conventional wisdom is that multiple dispatch is more expensive than single dispatch. In a single dispatch language, a single table lookup can find the best argument branch. With multiple dispatch, it may be more expensive to compute the branch of a multi-method that matches the arguments best, although various techniques have been designed to minimize the added expense [6, 23, 26]. However, in a language where the compiler can tell which argument positions need dispatching (as in CLOS), one can implement multi-method dispatch as a chain of single dispatches [36]. If this is done, then there is no extra cost for multiple dispatch in programs that do not use it; that is, in a multiple-dispatching language, programs that only use single dispatch have the same cost as in a single-dispatching language. Moreover, if a program in a single-dispatching language is written by using additional dispatching after methods are called to resolve problems caused by binary methods (as in Section 4.3), then such a program will be no faster than the equivalent multi-method program [21].

A final drawback of multi-methods à la CLOS is the difficulty of combining independently developed systems of multi-methods [24]. While other ways to solve this problem have been studied [22], the problem nearly disappears when multi-methods are combined with single dispatching, as described next.

### 4.2.2 Encapsulated multi-methods

To solve the encapsulation problems of multi-methods à la CLOS, we seek to emulate the Smalltalk model, where every method is the method of one object. Thus each method has a privileged receiver argument (self),
which is the only argument whose internal state can be accessed by the method. Instead of defining multi-methods as global functions, the idea is to use them to define the bodies of some methods in a class definition [17]. In this way a multi-method is always associated to a message \( m \) of a class \( C \). When \( m \) is sent to an object of class \( C \), it is dispatched to the corresponding method. If this method happens to be a multi-method, then the branch is selected according to the types of the further arguments of \( m \). Thus, the selection of the method is still based on the receiver, but the actual code is selected among several bodies that are encapsulated inside the object. Inside these bodies, the receiver is still denoted by the keyword `self`. Encapsulated multi-methods are to be distinguished from static overloading (as found in Ada, Haskell, C++, and other languages), because the selection of code must be made dynamically.

As an example of this technique, take the class `Point` as in Figure 1 and rewrite the class `ColorPoint` as in Figure 8. In that Figure (note we are using our original notation for classes again) there are two definitions for `equal`: the first is executed when the argument of `equal` is of type `Point`, the other when it is of type `ColorPoint`. The selection of the appropriate definition is done at run-time when the argument of `equal` has been fully evaluated and hence its run-time type is apparent. The selection is based on the type of the additional argument. In other words, we have transformed the method associated to `equal` into a multi-method, where arguments of different types are associated to different codes.

There are two differences from multi-methods à la CLOS. The first is that multi-methods are defined in particular classes, whereas in CLOS they are globally defined generic function names. This solves the encapsulation problems of CLOS multi-methods, because access to instance variables is restricted to the methods of a class, as only the receiver’s instance variables can be accessed. The second difference is that dispatch is not based on actual argument classes, but rather on argument types. This is possible because no privileged access is obtained to the additional arguments. However since types are not equated with classes, the technique cannot solve the problem of privileged access to other arguments discussed in Section 2.2: several different classes might implement the same type, so from the type alone it is unclear which class implementation the method should have access to.

The type of a multi-method is the set of the types of its different codes. Thus the type of an instance of `ColorPointClass` now becomes

\[
\text{ColorPoint} \equiv \text{OT} \langle x: \text{real}; \ y: \text{real}; \ c: \text{string}; \ \text{equal}: \{ \text{Point} \to \text{bool}, \text{ColorPoint} \to \text{bool} \} \rangle
\]

and `ColorPoint <: Point` holds, since, for subtyping, ordinary methods are considered as multi-methods with just one branch (their type is a singleton set) and in a type system for multi-methods (see [17]) one can deduce: \{Point \to bool, ColorPoint \to bool\} <: \{Point \to bool\}.

More precisely, the subtyping relation between sets of types states that one set of types is smaller than another if and only if for every type contained in the latter there exists a type in the former smaller than it. This fits the intuition that one multi-method can be replaced by another multi-method of different type when for every branch that can be selected in the former there is one branch in the latter that can replace it.

