Exact Unknown-State, Unknown-Input Reconstruction: A Geometric Framework for Discrete-Time Systems

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Abstract—The complete solution of the unknown-state, unknown-input reconstruction problem in systems with invariant zeros is intrinsically limited by the fact that for any invariant zero, at least one initial state exists, such that, when the mode associated to the invariant zero is suitably injected into the system, the corresponding output is zero. Although in the awareness of this restriction, the problem of reconstructing the initial state and the inaccessible inputs from the available measurements is the object of a fair amount of research activities because of its impact on a wide range of applications, specifically those dealing with the synthesis of enhanced-reliability control systems. In this context, the present paper contributes a geometric method aimed at solving the exact unknown-state, unknown-input reconstruction problem in discrete-time linear time-invariant multivariable systems with nonminimum-phase zeros. The case where all the system invariant zeros lie in the open set outside the unit disc of the complex plane is regarded as the basic one. The difficulties related to the presence of those invariant zeros are overcome by allowing a reconstruction delay commensurate to the invariant zero time constants. The same technique also applies to the case of systems without invariant zeros. In the latter circumstance, however, the reconstruction delay is related to the number of iteration required by the algorithm for the computation of a specific subspace to converge. Finally, the more general case where the problem is stated for a system whose invariant zeros lie both inside and outside the unit disc of the complex plane is reduced to the basic problem referred to a new system, derived from the original one through a procedure aimed at replacing the minimum-phase zeros with their mirror images with respect to the unit circle.

I. INTRODUCTION

Unknown-state, unknown-input reconstruction currently attracts noticeable interest, particularly for its impact on the synthesis of enhanced-reliability control systems. Indeed, there are strict connections between the possibility of solving the problem at issue and the capability of synthesizing sophisticated, fault tolerant control systems. Therefore, the motivations for research on this topic are provided by safety critical applications such as those in aerospace, underwater, power plants and chemical or petrochemical plants.

From the theoretical point of view, the unknown-state, unknown-input reconstruction problem has several aspects in common with classic problems of system and control theory, like unknown-input state observation and system inversion. The problem of the unknown-input observer (i.e., the problem of deriving an asymptotic estimate of the state in the presence of inaccessible inputs) has been widely investigated since the late 1960s and its theory, which relies on conditions that guarantee decoupling of the effects of the unknown inputs with respect to the state trajectories, is now completely settled. Although in unknown-state, unknown-input reconstruction, as in unknown-input observation, both the state and the input are unknown, the targets of the two problems are different: in fact, the object of unknown-state, unknown-input reconstruction is to achieve an exact estimate of both the initial state of the system and of the inaccessible input in a finite number of steps. Instead, the objective of left inversion is the reconstruction of the unknown inputs on the assumption that the system initial state is zero. Structural left inversion has been studied by several authors since the late 1960s, while left inversion with stability was first addressed in the middle 1970s. Since then, a great deal of research effort was spent on searching solutions to both right and left inversion problems in nonminimum-phase systems.

Although the so-called noncausal inversion is completely solved now, the transfer of the principal techniques through which noncausal inversion is attained (steering along zeros techniques, in particular) to the more general problem of input reconstruction when the system initial state is unknown is far from being accomplished. The aim of this work is, therefore, that of providing a contribution to unknown-state, unknown-input reconstruction in nonminimum-phase systems along these guidelines.

In [1], it was shown that a system \((A, B, C)\), where \(B\) is the unknown-input distribution matrix and \(C\) the measured-output distribution matrix, is completely unknown-state, unknown-input reconstructable if and only if the maximal \((A,\text{im} B)\)-controlled invariant subspace contained in \(\ker C\) is the origin of the state space. This condition, which, although stated in continuous-time systems, also holds in discrete-time systems, implies that the system under consideration is not allowed to have any invariant zeros in order to be completely unknown-state, unknown-input reconstructable. Nonetheless, the impact of the question on the issues of wide practical interest mentioned earlier has motivated further research, although with the awareness of this intrinsic constraint, as is shown by the recent papers [2], [3], [4], [5], [6]. The algorithms proposed in the abovementioned literature are based on extensive use of polynomial and algebraic approaches. They provide possibly-delayed estimates of the initial state and the unknown inputs either on the assumption that system has no invariant zeros or on the assumption that the system is minimum-phase. While, to the best of the authors’ knowledge, the available literature does not provide a satisfactory solution to the case of nonminimum-phase
systems, which, therefore, is the objective of this work.

