A Nested Computational Approach to $\ell_2$-Optimization of Regulation Transients in Discrete-Time LPV Systems

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Abstract

This article deals with the optimization, expressed as the minimization of the $\ell_2$ norm of the tracking error, of the regulation transients caused by instantaneous, wide parameter variations occurring in the regulated systems. The parameter-varying regulated system is modeled by a set of discrete-time, linear, time-invariant systems and the regulated system switching law is assumed to be completely known a priori. For each LTI system, a feedback regulator, including the exosystem internal model, is designed in order to guarantee closed-loop asymptotic stability and zero tracking error in the steady-state condition. The compensation scheme for the minimization of the regulation transients consists of feedforward actions on the regulation loop and a state switching policy for suitably setting the state of the feedback regulators at the switching times. Both the state switching policy and the feedforward actions are computed off-line: the former by exploiting some geometric properties of the multivariable autonomous regulator problem, the latter by resorting to a two-level, nested algorithm. The lower level includes a sequence of discrete-time, finite-horizon optimal control problems, each defined in the time interval between two consecutive switches. The upper level combines relevant data from the lower-level problems into a global, $\ell_2$-control problem. A significant feature of the approach to the lower-level problems is the original procedure providing the solution of the finite-horizon, optimal control problem stated for discrete-time stabilizable systems through the structural invariant subspaces of the associated singular Hamiltonian system.

Index Terms

Geometric and Structural Approaches; LPV Systems; Output Regulation; $\ell_2$-Optimization.

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I. INTRODUCTION

A wise treatment of the effects of fast, wide parameter variations affecting the regulated systems is a primary concern in many control applications. Hence, a variety of different approaches has been proposed to deal with the issue. Among them, the LPV approach has proved to be one of the most convenient, as is also documented by the huge amount of publications on the subject available in the main areas of control systems technology. In fact, the LPV approach is extensively adopted in flight control (see e.g. [27], [13], [12], [25] and the references therein), road vehicle control (see e.g. [10], [7]), process control (see e.g. [24], [21], [6]), power plant control (see e.g. [22]), and machine tool control as well (see e.g. [26]). In most of the cases considered in the abovementioned literature the parameter variations can be measured in real time. Nonetheless, in some circumstances, the variations in the system dynamics and the time of their occurrence can be known even in advance. For instance, in aircraft flight control, some manoeuvres are predetermined, which implies that the switching between the LTI systems modeling the aircraft dynamics at the different points of interest throughout the operational envelope can be specified a priori. In chemical processes, the dynamics of certain reactions, under different laboratory conditions, can be predicted. In control of machine tools, the profiles to be tracked, which are normally available ahead of their processing, can be obtained by means of a set of switched exosystems. In all those situations, the preview of the switching law can be exploited to achieve substantial improvements in the system performance. Incidentally, this idea generalizes the concept, known since the pioneering work by Tomizuka [28], that preview can be used to accomplish better tracking performance in LTI systems, particularly when nonminimum-phase systems are involved.

The problem of handling the transients of the regulated outputs induced by swift, relevant modifications occurring in the regulated system parameters was formerly examined from a strictly geometric perspective in [15]. In that paper, a set of sufficient conditions for perfect elimination of regulation transients was proved. On that basis, a control scheme was devised, which consists of a set of feedback regulators including the exosystem internal model (as was first established in [9] and [8]) and a set of feedforward compensators including a replica of the internal dynamics of the, so-called, maximum robust controlled invariant subspace (see e.g. [2]). However, since that set of geometric conditions encompasses a stabilizability condition which requires that all the to-be-controlled systems be minimum-phase, it turns
out to be too restrictive for many real engineering problems. Hence, the present work is still focused on the problem of facing the regulation transients due to sudden, large parameter changes, but the specification of perfect elimination is replaced with a less demanding one: namely, the minimization of the $\ell_2$ norm of the tracking error. Moreover, the preview of the switching law is explicitly taken into account in order to enhance the overall system behavior.

The control scheme involves a set of feedback regulators designed according to the internal model principle, a state switching policy for assigning their states at the switching times, and a set of feedforward dynamic units. However, differently from [15], a double feedforward action is needed: one action is aimed at steering the state of the to-be-controlled systems so as to attain the optimization objective, the other one is aimed at neutralizing the corresponding reaction of the feedback regulators. In these aspects, this article extends to output regulation in switched LPV systems some arguments recently developed in other contexts (like signal rejection, in particular). In fact, in [17] and [30] sets of geometric conditions for decoupling (in the sense of perfect rejection) of either inaccessible or measurable or even previewed signals were set forth, while in [18], the design goal was relaxed to that of minimizing the $H_2$ norm of the transfer function matrix from the, possibly previewed, external signals to the outputs.

The theoretical bases for the $\ell_2$-optimization of regulation transients discussed in this work are threefold. First, the geometric interpretation of the multivariable autonomous regulator problem originally presented for continuous-time systems in [14] is transferred to the discrete-time case by following some logical developments detailed in [19]. Consequently, the invariant subspaces and the state trajectories corresponding to the zero-error steady-state condition for each autonomous extended system (i.e., for each pair regulated system – feedback regulator) are neatly characterized. Second, a procedure similar to that introduced in [16] allows the minimal $\ell_2$-norm problem to be reduced to a two-level, nested problem. At the lower level, a sequence of finite-horizon optimal control problems, each defined in the time interval between two consecutive switches, is considered. At the upper level, significant information from the lower-level problems are combined in a global, $\ell_2$-control problem. Third, a technique, which extends some results formerly discussed in [20] on the finite-horizon optimal control problem with final state weighted by a generic quadratic function, provides a non-recursive solution of the analogous problem with fixed final state by exploiting the properties of the structural invariant subspaces of the associated singular Hamiltonian system.

As is clear from the previous considerations, the methodology illustrated herein entails an
extensive use of geometric and structural notions, in a framework also shared by several other, fairly recent, contributions. On the one hand, some geometric aspects of optimal control were first pointed out in [3]. Then, the connections between the solution of the $H_2$ optimal control problem and the invariant subspaces of the associated Hamiltonian system were discussed in [23] and [29] under standard assumptions, and in [11] under more general assumptions. On the other hand, the geometric approach has already furnished a very flexible environment for dealing with the smooth LPV models typically used, as is demonstrated in the related literature (see e.g. [5], [4], [1] and the references therein).

This work addresses stabilizable, discrete-time systems. First, relaxing the assumption of controllability, which, on the contrary, was a standing assumption in [14], is natural when dealing with switched LPV systems, since systems that change from one configuration to another are likely to include uncontrollable (stable or pre-stabilized) parts. Then, addressing discrete-time systems rather than continuous-time systems has several advantages. A major asset is that the $\ell_2$ optimal control problem is solvable on rather extensive assumptions, also including the singular case, while the treatment of the continuous-time counterpart is much more demanding. Moreover, the treatment in the discrete-time domain seems more appealing, since the solution, which, as mentioned earlier, includes both feedforward compensators and state switching strategies in addition to feedback regulators, is straightforwardly implementable with a single digital controller.

The remainder of this article is organized as follows. In Section II, the problem of the minimization of the $\ell_2$ norm of the tracking error caused by sudden, wide parameter variations is formulated. In Section III, the geometric interpretation of the multivariable regulator problem is revisited for discrete-time systems. In Section IV, the solution of the main problem and the corresponding control scheme are presented. This part also exploits the non-recursive solution of the discrete-time finite-horizon optimal control problem with fixed final state described in Appendix. In Section V, a numerical example is illustrated. A summary of the main results and some remarks in Section VI conclude the discussion.