Thus, if in writing a subclass one wants the type of the instances to be a subtype of the type of the instances of the superclass, then some care in overidding binary methods is required. Indeed, the rule of thumb for this approach is that to override a binary method one must use an (encapsulated) multi-method with (at least) two branches: one with a parameter whose type is the type of the instances of the class being defined, the other with a parameter whose type is the type of the instances of the original superclass in which the message associated with the binary method has been first defined. Thus, when a binary method is overridden in a new class, it is not enough to specify what the new method has to do with the objects of the new class. It is also necessary to specify what it has to do when the argument is an object of a superclass. Fortunately, this does not require a large amount of extra programming. The number of branches that suffice to override a binary (or \( n \)-ary) method in a type-safe manner is independent of both the size and the depth of the inheritance hierarchy; indeed, it is always equal to two. For example, suppose that we further specialize our `Point` hierarchy by adding further dimensions:

```plaintext
class 3DPointClass subclass of PointClass
  instance variables x3Value: real
  methods 

  class 4DPointClass subclass of 3DPointClass
  instance variables x4Value: real
  methods 

  ...

...and so on, up to a dimension \( n \). The new classes form a chain in the inheritance hierarchy. If we want to override `equal`, what do we have to do in order for this to be a chain of the subtyping hierarchy too (i.e., \( nDPoint <: \ldots <: 4DPoint <: 3DPoint <: Point \))?

If we want to override `equal` in `nDPointClass` (thus we want that in the description of `nDPointClass` a definition of the form `equal(p : nDPoint)`is \ldots appears), then the first idea is to write for the class `nDPointClass` a multi-method with \( n - 1 \) branches, one for each class in the chain\(^7\). This is possible, but for type safety a

\(^7\)Of course, if in the definition of `nDPointClass` we do not give any definition for `equal` then `nDPointClass` inherits the last (multi-)method defined for `equal` in the upper hierarchy. It is
multi-method with two branches is enough: one for arguments of type nDPoint, which is the one we want to define, and the other for arguments of type Point, which will handle all the arguments of a supertype of nDPoint. For example, in case of n = 4 one could define:

class 4DPointClass subclass of 3DPointClass
instance variables x4Value: real
methods
x4:real is return(x4Value)
equal(p: Point):bool is return(p.equal(self))
equal(p: 4DPoint):bool is return((xValue==p.x) && (yValue==p.y) && (x3Value==p.x3) && (x4Value==p.x4))
end class

Type safety stems from the fact that the subtyping condition is satisfied.9

A final remark is in order. The different branches that compose a single multi-method are not required to return the same type. For type safety it suffices to have the condition (1) as for multi-methods à la CLOS: for each pair of multi-method branches c1,c2 with the same name and number of arguments,10 if the parameter types of c1 are smaller than the corresponding parameter types of c2, then the result type of c1 must be smaller than the result type of c2 [49, 18].

Note that multi-methods can be considered as a kind of typecase construct enhanced by two features: (a) the selection of the case to apply uses subtyping instead of type equality; (b) all the cases are not required to return the same type (they are solely required to satisfy the condition (1)). This makes multi-methods more flexible than statically defined typecase statements as might be found in imperative languages: without (a), a binary method whose parameter type is guarded using a typecase would always have to be rewritten when new subclasses are added to the program; without (b), specialization of the result type of binary methods could not be handled. The only remaining problem that multi-methods and typecase share is that if the method should only be defined with a parameter of exactly the same type as the receiver, the multi-method user will be required to add a new method body with the original parameter type whose only purpose is to raise an error message. See the conclusion for further discussion of this issue.

Some further consistency conditions are required in case of multiple inheritance [31, 45, 18, 22].

One of the main advantages of this approach is that the extra branch required to assure type safety of subtyping can be generated in an automatic way. Therefore this technique can be embedded directly in the technology of the compiler, and used to “patch” the already existing code of languages that use covariant specialization, like Eiffel and O2. Thus, like the solution given in the next section, this solution can be directly applied to languages with covariant argument specialization without requiring any modification of the code: a recompilation of existing code will suffice (see [11]).

On the other hand this approach has some disadvantages. One disadvantage compared to multi-methods à la CLOS is that it does not solve the problem of obtaining privileged access to other arguments in a binary method. Another disadvantage of this approach is that in case of multiple inheritance additional type checking constraints are needed. The problem is that when multiple inheritance is used, the notion of a “best matching branch” to select or to inherit may be lost. Consequently, unconstrained use of multi-methods can break the modularity of programming [24], since the addition

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9In general, if we have a hierarchy of n classes whose instances have type S0 < ... < Sn and we want to define for each of them a binary method, respectively returning the type Tn < ... < T1 then according to the subtyping rule for multi-methods we have the following type inequalities: {Sn -> Tn, S1 -> Tn-1} < ... < {S1 -> T1, S1 -> T0} < ... < {S1 -> T1, S1 -> T0} < ... < {S1 -> T1}. This proves that two branches always suffice for binary methods. The declarations of the classes for points are a special case of this, where S1 = Point, for i ∈ [3..n] Si-1 = 3DPoint, and for i ∈ [1..n-1] Ti = bool.