Notation: The symbols $\mathbb{Z}, \mathbb{Z}_0^+, \mathbb{Z}^+$, $\mathbb{R}$ are used for the sets of integer numbers, nonnegative integer numbers, positive integer numbers, real numbers, respectively. The symbols $\mathbb{C}^0$, $\mathbb{C}^+$, $\mathbb{C}^0$, $\mathbb{C}^\circ$ are used for the unit circle, the open unit disc, the open unit disc without the origin, the open set outside the unit disc in the complex plane $\mathbb{C}$, respectively. Matrices and linear maps are denoted by slanted capital letters like $A$. The quotient space of a vector space $\mathcal{X}$ over a subspace $\mathcal{V} \subseteq \mathcal{X}$ is denoted by $\mathcal{X}/\mathcal{V}$. The restriction of a linear map $A$ to an $A$-invariant subspace $\mathcal{J}$ is denoted by $A|_{\mathcal{J}}$. The orthogonal complement of a subspace $\mathcal{V}$ is denoted by $\mathcal{V}^\perp$. The inverse image of a subspace $\mathcal{V}$ through a linear map $B$ is denoted by $B^{-1}\mathcal{V}$. The spectrum, the image, and the kernel of $A$ are denoted by $\sigma(A)$, $\text{im}A$, and $\ker A$, respectively. The symbols $A^{-1}, A^1$, and $A^\dagger$ are respectively used to denote the inverse, the Moore-Penrose inverse, and the transpose of $A$. The symbol $A^{-\top}$ is used to denote the inverse of the transpose of $A$. Some well-known geometric objects are used: these include the unit circle, the open unit disc in the complex plane, the set of the internal eigenvalues of $A$, the set of the internal unassignable eigenvalues of $A$, and the kernel of $A$, respectively. A subspace $\mathcal{V} \subseteq \mathcal{X}$ is denoted by the inverse, the Moore-Penrose inverse, and the kernel of $A$, respectively. A subspace $\mathcal{V} \subseteq \mathcal{X}$ is denoted by $\mathcal{V}$.

II. Geometric Background

Throughout this work, the discrete-time linear time-invariant system

$$x_{t+1} = A x_t + B u_t, \quad (1)$$

$$y_t = C x_t, \quad (2)$$

is considered, with $t \in \mathbb{Z}_0^+$ denoting the time variable, and $x \in \mathcal{X} = \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^q$ respectively denoting the state, the input, and the output. $A, B, C$ are constant real matrices of appropriate dimensions. $B$ and $C$ are assumed to be full-rank matrices. The symbols $B$ and $C$ respectively stand for $\text{im} B$ and $\ker C$. The set of all admissible input sequences is the set $\mathcal{U}_f$ of all bounded sequences with values in $\mathbb{R}^p$. Some well-known geometric objects are used: these are $\mathcal{V}^* = \text{max} \mathcal{V}(A, B, C)$, the maximal $(A, B)$-controlled invariant subspace contained in $\mathcal{C}$, $S^* = \min \mathcal{S}(A, C, B)$, the minimal $(A, C)$-conditioned invariant subspace containing $\mathcal{B}$, $\mathcal{R}_{\mathcal{V}^*} = \mathcal{V}^* \cap S^*$, the reachability subspace on $\mathcal{V}^*$. Some well-known geometric properties are also used. A subspace $\mathcal{V} \subseteq \mathcal{X}$ is an $(A, B)$-controlled invariant subspace if and only if there exists at least one matrix $F$ such that $(A + BF)\mathcal{V} \subseteq \mathcal{V}$. A subspace $\mathcal{S} \subseteq \mathcal{X}$ is an $(A, C)$-conditioned invariant subspace if and only if there exists at least one matrix $G$ such that $(A + GC)\mathcal{S} \subseteq \mathcal{S}$. It is also worth recalling that the subspace $S^*$ is computable as the last term of the sequence of subspaces defined through the recursive algorithm