**Notation:** The symbols $\mathbb{Z}^+_0$, $\mathbb{Z}^+$, $\mathbb{R}$ are used for the sets of nonnegative integer numbers, positive integer numbers, real numbers, respectively. The symbols $\mathbb{C}^\circ$, $\mathbb{C}^\circ$, $\mathbb{C}^\circ$ are used for the unit circle, the open set inside the unit circle, the open set outside the unit circle in the complex plane $\mathbb{C}$, respectively. Sets, vector spaces, and subspaces are denoted by capital script letters. The quotient space of a vector space $\mathcal{X}$ over a subspace $\mathcal{V} \subseteq \mathcal{X}$ is denoted by $\mathcal{X}/\mathcal{V}$. Matrices and linear maps are denoted by capital slanted letters. The restriction of a
II. OPTIMIZATION OF REGULATION TRANSIENTS: PROBLEM FORMULATION

The core of the control scheme for $\ell_2$-optimization of regulation transients is that of the multivariable autonomous regulator established by Francis in [8], with a replica of the exosystem eigenstructure for each regulated output. In addition, feedforward actions on the regulation loop and a switching policy for the state of the feedback regulators are considered (Fig. 1).

Fig. 1. Block diagram for optimization of regulation transients.

The symbol $\mathcal{I} = \{1, 2, \ldots, N\}$ denotes a finite index set. The symbol $\{\Sigma_{1i}, i \in \mathcal{I}\} = \{(A_{1i}, B_{1i}, C_{1i}), i \in \mathcal{I}\}$, with $A_{1i} \in \mathbb{R}^{n_1 \times n_1}$, $B_{1i} \in \mathbb{R}^{n_1 \times p}$, and $C_{1i} \in \mathbb{R}^{q \times n_1}$, denotes the set of the to-be-controlled systems. The pairs $(A_{1i}, B_{1i})$ are assumed to be stabilizable for all $i \in \mathcal{I}$. $\{\Sigma_{2i}, i \in \mathcal{I}\} = \{(A_{2i}, E_{2i}), i \in \mathcal{I}\}$, with $A_{2i} \in \mathbb{R}^{n_2 \times n_2}$, $E_{2i} \in \mathbb{R}^{q \times n_2}$, and $\sigma(A_{2i}) \subset \mathbb{C}^\circ \cup \mathbb{C}^\circ$, denotes the set of the exosystems. In particular, the matrices $A_{2i}$ and $E_{2i}$ should respectively...
have the structure

$$A_{2i} = \begin{bmatrix} J_i & 0 & \ldots & 0 \\ 0 & J_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & J_i \end{bmatrix} \right) \right}^1$$

$$E_{2i} = \begin{bmatrix} e_{1i}^\top & 0 & \ldots & 0 \\ 0 & e_{2i}^\top & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & e_{qi}^\top \end{bmatrix},$$

where $J_i \in \mathbb{R}^{n_J \times n_J}$, with $i \in \mathcal{I}$, denote the elementary Jordan blocks of the exosystem and $e_{ji} \in \mathbb{R}^{n_J}$, with $j = 1, \ldots, q$ and $i \in \mathcal{I}$, denote given vectors in $\mathbb{R}^{n_J}$, in order for the exosystem to generate, for any initial state, independent reference signals for each regulated output of $\Sigma_{1i}$.

$\{\Sigma_i, i \in \mathcal{I}\} = \{(A_i, B_i, E_i), i \in \mathcal{I}\}$ denotes the set of the regulated systems, defined as the connection of $\Sigma_{1i}$ and $\Sigma_{2i}$ such that, with $x = [x_1^\top \ x_2^\top]^\top$, $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $u \in \mathbb{R}^p$, and $e \in \mathbb{R}^q$, the state equations are

$$x(k + 1) = A_i x(k) + B_i u(k),$$

$$e(k) = E_i x(k),$$

where

$$A_i = \begin{bmatrix} A_{1i} & 0 \\ 0 & A_{2i} \end{bmatrix}, \quad B_i = \begin{bmatrix} B_{1i} \\ 0 \end{bmatrix}, \quad E_i = \begin{bmatrix} E_{1i} & E_{2i} \end{bmatrix},$$

with $E_{1i} = -C_{1i}$. The pairs $(A_i, E_i)$ are assumed to be observable for all $i \in \mathcal{I}$. $\{\Sigma_{ri}, i \in \mathcal{I}\} = \{(N_i, M_i, L_i, K_i), i \in \mathcal{I}\}$, with $N_i \in \mathbb{R}^{m \times m}$, $M_i \in \mathbb{R}^{m \times q}$, $L_i \in \mathbb{R}^{p \times m}$, and $K_i \in \mathbb{R}^{p \times q}$, denotes the set of the feedback regulators, designed according to the internal model principle, so that, with $z \in \mathbb{R}^m$ and $v_r \in \mathbb{R}^p$, the state equations are

$$z(k + 1) = N_i z(k) + M_i e(k),$$

$$v_r(k) = L_i z(k) + K_i e(k).$$

Moreover, $\{\hat{\Sigma}_i, i \in \mathcal{I}\} = \{(\hat{A}_i, \hat{E}_i), i \in \mathcal{I}\}$ denotes the set of the autonomous extended systems, defined as the connection of $\Sigma_i$ and $\Sigma_{ri}$ such that, with $\hat{x} = [x_1^\top \ x_2^\top \ z^\top]^\top$, the state equations are

$$\hat{x}(k + 1) = \hat{A}_i \hat{x}(k),$$

$$e(k) = \hat{E}_i \hat{x}(k),$$

where $\hat{A}_i \in \mathbb{R}^{n \times n}$ and $\hat{E}_i \in \mathbb{R}^{n \times q}$.
Fig. 2. Time schedule of active regulated systems.

where

$$
\hat{A}_i = \begin{bmatrix}
A_{1i} + B_{1i}K_iE_{1i} & B_{1i}K_iE_{2i} & B_{1i}L_i \\
0 & A_{2i} & O \\
M_iE_{1i} & M_iE_{2i} & N_i
\end{bmatrix},
\hat{E}_i = \begin{bmatrix}
E_{1i} & E_{2i} & O
\end{bmatrix}.
$$

(8)

The control time window is defined as the discrete time interval $[k_0, k_N)$, where $k_0 = 0$ and $k_N \geq N$ are assumed. The regulated system switching law is defined as a sequence $\varphi : [0, \infty) \rightarrow \mathcal{I}$, such that the set

$$
\mathcal{K} = \{k_i \in (0, k_N) : \varphi(k_i) \neq \varphi(k_i - 1), \text{ with } k_i < k_{i+1}, \ i = 1, 2, \ldots, N - 1\}
$$
defines the finite, ordered set of the switching times. The feedforward control on the to-be-controlled system is defined as a sequence $v : [0, k_N) \rightarrow \mathbb{R}^p$. The feedforward control on the feedback regulator is defined as a sequence $w : [0, k_N) \rightarrow \mathbb{R}^q$. The feedback regulator state switching policy is defined as a sequence $\psi : \mathcal{K} \rightarrow \mathbb{R}^m$. In the light of the above definitions, there is no loss of generality in assuming the regulated system $\Sigma_i$ as the active regulated system in the time interval $[k_{i-1}, k_i)$, with $i = 1, 2, \ldots, N$, and the regulated system $\Sigma_N$ as the active regulated system in the time interval $[k_N, \infty)$ also, as is depicted in Fig. 2.

The overall system outlined so far is subject to the following conditions:

**C1.** known ideal zero-error steady-state initial condition and known actual initial condition: i.e., the state of the regulated system corresponding to a zero-error steady-state condition at the time $k_0 = 0$ and the actual state of the regulated system at the time $k_0 = 0$ are known;

**C2.** consistency of the state of the regulated system: i.e., in particular, the state of the regulated system at the switching times $k_i \in \mathcal{K}$ is computable as

$$
x(k_i) = A_ix(k_i - 1) + B_iu(k_i - 1).
$$

In this context, the problem of $\ell_2$-optimization of regulation transients is stated as follows.
Problem 1: Given the set of regulated systems \( \{ \Sigma_i, i \in I \} \), the set of feedback regulators \( \{ \Sigma_{ri}, i \in I \} \), and the regulated system switching law \( \varphi \), find

(i) a feedforward control \( v \) on the to-be-controlled systems,
(ii) a feedforward control \( w \) on the feedback regulators,
(iii) a switching policy \( \psi \) of the state of the feedback regulators,

such that

\[
\| e \|_{L_2} = \left( \sum_{k=0}^{\infty} e(k)^\top e(k) \right)^{1/2}
\]

be minimal, on Conditions \( C1 \) and \( C2 \).