10Indeed multi-methods may have more than one parameter (this allows us to deal with n-ary methods), and the multi-method branches are not all required to have the same number of parameters.
4.3 Simulating Multi-methods in a Single-Dispatching Language

Ingalls offered a solution to what he called the problem of “multiple polymorphism” at the first OOPSLA conference [34]. His solution to the binary method problem, offered in the context of single-dispatching languages such as Smalltalk-80 [32], was to use two message dispatches, one to resolve polymorphism of each argument.

In the example of points, colored points, and equality, the equal method would be coded as in Figure 9. As usual, the class ColorPointClass inherits the method equalPoint from the class PointClass. Now the (mutually recursive) types of the instances of PointClass and ColorPointClass are:

\[
\text{Point} \equiv \text{OT} \langle (x: \text{real}; \ y: \text{real}); \ \text{equal}: \text{Point} \rightarrow \text{bool}; \ \text{equalPoint}: \text{Point} \rightarrow \text{bool}; \ \text{equalColorPoint}: \text{ColorPoint} \rightarrow \text{bool} \rangle
\]

\[
\text{ColorPoint} \equiv \text{OT} \langle (x: \text{real}; \ y: \text{real}; \ c: \text{string}); \ \text{equal}: \text{Point} \rightarrow \text{bool}; \ \text{equalPoint}: \text{Point} \rightarrow \text{bool}; \ \text{equalColorPoint}: \text{ColorPoint} \rightarrow \text{bool} \rangle
\]

Notice that, with this typing, ColorPoint is a subtype of Point. Also, equal in ColorPoint is a binary method, since by subsumption it can have argument type ColorPoint as well. This typing can be said to be more precise than the typing of ColorPoint given in the introduction; the general issue of the use of more precise typings is taken up in Section 4.4.

The solution offered by Ingalls is probably the best-known way to simulate multiple dispatch in a language with only single dispatch. With respect to true multiple-dispatch, the Ingalls simulation is more exact than the function simulation offered in Section 3.1, since it can arrange for equal with two arguments whose dynamic type is ColorPoint to always take color into account, regardless of the static types of the argument expressions. This is because of the second dynamic dispatch in the equal method. Such a result is not possible with the function simulation of Section 3.1: one will always be able to apply the eqPoint function to two ColorPoint objects and lose exact type information. This example is thus one case for which dynamic dispatch on binary methods occurs. In this respect, the simulation of multiple-dispatch with external functions is less faithful and flexible than the Ingalls simulation.

This translation can also be contrasted with encapsulated multi-methods as described in Section 4.2.2. Ingalls’ translation lacks modularity in that it requires equalColorPoint to be added to the PointClass class when ColorPointClass is defined. With multi-methods, modularity can be preserved since the redefinition of the equal method inside ColorPointClass does not require any modification of the code for PointClass; however, this introduces an unnatural asymmetry, since the redefinition of equal requires one to write code for how a ColorPoint behaves when its equal method is passed a Point, but not vice-versa. The natural symmetry cannot be restored except by breaking the modularity of the multi-method solution.

It should be pointed out that the above argument only holds if we require (multi-)methods to be written in classes, as in Section 4.2.2. For multi-methods à la CLOS there is no problem of asymmetry, although there is still a modularity problem. However, the multi-method approach still requires one to go back and add code for types that appeared to have been completed earlier.

Ingalls’ solution is surprisingly general—by overriding equalPoint in ColorPointClass, a different method can be executed for all four combinations of Point and ColorPoint. Ingalls’ solution could in fact be used as one technique for implementing encapsulated multi-methods in a compiler, provided the compiler had access to all of the code at compilation time.

Finally, for large inheritance hierarchies the number of cases required by Ingalls’ solution can, in principle, become quite cumbersome.