$$S_i = A(S_{i-1} \cap \mathcal{C}) + B, \quad i = 1, 2, \ldots, p,$$

with the initial condition $S_0 = \mathcal{B}$ and the terminal condition $S_{p+1} = \mathcal{S}_p$. Crucial notions to this paper purposes are those of right and left invertibility of a system. As was shown in [7], geometric conditions which are equivalent to the property of a system of being right invertible are $S^* + V^* = \mathbb{R}^n$, $S^* + C = \mathbb{R}^n$, and $CS^* = \mathbb{R}^q$, respectively. Duality arguments also prove that geometric conditions respectively equivalent to the property of a system of being left invertible are $V^* \cap S^* = \{0\}$, $V^* \cap \mathcal{B} = \{0\}$, and $B^{-1}V^* = \{0\}$. Moreover, the notion of system invariant zero plays a key role in the geometric solution of the unknown-state, unknown-input reconstruction problem presented in this work. In particular, we are interested in the geometric characterization of the system invariant zeros given, e.g., in [1] and recalled below for the reader’s convenience. As was proven in [1], for the given $\mathcal{V}^*$ there exists at least one constant real matrix $F$ of appropriate dimensions such that $(A + BF)\mathcal{V} \subseteq \mathcal{V}^*$. The internal eigenvalues of $\mathcal{V}^*$ are defined as the elements of the spectrum $\sigma((A + BF)\mathcal{V}^*)$. Moreover, for the same $F$, it is also $(A + BF)\mathcal{R}_{\mathcal{V}^*} \subseteq \mathcal{R}_{\mathcal{V}^*}$, with the spectra $\sigma((A + BF)\mathcal{R}_{\mathcal{V}^*})$ and $\sigma((A + BF)\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*})$ arbitrarily assignable and fixed, respectively. The former set is called the set of the internal assignable eigenvalues of $\mathcal{V}^*$, the second is called the set of the internal unassignable eigenvalues of $\mathcal{V}^*$. Incidentally, it is worth noting that these properties and definitions hold for any $(A, B)$-controlled invariant subspace contained in $\mathcal{C}$, not only for $\mathcal{V}^*$. The system invariant zeros, whose set is also denoted by $Z(A, B, C)$, are the internal unassignable eigenvalues of $\mathcal{V}^*$: i.e., $Z(A, B, C) = \sigma((A + BF)\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*})$.

III. Unknown-State, Unknown-Input Reconstruction: Problem Formulation

Consider system (1), (2), where $u$ denotes the unknown input and $y$ denotes the measured output. The initial state (i.e., the state at the time $t = 0$, also denoted by $x_0$) is supposed to be unknown.

The objective is to design a discrete-time linear time-invariant system, henceforth called the unknown-state, unknown-input reconstructor, such that, having the measured outputs of the original system as inputs, produces as outputs both the initial state and the input of the original system, possibly with a delay consisting of a finite number of steps.

The solution to the unknown-state, unknown-input reconstruction problem is presented under the following assumptions, concerning system (1), (2):

A1. no poles at the origin: i.e., $\sigma(A) \cap \{0\} = \{0\}$;

A2. no invariant zeros on the unit circle of the complex plane: i.e., $Z(A, B, C) \cap \mathbb{C}^0 = \{0\}$;

A3. left invertibility: i.e., $V^* \cap S^* = \{0\}$.

IV. Problem Solution for Systems with All The Invariant Zeros in $\mathbb{C}^0$

The solution of the unknown-state, unknown-input reconstruction problem is first presented for the case of systems whose invariant zeros are all in $\mathbb{C}^0$, the open set outside the unit disc of the complex plane. In particular, the solution is attained through the disentanglement of several subproblems that will be considered one at a time in the remainder of this section. The problem of the reconstruction of the unknown initial state is considered first. Then, the unknown input is derived with a delay of one step with respect to the reconstructed state trajectory through a straightforward use of the system equations.
Hence, the first subproblem that is tackled is that of reconstructing the unknown initial state in the presence of the unknown inputs. Indeed, the solution of this problem is attained through the solution of the dual problem, which is more intuitive in the geometric approach framework. In fact, the dual problem consists in determining the control sequence that drives the state of the dual system from the origin to an assigned final state, while maintaining the output identically equal to zero except at the last step. The next subsection consider a precise statement of this problem and provides a specific solution.

A. Control from the origin to an assigned final state with identically zero output until the last step but one

The algorithm described in this section is aimed to the design of a precompensator that generates the control sequence driving the state of the controlled system to a desired final state while maintaining the output at zero until the last step but one. As specified earlier, this problem is stated for the dual system \((A_d, B_d, C_d) = (A^T, C^T, B^T)\), which, by construction, satisfies the following assumptions:

- \(A_1\)’ no poles at the origin: i.e., \(\sigma(A_d) \cap \{0\} = \emptyset\;
- \(A_2\)’ no invariant zeros on the unit circle of the complex plane: i.e., \(Z(A_d, B_d, C_d) \cap \mathbb{C} \neq \emptyset\;
- \(A_3\)’ right invertibility: i.e., \(V_d^* + S_d^* = \mathbb{R}^n\), where \(V_d^* = \max V(A_d, B_d, C_d)\) and \(S_d^* = \min S(A_d, C_d, B_d)\).