III. A REVIEW OF THE GEOMETRIC APPROACH TO THE REGULATOR PROBLEM

The solution of Problem 1 that will be discussed in Section IV hinges on the geometric interpretation of the multivariable autonomous regulator problem first presented for continuous-time, controllable and observable systems in [14] and lately extended to discrete-time, stabilizable and detectable systems in [19]. Basic results proved in [19] are reviewed in this section with reference to the generic, \( i \)-th autonomous extended system \( \hat{\Sigma}_i \), modeled by (6)–(8) and also depicted in Fig. 3.

The multivariable autonomous regulator problem is stated in geometric terms as follows.

Problem 2: Given the regulated system (1)–(3), find a feedback regulator (4)–(5) such that, for the autonomous extended system (6)–(8), a \( \hat{A}_i \)-invariant subspace \( \hat{L}_i \) exists, which satisfies

(i) \( \hat{L}_i \subseteq \hat{E}_i \), where \( \hat{E}_i = \ker \hat{E}_i \),
(ii) \( \sigma(\hat{A}_i|_{\hat{X}/\hat{L}_i}) \subseteq \mathbb{C}^\circ \), where \( \hat{X} \) denotes the state space of system (6)–(8).
Sets of necessary and sufficient conditions for solvability of Problem 2 were proved in [19]. However, the sole Theorem 1, which follows, is functional to the later developments.

In order to state Theorem 1, the subspace \( \hat{P} \subseteq \hat{X} \) is introduced through the following definition:

\[
\hat{P} = \text{im} \hat{P} = \text{im} \begin{bmatrix} I & O & O \\ O & O & I \end{bmatrix}^\top.
\]

Since \( \hat{A}_i \hat{P} = \hat{P} \hat{S}_i \) holds with

\[
\hat{S}_i = \begin{bmatrix} A_{1i} + B_{1i}K_{1i}E_{1i} & B_{1i}L_i \\ M_{1i}E_{1i} & N_i \end{bmatrix},
\]

the subspace \( \hat{P} \) is \( \hat{A}_i \)-invariant and \( \sigma(\hat{A}_i|\hat{P}) = \sigma(\hat{S}_i) \): i.e., the internal eigenvalues of \( \hat{P} \) match the poles of the regulation loop.

**Theorem 1:** Problem 2 is solvable if and only if, for some feedback regulator (4)–(5), a \( \hat{A}_i \)-invariant subspace \( \hat{W}_i \subseteq \hat{X} \) exists, such that

(i) \( \hat{W}_i \subseteq \hat{E}_i \);
(ii) \( \hat{W}_i \oplus \hat{P} = \hat{X} \);
(iii) \( \sigma(\hat{A}_i|\hat{X}/\hat{W}_i) \subset \mathbb{C}^\circ \).

**Remark 1:** If Problem 2 is solvable, the \( \hat{A}_i \)-invariant subspace \( \hat{W}_i \) satisfying conditions (i)–(iii) of Theorem 1 is the minimal \( \hat{A}_i \)-invariant subspace satisfying conditions (i)–(ii) of Problem 2. Moreover, a basis matrix \( \hat{W}_i \) of \( \hat{W}_i \) has the structure \( \hat{W}_i = \begin{bmatrix} X_{1i}^\top & I & Z_i^\top \end{bmatrix}^\top. \square \)

In the light of Theorem 1, regulation without transients would require that, at any switching time \( k_i \in \mathcal{K} \), the state of the autonomous extended system were switched from the invariant subspace \( \hat{W}_i \), associated to \( \hat{S}_i \), active in \( [k_{i-1}, k_i) \), to the invariant subspace \( \hat{W}_{i+1} \) associated to \( \hat{S}_{i+1} \), active in \( [k_i, k_{i+1}) \). Hence, at any switching time \( k_i \in \mathcal{K} \), the state of the feedback regulator is forced to its correct value by the state switching policy \( \psi \). Namely, \( \psi \) is a correspondence from the set \( \mathcal{K} \) to the vector space \( \mathbb{R}^m \) such that any switching time \( k_i \) is associated to the last \( m \) components of the extended state corresponding to the zero-error steady-state condition on the invariant subspace \( \hat{W}_{i+1} \) at the time \( k_i \). The sequence \( \psi \) is exactly computable by virtue of Condition \( C1 \) and preview of the regulated system switching law \( \varphi \). On the contrary, the state of the regulated system is subject to Condition \( C2 \). In particular, by virtue of the peculiar structure of the basis matrices of the invariant subspaces \( \hat{W}_i \), with \( i \in \mathcal{T} \), the regulation transients are uniquely due to the difference between the ideal state of the to-be-controlled system, corresponding to the zero-error steady-state condition, and the
actual one. Hence, the state of the to-be-controlled system must be steered appropriately, in order to minimize the $\ell_2$ norm of the tracking error, by means of the feedforward action $v$. Simultaneously, the corresponding reaction of the feedback regulator must be neutralized by means of the feedforward action $w$.

IV. OPTIMIZATION OF REGULATION TRANSIENTS: PROBLEM SOLUTION

This section is focused on the synthesis of the feedforward actions $v$ and $w$. Since, as was observed in Section III, the minimization of the regulation transients is achievable by appropriately steering the state of the to-be-controlled systems, the following arguments will solely refer to them, hereafter also assumed to be asymptotically stable, observable, left-invertible, and without invariant zeros on $\mathbb{C}^\circ$. Indeed, the assumptions of asymptotic stability and left-invertibility are not restrictive with respect to those introduced in Section II (see e.g. [30] for details on pre-stabilization and left-invertibility in a geometric context).

As was mentioned in the Introduction, the methodology illustrated in this work solves the problem of minimizing the $\ell_2$ norm of the tracking error caused by switches by resorting to a two-level, nested algorithm. In this frame, the differences between the ideal states of the to-be-controlled systems at the switching times are treated as known disturbances. The control sequences and the state trajectories solving the lower-level, finite-horizon optimal control problems are given in analytic form by a procedure – outlined in Appendix and derived from results formerly presented in [20] – which exploits the structural invariant subspaces of the associated singular Hamiltonian system. The upper-level optimal control problem is solved by reduction to an equivalent mathematical programming problem, according to a technique introduced in [16]. The two-level, nested approach has the computational advantage of removing the dimensional constraint intrinsic in the use of the Moore-Penrose inverse to solve discrete-time finite-horizon control problems (more detailed considerations on this point can be found in [16]).

A. The Lower-Level Optimal Control Problems

In order to avoid notation clutter, let the $i$-th to-be-controlled system, with $i \in \mathcal{I}$, be henceforth modeled by

$$ x(k + 1) = A_i x(k) + B_i u(k), $$

$$ e(k) = C_i x(k), $$
where \( x \in \mathbb{R}^{n_1}, u \in \mathbb{R}^{p}, \) and \( e \in \mathbb{R}^{q} \). The system (9)–(10) is assumed to be asymptotically stable, observable, left-invertible, and without invariant zeros on \( \mathbb{C}^\circ \). Let \( x_0^{ai} \) and \( x_0^{bi} \) respectively denote the initial state and the final state relative to the \( i \)-th time interval. The state \( x_0^{bi} \) is assumed to be reachable from \( x_0^{ai} \) within the \( i \)-th time interval. The objective of the \( i \)-th, lower-level, optimal control problem is to find a control sequence \( u(k), \) with \( k \in [k_{i-1}, k_i) \), such that the cost functional

\[
J_i = \sum_{k=k_{i-1}}^{k_i-1} e(k)^\top e(k),
\]

be minimal, under the state constraints

\[
x(k_{i-1}) = x_0^{ai}, \quad x(k_i) = x_0^{bi}.
\]

The following algebraic manipulations are aimed at deriving an expression efficiently exploitable at the upper level for the optimal value \( J_i^o \) of the cost functional (11).