4.4 Precise Typings

It is sometimes advantageous to use more precise typings for methods. A binary method only needs its argument to have the methods that are explicitly used. Generally this is a weaker requirement than having the argument be an object of the current class, and it may allow for a “larger” (with respect to the <: relation)
class `PointClass`

... methods ...

```java
equal(p: Point): bool is return( p.equalPoint(self) )
equalPoint(p: Point): bool is return( xValue==p.x && yValue==p.y )
equalColorPoint(p: ColorPoint): bool is return( self.equalPoint(p) )
```

class `ColorPointClass` subclass of `PointClass`

... methods ...

```java
equal(p: Point): bool is return( p.equalColorPoint(self) )
    -- equalPoint is inherited
equalColorPoint(p: ColorPoint): bool is return( cValue==p.c && xValue==p.x && yValue==p.y )
```

Figure 9: Ingalls’ simulation of multi-methods.

type of the argument of this method; by the contravariant subtyping rule for functions this produces a smaller type for the method. The informal idea is thus to give methods smaller types [9, 10]. By subsumption, these types can always be lifted to “true binary” form, allowing objects of the same class to be passed as arguments to the method. Thus, specifying a smaller type of a method can only increase its usability.

Ingalls’ solution in Section 4.3 in fact depends on the use of precise types, for the key to its typability is the use of `Point` for the type of the argument of `equal` in `ColorPointClass`. This gives the method a smaller type than if the argument were of type `ColorPoint`. In this section we elaborate on this technique.

By way of illustration, consider the original `Point/ColorPoint` example of Figures 1 and 2. Since neither `equal` method calls `equal` recursively, the types

\[ \text{Point}_{\min} \equiv \]
\[ \text{OT} \langle x: \text{real}; y: \text{real}; \]
\[ \text{equal}: \text{OT} \langle x: \text{real}; y: \text{real} \rangle \rightarrow \text{bool} \rangle \]

\[ \text{ColorPoint}_{\min} \equiv \]
\[ \text{OT} \langle x: \text{real}; y: \text{real}; c: \text{string}; \]
\[ \text{equal}: \text{OT} \langle x: \text{real}; y: \text{real}; c: \text{string} \rangle \rightarrow \text{bool} \rangle \]

may also be given. These types are subtypes of the types given originally. Note that the objects passed to `equal` themselves require no `equal` method be present. Since `Point_{\min}` is a subtype of `OT \langle x: \text{real}; y: \text{real} \rangle` and similarly for `ColorPoint_{\min}`, it is easy to see that

\[ \text{Point}_{\min} \ll: \text{OT} \langle x: \text{real}; y: \text{real}; \]
\[ \text{equal}: \text{Point}_{\min} \rightarrow \text{bool} \rangle \]

\[ \text{ColorPoint}_{\min} \ll: \text{OT} \langle x: \text{real}; y: \text{real}; c: \text{string}; \]
\[ \text{equal}: \text{ColorPoint}_{\min} \rightarrow \text{bool} \rangle \]

so `equal` is indeed a binary method, and no typings are lost in this approach. In fact, something is gained over the matching interpretation described in Section 4.1: it is possible to invoke the `equal` method of a `Point_{\min}` with a `ColorPoint_{\min}` as argument. Typing this “heterogeneous” invocation is crucial for a class defining binary methods intended to be inherited without redefinition and able to take as arguments objects of any subclass [28]. In a type system based on matching, a method declared to take arguments of type `MyType` cannot, in general, accept an object of a subclass as argument; it is necessary to use bounded matching to realize this (see the discussion at the end of Section 4.1.1). Precise types here provide a simpler solution based on subtyping. Note that `ColorPoint_{\min}` does not match, nor is it a subtype of, `Point_{\min}`.

As shown in Section 4.3, typing Ingalls’ solution when `MyType` appears only in the types of method parameters is possible simply by using subsumption, e.g., to lift a `ColorPoint` to a `Point` in `cp1.equal(cp2)`, where both `cp1` and `cp2` are objects of type `ColorPoint`. However, this technique cannot be directly applied to binary methods with result of type `MyType` (or involving `MyType`), because subsumption on the type of the argument may cause loss of interesting type information. Consider the example in Figure 10, which defines points and colored points with a binary `max` method. Objects of `MPointClass` and `ColorMPointClass` could be given the following types, which are simple modifications of the types of `Points` in Section 4.3.