Moreover, since Section IV is focused on systems having all their invariant zeros in the open set outside the unit disc of the complex plane, Assumption \(A_2\)’ is restricted as follows: \(A_2''\) no invariant zeros in the unit disc of the complex plane: i.e., \(Z(A_d, B_d, C_d) \subseteq \mathbb{C} \setminus \mathbb{D}\).

Assumption \(A_3\)’ ensures the existence of an arbitrarily accurate solution of the control problem for any final state, provided that the control time interval be sufficiently large. Assumptions \(A_1\)’ and \(A_2''\) are technical assumptions that guarantee that the algorithm for the design of the precompensator applies in the simplest terms described in the remainder of this subsection.

The precompensator that solves the control problem consists of a finite impulse response system (henceforth, FIR system) activated by a pulse signal that carries the information on the final state that has to be reached. Namely, the input to the FIR system is of the type of \(h_t = x_f \delta_t\), \(t \in \mathbb{Z}_0^+\), where \(x_f \in \mathbb{X} = \mathbb{R}^n\) denotes the desired final state and \(\delta_t\), \(t \in \mathbb{Z}_0^+\), denotes the unit pulse signal: i.e., \(\delta_t = 1\) for \(t = 0\), and \(\delta_t = 0\) otherwise.

The input/output equation of the FIR system is of the type

\[
    u_t = \sum_{\ell=0}^{N-1} \Phi_{\ell} h_{t-\ell}, \quad t \in \mathbb{Z}_0^+, \tag{3}
\]

where \(N \in \mathbb{Z}^+\) denotes the number of steps of the FIR system time window and \(\Phi_{\ell}, \ell = 0, \ldots, N-1\), denotes the (time-varying) FIR system gain matrix.

The number of steps \(N\) of the FIR system time window is chosen in connection with the decaying time of the inverse of the dominant zero (i.e., the invariant zero with the least absolute value) and taking into account the constraint \(N > \rho\), where \(\rho\) is the last index of the sequence of subspaces converging to the minimal conditioned invariant subspace reviewed in Section II.

The FIR system described by (3) can also be described by input/state/output discrete-time equations where the matrices of the quadruple \((A_{FIR}, B_{FIR}, C_{FIR}, D_{FIR})\) have the structure

\[
    A_{FIR} = \begin{bmatrix} O & I & O & \cdots & O \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ O & \cdots & I & \ldots & O \end{bmatrix}, \quad B_{FIR} = \begin{bmatrix} O \\ \vdots \\ O \end{bmatrix},
\]

\[
    C_{FIR} = \begin{bmatrix} \Phi_{N-1} & \ldots & \Phi_1 \end{bmatrix}, \quad D_{FIR} = \Phi_0,
\]

with \(I\) and \(O\) denoting \(n \times n\) matrices.

The last part of this section is focused on the design of the FIR system gain matrix \(\Phi_\ell, \ell = 0, \ldots, N-1\). In order to avoid notation clutter, the subscript \(d\), which explicitly indicates that the whole procedure is supposed to be applied to the dual counterpart of system (1), (2), will be dropped.

From now on, a compact matrix notation is adopted. The matrix \(I\) denotes the main basis of the state space \(\mathbb{X} = \mathbb{R}^n\). Consequently, the sequence of matrices \(U(0), \ldots, U(N-1)\) denotes the sequence of the matrix control inputs: namely, the \(j\)-th column of the matrix \(U(\ell)\) is the input to be applied to the control system in order to obtain the \(j\)-th vector of the main basis of \(\mathbb{R}^n\) as the state of the controlled system at the time \(N\) (starting from the origin).

In order to compute the matrix control sequence \(U(0), \ldots, U(N-1), \ldots, U(N-1)\), two different control policies must be defined in connection with the respective motions on the subspaces \(S^*\) and \(V^*\). Let \(S\) and \(V\) respectively denote basis matrices of \(S^*\) and \(V^*\). Then the \(n \times n\) matrices \(X_S\) and \(X_V\) are defined by \(X_S = S\beta, X_V = V\alpha\), where \(\beta = [\beta^T, \alpha^T] = [S \ V]^T\).