In accordance with the procedure detailed in Appendix, \( J_i^o \) can be written as the quadratic form

\[
J_i^o = \begin{bmatrix} x_0^{ai} & x_0^{bi} \end{bmatrix} \begin{bmatrix} P_{1i} & P_{2i} \\ P_{2i}^\top & P_{3i} \end{bmatrix} \begin{bmatrix} x_0^{ai} \\ x_0^{bi} \end{bmatrix},
\]

where the matrices \( P_{1i}, P_{2i}, \) and \( P_{3i} \) are defined as stated in Corollary 4. Let \( A_i = A_i^{dk_i} \), where \( dk_i = k_i - k_{i-1} \). Let \( R_i \) denote a basis matrix of the reachable subspace of the pair \((A_i, B_i)\) in \( dk_i \) steps: i.e., a full column rank matrix such that

\[
\text{im } R_i = \text{im } \begin{bmatrix} A_i^{dk_i-1}B_i & A_i^{dk_i-2}B_i & \ldots & B_i \end{bmatrix}.
\]

In the light of (14), reachability of \( x_0^{bi} \) from \( x_0^{ai} \) in \( dk_i \) steps can be expressed as

\[
x_0^{bi} = A_i x_0^{ai} + R_i \gamma(i-1),
\]

where \( \gamma(i-1) \) denotes a suitable vector. Replacement of \( x_0^{bi} \) with its expression (15) in (13) gives

\[
J_i^o = \begin{bmatrix} x_0^{ai} & \gamma(i-1)\end{bmatrix} \begin{bmatrix} J_{1i} & J_{2i} \\ J_{2i}^\top & J_{3i} \end{bmatrix} \begin{bmatrix} x_0^{ai} \\ \gamma(i-1) \end{bmatrix},
\]

where

\[
J_{1i} = P_{1i} + P_{2i}A_i + A_i^\top P_{2i} + A_i^\top P_{3i}A_i,
\]

\[
J_{2i} = (P_{2i} + A_i^\top P_{3i})R_i,
\]

\[
J_{3i} = R_i^\top P_{3i}R_i.
\]
Moreover, let \([C_i \ D_i]\) denote a full row rank matrix such that
\[
\begin{bmatrix}
  C_i^T \\
  D_i^T
\end{bmatrix}
\begin{bmatrix}
  C_i \\
  D_i
\end{bmatrix}
= \begin{bmatrix}
  J_{1i} & J_{2i} \\
  J_{2i}^T & J_{3i}
\end{bmatrix},
\]
and let
\[
e(i - 1) = C_i x_{ai} + D_i \gamma(i - 1).
\]

Then, from (16), (20), and (21) it follows that
\[
J_i^o = e(i - 1)^T e(i - 1).
\]

**B. The Upper-Level Optimal Control Problem**

The concatenation of the lower-level problems, defined in the consecutive time intervals, requires that the differences between the ideal states of the to-be-controlled systems at the switching times be taken into account and, in particular, as mentioned earlier in this section, be treated as known disturbances.

In this perspective, let \(\delta x(0)\) denote the actual state of the to-be-controlled system at the initial time \(k_0 = 0\). The state \(\delta x(0)\) is known by virtue of Condition \(C1\). Let \(\delta x(i)\), with \(i \in \mathcal{I}\), denote the differences between the ideal states at the respective times \(k_i\). The state variations \(\delta x(i)\), with \(i \in \mathcal{I}\), are known due to Condition \(C1\), complete preview of the switching law \(\varphi\), and Remark 1. Hence, the initial states \(x_{ai}\), with \(i = 1, 2, \ldots, N + 1\), of the respective time intervals \([k_{i-1}, k_i)\), with \(i \in \mathcal{I}\), and \([k_{i-1}, \infty)\), with \(i = N + 1\), are given by
\[
x_{ai+1} = \begin{cases}
  \delta x(i), & \text{with } i = 0, \\
  A_i x_{ai} + R_i \gamma(i - 1) + \delta x(i), & \text{with } i \in \mathcal{I},
\end{cases}
\]
where (15) has also been considered. Moreover, let
\[
x(i) = x_{ai+1}, \quad \text{with } i = 0, 1, \ldots, N.
\]
Then, from (23) and (24) it follows that
\[
x(i) = \begin{cases}
  \delta x(i), & \text{with } i = 0, \\
  A_i x(i) + R_i \gamma(i - 1) + \delta x(i), & \text{with } i \in \mathcal{I},
\end{cases}
\]
and from (21) and (25) it ensues that
\[
e(i - 1) = C_i x(i - 1) + D_i \gamma(i - 1), \quad \text{with } i \in \mathcal{I}.
\]
In addition, let
\[
e(N) = Z x(N),
\]
where $Z$ denotes a full row rank matrix such that

$$Z^\top Z = M,$$  \hspace{1cm} (28)

with $M$ defined as the positive definite solution of the discrete Lyapunov equation

$$M = A_N^\top M A_N + C_N^\top C_N.$$  \hspace{1cm} (29)

Therefore, (22) and (27) – with $Z$ defined through (29) – provide the following expression for the square of the $\ell_2$ norm of the output

$$J = \sum_{i=0}^{N} e(i)^\top e(i).$$  \hspace{1cm} (30)

Equations (25)–(27) with the cost functional (30) define the upper-level optimal control problem.

The following arguments are aimed at deriving its solution by reducing it to an equivalent mathematical programming problem.

Let the sequences of the upper-level outputs and state variations be respectively organized in the vectors

$$e_N = \begin{bmatrix} e(0) \\ e(1) \\ \vdots \\ e(N-1) \\ e(N) \end{bmatrix}, \quad \delta x_N = \begin{bmatrix} \delta x(0) \\ \delta x(1) \\ \vdots \\ \delta x(N-1) \\ \delta x(N) \end{bmatrix},$$  \hspace{1cm} (31)

and let the sequence of the upper-level control inputs be organized in the vector

$$\gamma_N = \begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \vdots \\ \gamma(N-1) \end{bmatrix}.$$  \hspace{1cm} (32)

Then, the cost functional (30) can be written in compact form as

$$J = \| e_N \|_2^2,$$  \hspace{1cm} (33)

where

$$e_N = A_N \gamma_N + B_N \delta x_N,$$  \hspace{1cm} (34)
with

$$A_N = \begin{bmatrix} D_1 & O & \ldots & \ldots & O \\ C_N R_1 & D_2 & \ldots & \ldots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ Z \prod_{i=2}^N R_1 & Z \prod_{i=3}^N A_i & \ldots & Z A_N R_{N-1} & D_N \end{bmatrix}$$

(35)

and

$$B_N = \begin{bmatrix} C_1 & O & \ldots & \ldots & O \\ C_2 A_1 & C_2 & \ldots & \ldots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ Z \prod_{i=1}^N A_i & Z \prod_{i=2}^N A_i & \ldots & Z A_N & Z \end{bmatrix},$$

(36)

where matrix products are intended from right to left.