\[ \text{MPoint} \equiv \]
\[ \text{OT} \langle x: \text{real}; y: \text{real}; \]
\[ \text{max}: \text{MPoint} \rightarrow \text{MPoint}; \]
\[ \text{maxMPoint}: \text{MPoint} \rightarrow \text{MPoint}; \]
\[ \text{maxColorMPoint}: \text{ColorMPoint} \rightarrow \text{MPoint} \rangle \]

\[ \text{ColorMPoint} \equiv \]
\[ \text{OT} \langle x: \text{real}; y: \text{real}; c: \text{string}; \]
\[ \text{max}: \text{MPoint} \rightarrow \text{MPoint}; \]
\[ \text{maxMPoint}: \text{MPoint} \rightarrow \text{MPoint}; \]
\[ \text{maxColorMPoint}: \text{ColorMPoint} \rightarrow \text{ColorMPoint} \rangle \]
class ColorMPointClass

... methods ...

max(p: MPoint): MPoint is return( p.maxMPPoint(self) )
maxMPPoint(p: MPoint): MPoint is
    if xValue **2 + yValue **2 < p.x**2 + p.y**2
    then return(p) else return(self)
maxColorMPPoint(p: ColorMPoint): MPoint is return( self.maxMPPoint(p) )

class ColorMPointClass subclass of MPointClass

... methods ...

max(p: MPoint): MPoint is return( p.maxColorMPPoint(self) )
    -- maxMPPoint is inherited
maxColorMPPoint(p: ColorMPoint): ColorMPoint is
    if (xValue **2 + yValue **2) * brightness(cValue)
    < (p.x**2 + p.y**2) * brightness(p.c)
    then return(p) else return(self)

Figure 10: Ingalls simulation of points with a max method: first attempt.

The subtyping ColorMPoint <: MPoint still holds, but note that the result of method max of ColorMPointClass is of type MPoint; type checking would fail if we assigned this method the type MPoint -> ColorMPoint. Thus the static type of taking the max of two ColorMPoints will have to be merely MPoint (unless the method maxColorMPPoint was used explicitly). True multi-methods do not suffer from this shortcoming.

We can overcome this problem in a more expressive type system that provides for polymorphism in addition to recursive types. The idea is to make the type of max more precise, and in this case, polymorphic. The code for the max methods with their new type annotations is given in Figure 11. This modification yields types for the objects of MPointClass and ColorMPointClass described in Figure 12. If p is a MPoint, its max method can still be specialized to a binary method: p.max[MPoint] is of type MPoint -> MPoint, and similarly for the max method of a ColorMPoint. The relation with the types of the “true binary” methods is more direct in an implicitly typed language, where the precise types are smaller [30, 28].

With this typing, taking the max of two elements of ColorMPoint returns a ColorMPoint; any other combination returns a MPoint, the best static type possible. Note that ColorMPoint is still a subtype of MPoint in a system with implicit unfolding of recursive types. So, this typing has all the desired properties of the typing via pure multi-methods of Section 4.2, giving more situations in which Ingalls' method may be usefully applied.

Soop and PolyTOIL are two languages in which all of the precise typings of this section may be expressed. Precise types are complex, however, and it is difficult to imagine programmers writing them routinely. A solution to this problem is to automatically infer minimal types. See [28] for a type inference algorithm for the I-Loop object-oriented language. The algorithm infers a form of F-bounded polymorphic type for classes and objects. It infers minimal types for the original Point/ColorPoint example that are very similar to the “small” types presented above. The types inferred for objects of MPointClass and ColorMPointClass are slightly more general than the form above.

To summarize some of the advantages and disadvantages of precise typing:

- Precise types allow more flexibility in typing than matching alone. They may be expressed using bounded matching, but bounded matching requires explicit quantification and instantiation where subtyping alone may suffice.

- Precise types are a critical component of a typed version of Ingalls’ solution.

- More precise types in module interfaces can be used to overcome some of the limitations of matching.

- The generally more complicated form of the precise types suggests that a type inference algorithm may be the only practical alternative.

- In defining a subclass, one may have to go back and modify the type annotations of the superclass (and, in general, the superclass of the superclass, etc.) in order to generate subtypes. This may be seen as another argument in favor of type inference, since no modifications will be required in an implicitly typed language.
Figure 11: Ingalls simulation of points with a `max` method: precise typing.