It is known from [1] (see also [7] for applications in the context of perfect tracking), that any state \(x_S \in S^*\) can be reached from the origin through a state trajectory of \(\rho\) steps at most that corresponds to identically zero output until the last step but one, by means of a suitable control action. Hence, this assertion holds for any column vector of the matrix \(X_S\), in particular. Algorithm 1 provides the contribution \(U_S(0), \ldots, U_S(\rho-1)\), relative to the motion on \(S^*\), to the matrix control input \(U(0), \ldots, U(N-1)\).

It is also known from [1], that any state \(x_V \in V^*\), where the internal eigenvalues of \(V^*\) are unstable, can be driven asymptotically to the origin backward in time along trajectories in \(V^*\) (hence, invisible at the output). Hence, there exist control input sequences driving the state from the origin to \(x_V\) along trajectories in the zero dynamics up to an arbitrary accuracy, provided that the control interval is sufficiently large. Hence, this holds, in particular, for any column vector of the matrix \(X_V\). Algorithm 2 provides the contribution \(U_V(0), \ldots, U_V(N-1)\), relative to the motion on \(V^*\), to the matrix control input \(U(0), \ldots, U(N-1)\).
Algorithm 1 (Motion on $S^*$): Let $S_0, S_1, ..., S_\rho$ be basis matrices of the respective subspaces of the sequence $S_0, S_1, ..., S_\rho$ converging to $S^*$, according to the recursive algorithm reviewed in Section II. Let $M_0, M_1, ..., M_{\rho-1}$ be basis matrices of the respective subspaces of the sequence $\delta_0 \cap \mathbb{C}, ..., \delta_{\rho-1} \cap \mathbb{C}$. Let $\alpha_\rho = S_\rho^\top X$. Then, the sequence $U_S(0), ..., U_S(\rho-1)$ is provided by the recursive algorithm

$$\begin{bmatrix} \nu \omega_i \end{bmatrix} = \begin{bmatrix} B & AM_{i-1} \end{bmatrix}^\top S_i \omega_i, \ i = \rho, \rho-1, ..., 1.$$

Algorithm 2 (Motion on $V^*$): Let $F$ be a matrix such that $(A + BF)V^* \subseteq V^*$ and let $F_1 = FV$. Let $\alpha_N = \alpha$. Apply the similarity transformation $T = [V \ S]$, so that

$$A'_F = T^{-1}(A + BF)T = \begin{bmatrix} A'_{F11} & A'_{F12} \\ O & A'_{F22} \end{bmatrix},$$

where $A'_{F11}$ is a $\nu \times \nu$ matrix, with $\nu = \dim V^*$. Then, the sequence $U'_{V}(0), ..., U'_{V}(N-1)$, is given by the recursive algorithm

$$\begin{align*}
\alpha_{i-1} &= (A'_{F11})^{-1} \alpha_i, \ i = N, N-1, ..., 1, \\
U'_{V}(i-1) &= F_1 \alpha_{i-1}.
\end{align*}$$

Note that $A'_{F11}$ is invertible since its eigenvalues are the internal unassignable eigenvalues of $V^*$, which are in $\mathbb{C}^\oplus$ by assumption.

The composition of the matrix control sequences $U_S(i), i = 0, ..., \rho - 1$, and $U_V(i), i = 0, ..., N - 1$, is made as follows:

$$U(N-\ell) = \begin{cases} 
U_S(\rho - \ell) + U_V(N - \ell), \ &\ell = 1, ..., \rho, \\
U_V(N - \ell), \ &\ell = \rho + 1, ..., N.
\end{cases}$$

Finally, the FIR system gain matrix $\Phi_\ell$, $\ell = 0, 0, ..., N - 1$, is defined by $\Phi_\ell = U(\ell)$, $\ell = 0, 0, ..., N - 1$. It is worth noting that in the special case where $S^* = \mathcal{X}$ (i.e., $V^* = \{0\}$), the matrix control sequence reduces to the sole $U_S(i), i = 0, ..., \rho - 1$, and the FIR system gain matrix is defined consequently.