An optimal solution (corresponding to $\xi_N$ with the minimal Euclidean norm) can be obtained by pseudoinversion of the matrix $A_N$ in the expression (34) of $\xi_N$: i.e.,

$$\gamma_N = -A_N^\dagger B_N \delta \xi_N.$$  

(37)

Consequently, the states $\xi(i)$, with $i = 0, 1, \ldots, N$, are obtained by replacing the components, defined as in (32), of the optimal $\gamma_N$ derived in (37) in expression (25). Then, the initial states $\xi_{ai}$ and the final states $\xi_{bi}$, with $i = 1, 2, \ldots, N$, of the lower-level control problems are derived from (24) and (15), respectively. Once the boundary states of the lower-level problems have been evaluated, the results presented in Appendix enable the computation of the feedforward actions $v(k)$ and $w(k)$, with $k \in [k_0, k_N)$. In fact, $v(k)$, with $k \in [k_{i-1}, k_i)$ and $i \in \mathcal{I}$, is given by Corollary 3. Finally, $w(k)$, with $k \in [k_{i-1}, k_i)$ and $i \in \mathcal{I}$, is necessary in order to compensate the effect of feedback when the input sequences defined to minimize the regulation transients are applied to the to-be-controlled system. That correction is defined as $w(k) = C_i x(k)$, with $k \in [k_{i-1}, k_i)$ and $i \in \mathcal{I}$, where the state trajectories are given by Corollary 2.
V. An Illustrative Example

Let $I = \{1, 2\}$. Let the set of the to-be-controlled systems $\{\Sigma_{1i}, i \in I\} = \{(A_{1i}, B_{1i}, C_{1i}), i \in I\}$, be defined by the sampled-data systems

\[
A_{11} = \begin{bmatrix}
-0.0206 & 0.3794 & 0 & 0 \\
-0.1265 & -0.5265 & 0 & 0 \\
0.5291 & 1.6770 & -0.0034 & 0.1338 \\
-0.3921 & -1.2050 & -0.1190 & -0.2709
\end{bmatrix}, \quad B_{11} = \begin{bmatrix} 0 \ 0 \ 0 \ 1 \end{bmatrix},
\]

\[
A_{12} = \begin{bmatrix}
0.3705 & 0.1361 & -0.6402 & 0.4804 \\
-0.4083 & 0.9149 & -2.2450 & 1.5950 \\
0 & 0 & 0.1022 & 0.3166 \\
0 & 0 & -0.3560 & 0.8142
\end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0.0163 & 0.0419 \\
-0.1077 & -0.0422 \\
0.0303 & -0.0371 \\
-0.0676 & -0.0740
\end{bmatrix}.
\]

\[
C_{11} = C_{12} = \begin{bmatrix} 1 & 0 & 0 & 1 \\
10 & 0 & 1 & 0
\end{bmatrix}.
\]

The systems $\Sigma_{11}$ and $\Sigma_{12}$ are asymptotically stable, observable, left-invertible and without invariant zeros on $\mathbb{C}^o$. In particular, the sets of their invariant zeros respectively are $Z_{11} = \{-0.1471, -0.4\}$ and $Z_{12} = \{-2.5794, 0.5723\}$, so that $\Sigma_{11}$ is minimum-phase, while $\Sigma_{12}$ is nonminimum-phase. Let the set of the exosystems $\{\Sigma_{2i}, i \in I\} = \{(A_{2i}, E_{2i}), i \in I\}$ be defined by

\[
A_{21} = A_{22} = \begin{bmatrix} 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad C_{21} = C_{22} = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}.
\]

Let the set of the feedback regulators $\{\Sigma_{ri}, i \in I\} = \{(N_i, M_i, L_i, K_i), i \in I\}$, including the exosystem internal model, be defined by

\[
N_1 = \begin{bmatrix}
0.6194 & 0.3794 & 0.0784 & -0.1437 & 0.1437 & 0 & -0.0784 & 0 \\
-2.3725 & -0.5265 & -0.2139 & -0.1070 & 0.1070 & 0 & 0.2139 & 0 \\
10.6966 & 1.6773 & 1.7400 & 1.8574 & -1.8574 & 0 & -0.7400 & 1 \\
-0.7467 & -1.2053 & -0.2346 & 1.9917 & -0.9917 & 1 & 0.2346 & 0 \\
0.1616 & 0 & 0.0074 & 0.0875 & 0.9125 & 1 & -0.0074 & 0 \\
0.0127 & 0 & 0.0006 & 0.0065 & -0.0065 & 1 & -0.0006 & 0 \\
0.2048 & 0 & 0.0227 & -0.0226 & 0.0226 & 0 & 0.9773 & 1 \\
0.0080 & 0 & 0.0010 & -0.0017 & 0.0017 & 0 & -0.0010 & 1
\end{bmatrix}
\]
\[
M_1 = \begin{bmatrix}
-0.1437 & -0.1070 & 1.7674 & 1.0917 & 0.0875 & 0.0065 & -0.0226 & -0.0017 \\
0.0784 & -0.2139 & 0.8400 & -0.1446 & 0.0074 & 0.0006 & 0.0227 & 0.0010
\end{bmatrix}^T
\]
\[
L_1 = \begin{bmatrix}
0 & 0 & 0.0290 & 1.1709 & 0.1000 & 1.0900 & 0 \\
0 & 0 & 0.9034 & -0.0438 & -0.0900 & 0 & 0.1000 & 1
\end{bmatrix}
\]
\[
K_1 = O_{2 \times 2},
\]
and
\[
N_2 = \begin{bmatrix}
1.7854 & -0.3179 & -0.3045 & 0.8133 & -0.0144 & -0.1060 & 0.0047 & 0.0738 \\
-0.3728 & 0.9847 & -3.0503 & 2.2546 & -0.0602 & 0.1999 & -0.0229 & -0.0162 \\
-1.6588 & 0.5922 & 0.0574 & -0.3854 & -0.0323 & 0.0481 & -0.0182 & -0.0939 \\
-1.8217 & 0.6363 & -1.1416 & 0.6463 & -0.0026 & 0.2267 & -0.0218 & -0.1056 \\
0.1403 & 0 & -0.0261 & 0.4015 & 0.5985 & 1 & 0.0261 & 0 \\
0.0052 & 0 & -0.0032 & 0.0371 & -0.0371 & 1 & 0.0032 & 0 \\
2.3301 & 0 & 0.2342 & -0.0116 & 0.0116 & 0 & 0.7658 & 1 \\
0.1354 & 0 & 0.0135 & 0.0003 & -0.0003 & 0 & -0.0135 & 1
\end{bmatrix}
\]
\[
M_2 = \begin{bmatrix}
0.0090 & 0.0789 & 0.0304 & 0.0177 & 0.4015 & 0.0371 & -0.0116 & 0.0003 \\
0.0038 & 0.0180 & 0.0089 & 0.0083 & -0.0261 & -0.0032 & 0.2342 & 0.0135
\end{bmatrix}^T
\]
\[
L_2 = \begin{bmatrix}
-12.6736 & 4.2530 & 5.3613 & -9.9533 & -0.1468 & -1.0202 & -0.0397 & -0.6382 \\
37.5844 & -12.4930 & 5.8314 & 11.6126 & -0.0700 & -2.1319 & 0.2194 & 2.0113
\end{bmatrix}
\]
\[
K_2 = O_{2 \times 2}.
\]

The control time window is defined as the discrete time interval \([0, 12)\). The regulated system switching law is defined as the sequence \(\varphi: [0, \infty) \rightarrow \mathcal{I}\), such that \(\mathcal{K} = \{k_1, k_2\}\), with \(k_1 = 4\) and \(k_2 = 10\). \(\Sigma_1\) is the active regulated system in the time intervals \([0, 4)\) and \([10, \infty)\), while \(\Sigma_2\) is the active regulated system in the time interval \([4, 10)\).