\[
\begin{align*}
MPoint & \equiv \\
& \text{OT } \langle \langle x: \text{real}; y: \text{real}; \\
& \quad \text{max}: \text{All}(X) \text{ OT } \langle \langle \text{maxMPoint: } \text{MPoint} \rightarrow X; \text{maxColorMPoint: } \text{ColorMPoint} \rightarrow X \rangle \rangle X; \\
& \quad \text{maxMPoint: } \text{MPoint} \rightarrow \text{MPoint}; \\
& \quad \text{maxColorMPoint: } \text{ColorMPoint} \rightarrow \text{ColorMPoint} \rangle \\
\end{align*}
\]

\[
\begin{align*}
\text{ColorMPoint} & \equiv \\
& \text{OT } \langle \langle x: \text{real}; y: \text{real}; c: \text{string}; \\
& \quad \text{max}: \text{All}(X) \text{ OT } \langle \langle \text{maxColorMPoint: } \text{ColorMPoint} \rightarrow X \rangle \rangle X; \\
& \quad \text{maxMPoint: } \text{MPoint} \rightarrow \text{MPoint}; \\
& \quad \text{maxColorMPoint: } \text{ColorMPoint} \rightarrow \text{ColorMPoint} \rangle \\
\end{align*}
\]

Figure 12: Types for object of `MPointClass` and `ColorMPointClass`

5 Privileged Access to Object Representations

In Section 2.2, we saw that the problems of typing binary methods are often accompanied by difficulties in implementing binary operations without exposing object internals to public view. This section sketches a technique whereby such “overexposed objects” can be wrapped in an additional layer of abstraction, creating a limited scope in which their internal structure is visible. The technique was developed by Pierce and Turner [47] and by Katiyar, Luckham, and Mitchell [35]; we refer the reader to these papers for further details. In particular, [47] demonstrates that the mechanism shown here is compatible with inheritance (though it requires some additional machinery). These ideas give a semantic basis for some aspects of the encapsulation via friends found in C++ and the encapsulation in Cecil [20]. Returning to the example of integer set objects (and dropping the `union` method, for brevity), it is clear that the typing

\[
\text{IntSet} \equiv \text{OT } \langle \langle \text{add: int} \rightarrow \text{unit}; \\
& \quad \text{member: int} \rightarrow \text{bool}; \\
& \quad \text{superSetOf: IntSet} \rightarrow \text{bool} \rangle \\
\]

does not provide a sufficiently rich protocol to allow the `superSetOf` method to be implemented: there is no way to find out what are the elements of the other set (the one provided as argument to `superSetOf`). We have no choice but to extend the interface of set objects with a method that provides access to this information; let us call it `rep`, as a reminder that, in general, it may need to provide access to the whole internal representation of the object.

\[
\text{IntSetExposed} \equiv \langle \langle \text{add: int} \rightarrow \text{unit}; \\
& \quad \text{member: int} \rightarrow \text{bool}; \\
& \quad \text{superSetOf: IntSetExposed} \rightarrow \text{bool} \rangle \\
& \quad \text{rep: IntList} \rangle \\
\]

Now we can easily implement all the methods of `IntSetExposedClass`, as shown in Figure 13.

It remains to show how to package the class `IntSetExposedClass` so that the `rep` method can only be called by other instances of the same class. For this, we generalize Mitchell and Plotkin’s motto that “abstract types have existential type” [44], combining it with the idea of object interfaces as type operators from Cardelli and Wegner’s `partially abstract types` [16].

The interface of the exposed integer set objects can be written as follows.

\[
\text{IntSetExposedOperator}(S) \equiv \langle \langle \text{add: int} \rightarrow \text{unit}; \\
& \quad \text{member: int} \rightarrow \text{bool}; \\
& \quad \text{superSetOf: S} \rightarrow \text{bool} \rangle \\
& \quad \text{rep: IntList} \rangle \\
\]
we need only check that when the hidden “witness type” \( \text{IntSetExposedOperator} \) is replaced by the abstract placeholder \( \text{ISOp} \) in the type of the body of the

\[
\text{intSetPackage} = \\
\text{pack} \\
\begin{array}{l}
\text{procedure } \text{newIntSet}() \text{ is} \\
\text{var} \\
\text{nuSet} : \text{Fix}(\text{IntSetExposedOperator}) \\
\text{begin} \\
\text{nuSet} := \text{new } \text{IntSetExposedClass}(); \\
\text{return}(\text{nuSet}) \\
\text{end} \\
\text{as} \\
\text{Some}(\text{ISOp} <: \text{IntSetOperator}) \\
\text{OT } \langle \langle \text{newIntSet} : \text{Fix}(\text{ISOp}) \rangle \rangle \\
\text{hiding} \\
\text{IntSetExposedOperator} \\
\end{array}
\]

In the remainder of the program, objects created using \text{newIntSet} have type \( \text{IntSet} \). In particular, they can be sent the \( \text{superSetOf} \) message.