B. The unknown-state, unknown-input reconstructor

As explained at the beginning of Section IV, the FIR system provided by the procedure of Section IV-A solve the control problem which is the dual counterpart of the problem of reconstructing the system initial state in the presence of unknown inputs. Hence, the FIR system that solves the original problem is simply obtained as the dual counterpart of that devised in Section IV-A. Namely, the FIR system that solves the reconstruction problem is given by the quadruple $(A'_{FIR}, C'_{FIR}, B'_{FIR}, D'_{FIR})$.

Finally, as far as the input reconstruction is concerned, since, from the time $t = N$ on, the state estimate $\hat{x}_t$ matches the state $x_t-N$, from that time on, the input can be reconstructed from the state trajectory with a further delay of one step by $\hat{u}_t = B^\top (\hat{x}_{t+1} - A\hat{x}_t), t \geq N$. The solution presented in this section has led to an unknown-state, unknown-input reconstructor which consists of a FIR system which provides a delayed estimate of the state and an input reconstructor that provides the estimate of the unknown input. The scheme is shown in Fig. 1.

V. PROBLEM SOLUTION FOR SYSTEMS WITH THE INVARIANT ZEROS IN $\mathbb{C}^\oplus \cup \mathbb{C}^\ominus$

The scope of this section is to show how the unknown-state, unknown-input reconstruction problem stated for a generic system whose invariant zeros are allocated both inside and outside the unit circle can be reduced to an equivalent problem stated for a new system which is derived from the original one through a geometric procedure that preserves some specific properties of the original system, while it replaces the original minimum-phase zeros with their inverses. The procedure for obtaining the new system is described in Section V-A, that follows. When the new triple, denoted by $(A, B, C)$, has been obtained, a model-matching procedure of the type of that described in [8] applied to the dual counterpart $(A^\top, B^\top, C^\top)$ of the original system $(A, B, C)$ and the dual counterpart $(A^\top, C_1^\top, B^\top)$ of the modified system $(A, B, C)$ provides a precompensator $(A'_{F}, C'_{F}, B'_{F}, D'_{F})$ such that, when its dual counterpart $(A'_{F}, B'_{F}, C'_{F}, D'_{F})$ is connected to the output of the original system $(A, B, C)$, the cascade of the two behaves as the modified system $(A, B, C_1)$. Hence, the unknown-state, unknown-input reconstructor designed with respect to the modified system, according to the procedure illustrated in Section IV, actually solves the problem stated for the original system.

A. Geometric procedure to derive a new system with all its invariant zeros in $\mathbb{C}^\ominus$

The algorithm described in this section is aimed at deriving a new discrete-time linear time-invariant system $(A, B, C_1)$ which, with respect to the original system $(A, B, C)$, has the following properties:

$P_1$. the same number of outputs: i.e., $C_1 \in \mathbb{R}^q \times n$;

$P_2$. the same steady-state gain: i.e., $C(zI - A)^{-1}B|_{z=1} = C_1(zI - A)^{-1}B|_{z=1}$;

$P_3$. its invariant zeros are the nonminimum-phase zeros and, respectively, the inverses of the minimum-phase zeros.
of \((A, B, C)\): i.e., \(Z(A, B, C_1) = Z_{NMP}(A, B, C) \uplus Z_{M}(A, B, C)\), where \(Z_{NMP}(A, B, C)\) denotes the set of the invariant zeros of \((A, B, C)\) contained in \(\mathbb{C}^\circ\) and \(Z_{M}(A, B, C)\) denotes the set of the inverses of the invariant zeros of \((A, B, C)\) contained in \(\mathbb{C}^\circ\).

The triple \((A, B, C_1)\) is derived through a procedure that exploits the geometric properties of the Hamiltonian system associated to the triple \((A, B, C)\). Indeed, the geometric properties of the Hamiltonian system associated to a discrete-time linear time-invariant system were deeply investigated, although with different purposes, in [9] and [10] and are reviewed in the first part of this section for the reader’s convenience.