The invariant subspaces respectively associated to the autonomous extended systems \(\hat{\Sigma}_1\) and \(\hat{\Sigma}_2\) are...
and $\hat{\Sigma}_2$ are given by

\[
\hat{W}_1 = \text{im}
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

and

\[
\hat{W}_2 = \text{im}
\begin{bmatrix}
-0.0777 & 0.0568 & 0.1103 & 0.0043 \\
1.7755 & 0.4500 & 0.1039 & -0.0742 \\
0.7775 & -0.5677 & -0.1031 & -0.0427 \\
1.0777 & -0.0568 & -0.1103 & -0.0043 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-0.0777 & 0.0568 & 0.1103 & 0.0043 \\
1.7755 & 0.4500 & 0.1039 & -0.0742 \\
0.7775 & -0.5677 & -0.1031 & -0.0427 \\
1.0777 & -0.0568 & -0.1103 & -0.0043 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

The extended states $\hat{x}_1$ and $\hat{x}_2$ corresponding to the zero-error steady-state conditions at
the times $k_0 = 0$, $k_1 = 4$ and $k_2 = 10$ are respectively given by

\[
\begin{align*}
\hat{x}_1(0) &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}, \\
\hat{x}_1(4) &= \begin{bmatrix} 0 \\ 0 \\ 4 \\ 4 \\ 4 \\ 4 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}, \\
\hat{x}_1(10) &= \begin{bmatrix} 0 \\ 0 \\ 4 \\ 4 \\ 4 \\ 4 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}, \\
\end{align*}
\]
The autonomous extended system $\hat{\Sigma}_1$ is assumed to be in the zero-error steady-state condition at the time $k_0 = 0$, so that its actual state matches its ideal state $\hat{x}_1(0)$. The state switching policy $\psi: \{4, 10\} \to \mathbb{R}^8$ assigns the correct state of the feedback regulators at the switching times. Hence,

$$
\hat{x}_2(0) = \begin{bmatrix} 0.0061 \\ 0.0376 \\ -0.0610 \\ -0.0061 \\ 0 \\ 0.1 \\ 0 \\ 0.1 \end{bmatrix}, \quad \hat{x}_2(4) = \begin{bmatrix} 0.1363 \\ 7.5554 \\ 2.6367 \\ 3.8637 \\ 0 \\ 0.1 \\ 0 \\ 0.1 \end{bmatrix}, \quad \hat{x}_2(10) = \begin{bmatrix} 0.3317 \\ 18.8320 \\ 6.6832 \\ 9.6683 \\ 0 \\ 0.1 \\ 0 \\ 0.1 \end{bmatrix},
$$

$$
\psi(4) = \begin{bmatrix} 0.1363 \\ 7.5554 \\ 2.6367 \\ 3.8637 \\ 4 \\ 0.1 \\ 4 \\ 0.1 \end{bmatrix}, \quad \psi(10) = \begin{bmatrix} 0 \\ 0 \\ 10 \\ 10 \\ 10 \\ 0.1 \\ 10 \\ 0.1 \end{bmatrix}.
$$

The variations of the states of the to-be-controlled systems at the switching times respectively
are

\[ \delta x(4) = \begin{bmatrix} 0.1363 \\ 7.5554 \\ -1.3633 \\ -0.1363 \end{bmatrix}, \quad \delta x(10) = \begin{bmatrix} -0.3317 \\ -18.8320 \\ 3.3168 \\ 0.3317 \end{bmatrix}. \]

The initial and final variations \( \delta x(0) \) and \( \delta x(12) \) respectively are equal to zero. The solution \( M \) of the discrete Lyapunov equation related to the free evolution of the to-be-controlled system \( \sum_{11} \) is

\[ M = \begin{bmatrix} 102.0580 & 4.9907 & 10.0674 & 1.1957 \\ 4.9907 & 41.3147 & 0.1470 & 1.0992 \\ 10.0674 & 0.1470 & 1.0156 & 0.0344 \\ 1.1957 & 1.0992 & 0.0344 & 1.0961 \end{bmatrix}. \]

Then, all the elements for the computation of the solution of the upper level problem have now been set. The solution of the lower-level problems, hence the feedforward actions on the to-be-controlled systems and the feedback regulators, are derived consequently. In particular, they are plotted in Figs. 4 and 5. Figure 6 shows the regulated output without correction, to be compared with Fig. 7, showing the regulated output with correction. Figure 8 shows the tracking error with correction: its plot is exactly the opposite of that representing the feedforward correction of the feedback regulators. In fact, this is the action required to neutralized the reaction of the feedback regulators. It is worth noting that the implemented regulation scheme acts in such a way that the second jump, in particular, does not cause any transient of the regulated output. This happens in general when the switch occurs from a nonminimum-phase system to a minimum-phase system, according to the geometric results formerly proved in [15]. In this case, the correction of the feedback regulator is zero, since the corresponding tracking error is zero.

VI. Conclusions

The results presented in this work hinge on some basic concepts of the geometric approach to control theory: these are, in particular, the invariance of suitable subspaces in the multivariable autonomous regulator problem and the solution by Moore-Penrose inverse of mathematical programming problems respectively equivalent to optimal control problems. Moreover, in the solution of the finite-horizon optimal control problems, a crucial role is played by the extension of some results derived from the generalized Riccati theory, like,
more specifically, the geometric characterization of the structural invariant subspaces of the associated singular Hamiltonian system. Effectiveness of the suggested methodology can be tested by analyzing switches between minimum-phase systems and nonminimum-phase systems: i.e., typical cases where the geometric conditions for perfect elimination of regulation transients are not normally satisfied.

APPENDIX

In this section, an original, non-recursive solution of the discrete-time, finite-horizon optimal control problem is presented, which is based on a characterization of the structural invariant subspaces of the associated singular Hamiltonian system holding on quite general assumptions. Former results on this topic were presented in [20].

Consider the discrete time-invariant linear system

\[ x(k + 1) = Ax(k) + Bu(k), \]

\[ e(k) = Cx(k) + Du(k), \]

with state \( x \in \mathbb{R}^n \), input \( u \in \mathbb{R}^p \), output \( e \in \mathbb{R}^q \). The following assumptions are introduced:
\( \mathcal{A}_1 \). \((A, B)\) stabilizable;

\( \mathcal{A}_2 \). \((A, B, C, D)\) left invertible;

\( \mathcal{A}_3 \). \(\mathcal{Z}(A, B, C, D) \cap \mathbb{C}^o = \emptyset\), where \(\mathcal{Z}(A, B, C, D)\) denotes the set of the invariant zeros of \((A, B, C, D)\).

**Problem 3:** Refer to system (38)–(39), with \( k \in [0, k_f) \). Let the initial state \( x_0 \) and the final state \( x_f \) be assigned. Find a control sequence \( u(k) \), with \( k \in [0, k_f) \), driving the state from \( x_0 \) to \( x_f \), and minimizing the cost functional

\[
J = \sum_{k=0}^{k_f-1} e(k)^\top e(k).
\]  

(40)

**Remark 2:** Problem 3 is assumed to be well-posed: namely, the final state \( x_f \) is reachable from the initial state \( x_0 \) within the considered time interval, or, equivalently, there exists a vector \( \gamma \) of suitable dimension such that

\[
x_f = A^{k_f}x_0 + R\gamma,
\]

where \( R \) denotes a basis matrix of the reachable subspace in \([0, k_f)\).
Remark 3: The cost functional (40) can also be written as

\[ J = \sum_{k=0}^{k_f-1} \left[ x(k)^\top Q x(k) + 2x(k)^\top S u(k) + u(k)^\top R u(k) \right], \]

where

\[
\begin{bmatrix}
Q & S \\
S^\top & R
\end{bmatrix} =
\begin{bmatrix}
C^\top \\
D^\top
\end{bmatrix}
\begin{bmatrix}
C \\
D
\end{bmatrix}.
\]

The well-known Lagrange multiplier approach leads to a two-point boundary value problem defined by the state equations, the costate equations, the stationarity condition and the boundary conditions. The results listed below defines a non-recursive procedure to solve this problem based on a characterization of the structural invariant subspaces of the associated singular Hamiltonian system holding on the general assumptions A1–A3.

The difference equations of the abovementioned two-point boundary-value problem can
also be written as the state-space generalized system

\[
\begin{bmatrix}
I & O & O \\
O & -A^T & O \\
O & -B^T & O
\end{bmatrix}
\begin{bmatrix}
x(k + 1) \\
p(k + 1) \\
u(k + 1)
\end{bmatrix}
=
\begin{bmatrix}
A & O & B \\
Q & -I & S \\
S^T & O & R
\end{bmatrix}
\begin{bmatrix}
x(k) \\
p(k) \\
u(k)
\end{bmatrix},
\] (41)

also called the singular Hamiltonian system. The matrix on the left-hand side of (41) will be
denoted by \(M\), that on the right-hand side will be denoted by \(N\). The matrix pencil \(\lambda M - N\)
is assumed to have non-vanishing determinant, i.e. \(\det (\lambda M - N) \neq 0\).