In effect, what we have accomplished is to blend object- and ADT-style abstraction mechanisms. The primary mechanism is objects: both ordinary (unary) operations like \( \text{add} \) and binary operations like \( \text{superSetOf} \) are methods of objects rather than free-standing procedures. The extra layer of packaging guarantees that elements of \( \text{IntSet} \) can only be created by calling \text{newIntSet}—i.e., that every element of \( \text{IntSet} \) is actually an instance of \( \text{IntSetExposedClass} \), and hence supports the \( \text{rep} \) message.

### 6 Summary and Conclusions

Binary methods pose real problems in object-oriented programming languages. There is a typing problem because types with binary methods have few interesting subtypes, and there is a problem obtaining privileged access to additional arguments in binary methods.

We discussed the following solutions to the typing problem for binary methods.

- Avoiding binary-methods completely. We proposed several techniques for achieving similar effects.

- Using a notion of matching, which is weaker than subtyping. This allows more polymorphism in the presence of types with binary methods. However, it seems to require programmers to plan ahead more than they would using subtyping, and its flexibility is not as great as with multi-methods.

- Using multi-methods, either as a basis for object-oriented programming, or as a solution within the framework of single-dispatched languages. This gives the programmer more flexibility in programming binary methods, and consequently allows more subtyping. However, there are modularity and efficiency problems with these approaches.
To solve the problem of privileged access to additional arguments, we discussed adding additional layers of abstraction. However, this by itself does not solve the typing problems of binary methods; it must be combined with one of the previous solutions.

Matching, multi-methods, and precise typings offer three different solutions to the binary methods problem. Matching insists that binary methods have the type of both arguments (the receiver and the extra argument) exactly the same, and statically enforces this property. Multi-methods allow all heterogeneous invocations of binary methods, so for instance the equal method of a \texttt{ColorPoint} may be passed a \texttt{Point} as argument. Precise typings lie somewhere between the two: they do not insist that binary methods have the same type for both arguments, but also do not allow all safe heterogenous invocations of binary methods.

To illustrate a weakness of multi-methods, consider the binary methods of 	exttt{DoubleLinkClass} in Figure 5. Using multi-methods, there is a problem if the \texttt{setNext} method of \texttt{DoubleLink} is invoked with a \texttt{Link} node as argument: the \texttt{Link} should point back to the \texttt{DoubleLink}, but it cannot. The best solution is to define a multi-method case here to flag a run-time error. In the case of matching, the receiver and argument must be the same type and the type system safely precludes such a message send. Precise typings produce a solution between the two: a \texttt{Link} may point to a \texttt{DoubleLink}, but a \texttt{DoubleLink} may not point to a \texttt{Link}. The latter restriction, imposed by the type system, prevents the above run-time error from arising.

To illustrate a lack of expressiveness of matching, consider a graphical user interface in which an \texttt{AlertWindowClass} has been defined by inheritance from \texttt{WindowClass}. A binary method \texttt{overlap} should be able to compare plain windows with alert windows. Note also that the \texttt{overlap} method should be specialized in \texttt{AlertWindowClass} in order to take into account some priority of the alerts (thus \texttt{overlap} is a binary method). This could be programmed with multi-methods and precise typings, but not with matching.

A weakness of precise typings is illustrated by the need to use Ingalls’ solution to simulate multiple dispatch. This solution is an \textit{ad hoc} implementation of multiple dispatch. All three solutions thus have strengths and weaknesses. This suggests that the integration of different solutions into a single object-oriented language is a task worthy of study.

So, which solution is the best? None of the solutions discussed above are perfect. Some work also remains in determining if some of the solutions will scale up to all object-oriented languages. For practical programming languages the bottom line may be the empirical question of what sort of inconvenience the programmer is most likely to tolerate. It is our hope that further research will uncover better solutions, perhaps using some combination of the techniques discussed in this paper.

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**References**