The singular Hamiltonian system associated to the triple \((A, B, C)\) is the implicit discrete-time linear time-invariant system
\[
\begin{bmatrix}
\begin{array}{ccc}
I & O & O \\
O & -A^T & O \\
O & -B^T & O
\end{array}
\end{bmatrix}
\begin{bmatrix}
x_{t+1} \\
p_{t+1} \\
u_{t+1}
\end{bmatrix}
= \begin{bmatrix}
A & O & B \\
Q & -I & O \\
O & O & O
\end{bmatrix}
\begin{bmatrix}
x_t \\
p_t \\
u_t
\end{bmatrix}
\]

where \(p \in \mathbb{R}^m\) denotes the so-called costate. Since matrix \(A\) is nonsingular by Assumption A1, any generalized trajectory \([x_t, p_t, u_t]^T\), \(t \in Z\), that satisfies the equations of the singular Hamiltonian system, corresponds to a control sequence \(u_t, t \in Z\), and a state trajectory \([x_t, p_t]^T, t \in Z\), that solve the exact decoupling problem for the nonimplicit Hamiltonian system:
\[
\begin{bmatrix}
x_{t+1} \\
p_{t+1}
\end{bmatrix}
= \begin{bmatrix}
A & O \\
-AT & -A
\end{bmatrix}
\begin{bmatrix}
x_t \\
p_t
\end{bmatrix}
+ \begin{bmatrix}
B & O
\end{bmatrix}
u_t,
\]

where \(\eta \in \mathbb{R}^q\) denotes the output to be maintained identically equal to zero. Let \(\hat{A}, \hat{B}, \hat{C}\) respectively denote the system matrix, the input distribution matrix, and the output distribution matrix of the nonimplicit Hamiltonian system. Let \(\hat{X}\) denote the state space of the nonimplicit Hamiltonian system. As is known from the geometric approach, the locus of the state trajectories corresponding to identically zero output is the maximal \((\hat{A}, \hat{B})\)-controlled invariant subspace contained in \(\hat{C}\), namely \(V^* = \max V(\hat{A}, \hat{B}, \hat{C})\). Similar arguments yield the nonimplicit reverse-time Hamiltonian system:
\[
\begin{bmatrix}
x_t \\
p_t
\end{bmatrix}
= \begin{bmatrix}
A^{-1} & O \\
QA^{-1} & A^{-1}
\end{bmatrix}
\begin{bmatrix}
x_{t+1} \\
p_{t+1}
\end{bmatrix}
+ \begin{bmatrix}
-A^{-1}B & -QA^{-1}B
\end{bmatrix}
u_t,
\]

where \(\mu \in \mathbb{R}^q\) denotes the output. Let \(\hat{A}_r, \hat{B}_r, \hat{C}_r\) respectively denote the system matrix, the input distribution matrix, and the output distribution matrix of the nonimplicit reverse-time Hamiltonian system. Then, the subspace resolving the exact decoupling problem for the nonimplicit reverse-time Hamiltonian system is \(\hat{V}_r^* = \max V(\hat{A}_r, \hat{B}_r, \hat{C}_r)\).

As was shown in [9], the intersection of subspaces \(\mathcal{V} = \mathcal{V}\cap \mathcal{V}_r^*\) is an \((\hat{A}, \hat{B})\)-controlled invariant subspace contained in \(\hat{C}\) and also an \((\hat{A}_r, \hat{B}_r)\)-controlled invariant subspace contained in \(\hat{C}_r\). Let \(F\) be such that \((\hat{A} + \hat{B}F)\hat{V} \subseteq \mathcal{V}\) and let \(\hat{A}_F = \hat{A} + \hat{B}F\). Consider the similarity transformation defined by the matrix \(T = [\hat{T}_1 \hat{T}_2]\), where \(\hat{T}_1\) denotes a basis matrix of \(\hat{V}\) and \(\hat{T}_2\) is such that \(\hat{T}\) is nonsingular. The matrix \(\hat{A}_{F,11} = T^{-1}\hat{A}_F T\) has the structure
\[
\hat{A}_{F,11} = \begin{bmatrix}
\hat{A}_{F,11} & \hat{A}_{F,12} \\
\hat{A}_{F,12} & \hat{A}_{F,22}
\end{bmatrix},
\]

where \(\hat{A}_{F,11} \in \mathbb{R}^{\hat{d}\times \hat{d}}\), with \(\hat{d} = \dim \hat{V}\), and \(\hat{A}_{F,12}, \hat{A}_{F,22}\) have consistent dimensions. Consider the similarity transformation defined by the matrix \(T = [T_1 T_2]\), where \(T_1\) denotes a basis matrix of the subspace of the unstable modes of \(\hat{A}_{F,11}\) and \(T_2\) is such that \(T\) is nonsingular. The matrix \(\hat{A}_{F,11} = T^{-1}\hat{A}_{F,11} T\) has the structure
\[
\hat{A}_{F,11} = \begin{bmatrix}
X_{11} & X_{12} \\
X_{12} & X_{22}
\end{bmatrix},
\]