Assumptions \(A1-A3\) are sufficient to guarantee the existence and uniqueness of the
stabilizing solution of the discrete algebraic Riccati equation

\[
P = -(A^T PB + S)(R + B^T PB)^{-1}(B^T PA + S^T) + A^T PA + Q,
\] (42)

\[
R + B^T PB > 0,
\] (43)

see e.g. [23]. The stabilizing solution, henceforth denoted as \(P_+\), is also positive semi-definite.
and is the largest real symmetric solution of (42)–(43). Let

$$K_+ = (R + B^T P_+ B)^{-1}(B^T P_+ A + S^T),$$  \hspace{1cm} (44)$$

$$A_+ = A - BK_+.$$  \hspace{1cm} (45)

The condition $\sigma(A_+) \subset \mathbb{C}^\circ$, implied by (44)–(45), is sufficient to guarantee the existence and uniqueness of the solution $W$ of the discrete Lyapunov equation

$$A_+ W A_+^T - W + B(R + B^T P_+ B)^{-1}B^T = 0,$$  \hspace{1cm} (46)

see e.g. [11].

The following properties provide a geometric characterization of a pair of structural invariant subspaces related to the singular Hamiltonian system.

**Property 1:** The subspace

$$\mathcal{V}_1 = \text{im} \mathcal{V}_1 = \text{im} \left[ \begin{array}{c} I \\ P_+ \\ -K_+ \end{array} \right],$$  \hspace{1cm} (47)
is a deflating subspace of the matrix pencil $\lambda M - N$. The spectrum of the pencil restricted to the subspace $\mathcal{V}_1$ is such that

$$(\lambda M - N)|_{\mathcal{V}_1} \equiv (\lambda I - A_+).$$

**Proof:** It is implied by

$$MV_1 S = NV_1 \text{ with } S = A_+.$$ 

In fact, the previous equation, written according to the partition introduced in (41) and (47), produces the identities listed below, where (42)–(45) have been taken into account. The row blocks, from the first to the third, are considered in order.

- **1st row block:**
  $$A_+ = A - BK_+.$$

- **2nd row block:**
  $$-A^T P_+ A_+ = Q - P_+ - SK_+,$$
  $$-A^T P_+ A + A^T P_+ BK_+ = Q - P_+ - SK_+,$$
  $$P_+ = -(A^T P_+ B + S)(R + B^T P_+ B)^{-1}(B^T P_+ A + S^T) + A^T P_+ A + Q.$$

- **3rd row block:**
  $$-B^T P_+ A_+ = S^T - RK_+,$$
  $$-B^T P_+ A + B^T P_+ BK_+ = S^T - RK_+,$$
  $$(B^T P_+ B + R)(R + B^T P_+ B)^{-1}(B^T P_+ A + S^T) = B^T P_+ A + S^T,$$
  $$B^T P_+ A + S^T = B^T P_+ A + S^T.$$

Let

$$K_+ = (R + B^T P_+ B)^{-1}(B^T - B^T P_+ A W A^T_+ - S^T W A^T_+).$$

**Property 2:** The subspace

$$\mathcal{V}_2 = \text{im} \mathcal{V}_2 = \text{im} \left[ \begin{array}{c} W A^T_+ \\ (P_+ W - I) A^T_+ \\ K_+ \end{array} \right],$$

(49)

is a deflating subspace of the matrix pencil $\lambda N - M$. The spectrum of the pencil restricted to the subspace $\mathcal{V}_2$ is such that

$$(\lambda N - M)|_{\mathcal{V}_2} \equiv (\lambda I - A^T_+).$$
Proof: It is implied by

\[ NV_2 S = MV_2 \text{ with } S = A_+^T. \]

In fact, the previous equation, written according to the partition introduced in (41) and (49), produces the identities listed below, where (42)–(46) and (48) have been taken into account. Again, the row blocks are considered in order.

- 1st row block:
  \[
  AW A_+^T + B K_+ = W,
  \]
  \[
  AW A_+^T + B(R + B^T P_+ B)^{-1} B^T - BK_+ W A_+^T = W,
  \]
  \[
  A_+ W A_+^T - W + B(R + B^T P_+ B)^{-1} B^T = 0.
  \]

- 2nd row block:
  \[
  QW A_+^T - (P_+ W - I) A_+^T + S K_+ = -A^T (P_+ W - I),
  \]
  \[
  QW A_+^T - P_+ W A_+^T - K_+ B^T + S K_+ = -A^T P_+ W,
  \]
  \[
  QW A_+^T - P_+ W A_+^T - A^T P_+ B(R + B^T P_+ B)^{-1} B^T
  \]
  \[
  - S(R + B^T P_+ B)^{-1} (B^T P_+ A + S^T) W A_+^T = -A^T P_+ W,
  \]
  \[
  Q - P_+ - S(R + B^T P_+ B)^{-1} (B^T P_+ A + S^T) = -A^T P_+ A_+,
  \]
  \[
  P_+ = -(A^T P_+ B + S)(R + B^T P_+ B)^{-1} (B^T P_+ A + S^T) + A^T P_+ A + Q.
  \]

- 3rd row block:
  \[
  S^T W A_+^T + R K_+ = -B^T (P_+ W - I),
  \]
  \[
  S^T W A_+^T + R(R + B^T P_+ B)^{-1} B^T - R(R + B^T P_+ B)^{-1} (B^T P_+ A + S^T) W A_+^T =
  \]
  \[
  -B^T P_+ A_+ W A_+^T + B^T - B^T P_+ B(R + B^T P_+ B)^{-1} B^T,
  \]
  \[
  S^T - R(R + B^T P_+ B)^{-1} (B^T P_+ A + S^T) = -B^T P_+ A_+,
  \]
  \[
  B^T P_+ A + S^T = B^T P_+ A + S^T.
  \]

The following theorem provides a geometric characterization of the admissible trajectories of the singular Hamiltonian system.

**Theorem 2:** A trajectory

\[
\xi(k) = \begin{bmatrix} x(k) \\ p(k) \\ u(k) \end{bmatrix}, \text{ with } k \in [0, k_f),
\]

is admissible for the singular Hamiltonian system (41) if and only if

\[
\xi(k) = V_1 A_+^k \alpha + V_2 (A_+^T)^{k_f-k-1} \beta, \text{ with } k \in [0, k_f), \tag{50}
\]
where \( V_1, V_2 \) are respectively defined as in (47), (49), and \( \alpha, \beta \in \mathbb{R}^n \) are parameters.

**Proof:** If. The trajectory \( \xi(k) \), with \( k \in [0, k_f) \), satisfies

\[
M \xi(k + 1) = N \xi(k), \quad \text{with} \quad k \in [0, k_f),
\]

for any \( \alpha, \beta \in \mathbb{R}^n \). In fact, by virtue of (41), (47), (49), (50), the previous equation can also be written as

\[
\begin{bmatrix}
I + A\top P_+ + B\top P_+ + A\top (P_+ W - I) A\top + (A\top)^{k_f - k - 2} \beta = \\
-A\top P_+ + A\top (P_+ W - I) A\top + (A\top)^{k_f - k - 2} \beta = \\
A - B K_+ + A\top P_+ A_+ - Q + P_+ S K_+ + S\top - R K_+ \end{bmatrix}
\begin{bmatrix}
W A\top_+ + B K_+ \\
Q W A\top_+ - (P_+ W - I) A\top_+ + SK_+ + S\top W A\top_+ + R K_+
\end{bmatrix}
\]

or, equivalently, as

\[
\begin{bmatrix}
A_+ - A + B K_+ \\
-A\top P_+ A_+ - Q + P_+ S K_+ + S\top + R K_+ \\
-B\top P_+ A_+ - S\top + R K_+
\end{bmatrix}
\begin{bmatrix}
A^k_+ \alpha = \\
W + A W A\top_+ + B K_+ \\
A\top (P_+ W - I) + Q W A\top_+ - (P_+ W - I) A\top_+ + SK_+ + S\top W A\top_+ + R K_+
\end{bmatrix}
\]

The previous equalities hold for any \( \alpha, \beta \in \mathbb{R}^n \), since each row block of the matrix on the left is equal to zero due to the respective identities shown in the proof of Property 1 and each row block of the matrix on the right is equal to zero due to the respective identities shown in the proof of Property 2.

**Only if.** It follows from the structure of (41) (which derives from the Lagrange-multiplier approach), Property 1, and Property 2.

The following corollaries provide the solution of Problem 3 through the geometric characterization of the solutions of the singular Hamiltonian system introduced in Theorem 2.

In the light of Theorem 2, the state and costate trajectories can be written as

\[
\begin{bmatrix}
x(k) \\
p(k)
\end{bmatrix} = \begin{bmatrix}
I \\
P_+
\end{bmatrix} A^k_+ \alpha + \begin{bmatrix}
W \\
P_+ W - I
\end{bmatrix} (A\top_+)^{k_f - k - 1} \beta, \quad \text{with} \quad k \in [0, k_f].
\]
Hence, Corollary 1 and Remark 4, which follow, provide the criterion to select the trajectories of the singular Hamiltonian system solving the original, two-point boundary-value problem.

**Corollary 1:** Let \([x_0^\top \ x_f^\top]^\top \in \text{im } \Phi\), where
\[
\Phi = \begin{bmatrix}
I & W(A_+^\top)^{k_f} \\
A_+^{k_f} & W
\end{bmatrix}.
\]

A trajectory \(\xi(k) = [x(k)^\top \ p(k)^\top \ u(k)^\top]^\top\), with \(k \in [0, k_f)\), of the singular Hamiltonian system (41), satisfying the boundary conditions
\[
x(0) = x_0, \quad x(k_f) = x_f,
\]
is determined by
\[
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix} = \Phi^\dagger \begin{bmatrix}
x_0 \\
x_f
\end{bmatrix}.
\]

**Proof:** From (51) it follows that
\[
x(0) = \alpha + W(A_+^\top)^{k_f}\beta, \quad x(k_f) = A_+^{k_f}\alpha + W\beta.
\]

Hence, (53) can also be written as
\[
\alpha + W(A_+^\top)^{k_f}\beta = x_0, \quad A_+^{k_f}\alpha + W\beta = x_f,
\]
respectively, or, in a more compact form, as
\[
\begin{bmatrix}
I & W(A_+^\top)^{k_f} \\
A_+^{k_f} & W
\end{bmatrix} \begin{bmatrix}
\alpha \\
\beta
\end{bmatrix} = \begin{bmatrix}
x_0 \\
x_f
\end{bmatrix},
\]
which, with definition (52), completes the proof.

**Remark 4:** If \(x_f\) is reachable from \(x_0\) within the considered time interval (see Remark 2), then \([x_0^\top \ x_f^\top]^\top \in \text{im } \Phi\) and the two-point boundary-value problem is solvable: i.e., there exists a trajectory \(\xi(k)\), with \(k \in [0, k_f)\), of the singular Hamiltonian system (41), satisfying the boundary conditions (53). In particular, if \(\Phi\) is singular, the set of all admissible values of \(\alpha, \beta \in \mathbb{R}^n\) is characterized by
\[
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix} = \Phi^\dagger \begin{bmatrix}
x_0 \\
x_f
\end{bmatrix} + \Omega\gamma,
\]
where \(\Omega\) denotes a basis matrix of \(\ker \Phi\) and \(\gamma \in \mathbb{R}^\nu\), with \(\nu = \dim (\ker \Phi)\), denotes a free parameter vector.
Hence, let $[x_0^T \ x_f^T]^T \in \text{im } \Phi$ and let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}^{n \times n}$ be such that

$$
\begin{bmatrix}
\alpha_1 & \alpha_2 \\
\beta_1 & \beta_2
\end{bmatrix} = \Phi^\dagger,
$$

where $\Phi^\dagger$ is assumed to be partitioned according to (54). Then, $\alpha, \beta \in \mathbb{R}^n$, determined according to Corollary 1, can be expressed as

$$
\alpha = \alpha_1 x_0 + \alpha_2 x_f, \quad \beta = \beta_1 x_0 + \beta_2 x_f.
$$

(55)

Consequently, the state trajectories, the control input sequences, and the optimal value of the cost functional solving the finite-horizon optimal control problem can be expressed as functions of the initial state $x_0$ and the final state $x_f$ as detailed in the following statements.

**Corollary 2:** An optimal state trajectory $x(k)$, with $k \in [0, k_f]$, for the finite-horizon optimal control problem defined by (38)–(39) with cost functional (40) and boundary conditions (53) is

$$
x(k) = X_{0_k} x_0 + X_{f_k} x_f,
$$

where

$$
X_{0_k} = A_+^k \alpha_1 + W (A_+^T)^{k_f-k} \beta_1,
$$

$$
X_{f_k} = A_+^k \alpha_2 + W (A_+^T)^{k_f-k} \beta_2,
$$

with $k \in [0, k_f]$.

**Proof:** It follows from (51) and (55).

**Corollary 3:** An optimal control law $u(k)$, with $k \in [0, k_f]$, for the finite-horizon optimal control problem defined by (38)–(39) with cost functional (40) and boundary conditions (53) is

$$
u(k) = U_{0_k} x_0 + U_{f_k} x_f,
$$

where

$$
U_{0_k} = -K_+ A_+^k \alpha_1 + \bar{K}_+ (A_+^T)^{k_f-k-1} \beta_1,
$$

$$
U_{f_k} = -K_+ A_+^k \alpha_2 + \bar{K}_+ (A_+^T)^{k_f-k-1} \beta_2,
$$

with $k \in [0, k_f]$.

**Proof:** In the light of Theorem 2, the parametric form of the control law satisfying (41) is

$$
u_k = -K_+ A_+^k \alpha + \bar{K}_+ (A_+^T)^{k_f-k-1} \beta, \quad \text{with } k \in [0, k_f).$$
Hence, the thesis follows by virtue of (55).

**Corollary 4:** The optimal cost for the finite-horizon optimal control problem defined by (38)–(39) with cost functional (40) and boundary conditions (53) is

\[
J^o = \left[ x_0^T \ x_f^T \right] \left[ \begin{array}{cc} P_1 & P_2 \\ P_2^T & P_3 \end{array} \right] \left[ \begin{array}{c} x_0 \\ x_f \end{array} \right],
\]

where

\[
P_1 = P_+ \alpha_1 + (P_+ W - I)(A_+^T)^{k_f} \beta_1,
\]

\[
P_2 = \frac{1}{2} \left( (P_+ \alpha_2 + (P_+ W - I)(A_+^T)^{k_f} \beta_2) - (P_+ A_+^{k_f} \alpha_1 + (P_+ W - I) \beta_1)^T \right)
\]

\[
P_3 = -(P_+ A_+^{k_f} \alpha_2 + (P_+ W - I) \beta_2).
\]

**Proof:** First, note that

\[
J^o = \sum_{k=0}^{k_f-1} e(k)^T e(k) = x(0)^T p(0) - x(k_f)^T p(k_f).
\]

Moreover, \(p(0)\) and \(p(k_f)\) are respectively given by (51) with (55): i.e.,

\[
p(0) = (P_+ \alpha_1 + (P_+ W - I)(A_+^T)^{k_f} \beta_1) x_0 + (P_+ \alpha_2 + (P_+ W - I)(A_+^T)^{k_f} \beta_2) x_f,
\]

\[
p(k_f) = \left( P_+ A_+^{k_f} \alpha_1 + (P_+ W - I) \beta_1 \right) x_0 + \left( P_+ A_+^{k_f} \alpha_2 + (P_+ W - I) \beta_2 \right) x_f.
\]

Then, the thesis follows from (60) by performing simple algebraic manipulations which take (53), (61), (62), and (57)–(59) into account.

---

**References**


[26] W. Symens, H. Van Brussel, and J. Swevers. Pole-placement vs. loop-shaping design for gain-scheduling control of