where \(X_{11}\) is a square matrix with the dimension equal to the dimension of the subspace of the unstable modes of \(\hat{A}_{F,11}\), and with \(X_{12}, X_{22}\) matrices of consistent dimensions. Let the subspace \(\hat{V}_u \subseteq \hat{X}\) be defined by \(\hat{V}_u = \text{im} \hat{V}_{u} = \text{im} [\hat{X}_{11}^\perp O]^\perp\). Let \(\hat{V}_{u} = \hat{V}_{u}\). The subspace \(\hat{V}_u \subseteq \hat{X}\) be defined by \(\hat{V}_u = \text{im} \hat{V}_{u} = \text{im} [\hat{Y}_{u}^\perp O]^\perp\). Let \(\hat{V}_u = \hat{V}_{u}\) be partitioned into \([\hat{V}_{u}^\perp \hat{X}_{11}^\perp]^\perp\) according to the same partition that defines \(\hat{V}_{u}\) and let \(\hat{V}_u \subseteq \hat{X}\) be defined by \(\hat{V}_u = \text{im} \hat{V}_{u}\). Then, consider the subspace \(\hat{V}_c = \hat{V}_u + (S^\perp \cap \hat{C})\) and let the matrix \(\hat{C}\) be such that of \(\text{im} \hat{C}^\perp = \hat{V}_{c}\).

Finally, let \(G\) and \(\hat{G}\) respectively denote the steady-state gain of the triples \((A, B, C)\) and \((A, B, \hat{C})\): i.e., \(G = \hat{C} (zI - A)^{-1} \hat{B}\) and \(\hat{G} = \hat{G} (zI - \hat{A})^{-1} \hat{B}\). Then, the matrix \(C_1\) of the triple \((A, B, C_1)\) is defined by \(C_1 = G \hat{G}^\perp C\).

VI. AN ILLUSTRATIVE EXAMPLE

Consider system (1), (2), with \(A, B, C\) given by
\[
A = \begin{bmatrix}
0.6 & -0.3 & 0 & 0 \\
0.1 & 1 & 0 & 0 \\
-0.4 & -1.5 & 0.4 & -0.3 \\
0.3 & 1.1 & 0.2 & 0.9
\end{bmatrix},
\]
\[
B = \begin{bmatrix}
0 & 0.4 \\
0 & 0 \\
0 & -0.1 \\
0.1 & 0.1
\end{bmatrix},
\]
\[
C = \begin{bmatrix}
2.8789 & 27.8999 & 10.5156 & 4 \\
1.4787 & -6.1858 & 2.9148 & 6
\end{bmatrix}.
\]

The system is stable, since \(\sigma(A) = \{0.6, 0.7, 0.7, 0.9\}\), with no poles at the origin. The system is left invertible since \(\mathcal{V}_r^* \cap \mathcal{S}_r^* = \{0\}\). The system invariant zeros are \(Z(A, B, C) = \{1.9928, 1.6469\} \subset \mathbb{C}^\circ\). The chosen length of the FIR system time window is \(T = 30\), so that the estimation error is of order \(10^{-6}\), according to the graph shown in Fig. 2. The simulation results are shown in Fig. 3 and Fig. 4. The initial state, which is supposed to be unknown, is \(x_0 = [1 \ 2 \ 3 \ 4]^T\). The unknown input is the random sequence shown in Fig. 4. The initial state is reconstructed as the output of the FIR system with a delay of \(T = 30\) steps and the consistent accuracy. Fig. 3 shows that the state trajectory from the initial
time $t = 0$ is reproduced at the output of the FIR system with the delay $T = 30$. Similarly, the input from the initial time $t = 0$ is reproduced at the output of the reconstructor, as is shown in Fig. 4. The computational support for this example consists of the basic routines of the geometric approach published with [1] and also available online.

VII. CONCLUSION

This work contributes a solution to the problem of the unknown-state, unknown-input reconstruction in discrete-time, linear, time-invariant multivariable systems with nonminimum-phase zeros. The case of systems with invariant zeros in sole open set outside the unit disc is considered first. The same procedure also applies unmodified to the case of systems with no invariant zeros. Then, the proposed solution is extended to the more general case of systems with invariant zeros generically allocated in the complex plane with the sole exception of the unit circle. The methodological support is strictly geometric. The presentation makes an extensive use of duality arguments. An illustrative example shows the effectiveness of the proposed framework.

REFERENCES