Suboptimal solutions to team optimization problems with stochastic information structure

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Decentralized optimization problems with cooperating decision makers

- **Static team of** $n$ decision makers (DMs), $i = 1, \ldots, n$ with the **same** objective, but different available information.

- $x \in X \subseteq \mathbb{R}^{d_0}$: “state of the world” (random variable describing a stochastic environment).

- **Stochastic information structure**: the information that the $n$ DMs have on the state of the world $x$ is modelled by an $n$-tuple of random variables $y_1, \ldots, y_n$.

- Joint probability density $\rho(x, y_1, \ldots, y_n)$.

- $y_i \in Y_i \subseteq \mathbb{R}^{d_i}$: information that the DM $i$ has about $x$. 
Strategies and team utility function

- \( s_i : Y_i \rightarrow A_i \subseteq \mathbb{R}^{l_i} \): strategy of the \( i \)-th DM.

- \( a_i = s_i(y_i) \): decision that the DM \( i \) chooses on the basis of the information \( y_i \).

- \( u : X \times \prod_{i=1}^n Y_i \times \prod_{i=1}^n A_i \rightarrow \mathbb{R} \): team utility function.
Static teams versus dynamic teams

- **Static teams**: the information of each DM does not depend on the decision of any DM.
  - First investigated in (Marschak, 1955), (Radner, 1962), and (Marschak & Radner, 1972). They derived optimal solutions in closed-form for some particular cases (e.g., Radner’s Theorem under LQG assumptions).

- **Dynamic teams**: each DM’s information can be affected by the decisions of other members of the team (Ho and Chu, 1972). Causality conditions are also usually imposed.

- Interest for static teams (apart from problems arising directly as static team optimization ones): many dynamic team optimization problems can be reformulated in terms of equivalent static ones, with the same sets of strategies but a different team utility function (Witsenhausen, 1988).
Problem TO: static team optimization with stochastic information

Find
\[(s_1^\circ, \ldots, s_n^\circ) \in \arg\max_{s_i \in M(Y_i, A_i), i=1, \ldots, n} v(s_1, \ldots, s_n),\]
where
\[M(Y_i, A_i) = \text{set of bounded and measurable functions from } Y_i \text{ to } A_i,\]
\[v(s_1, \ldots, s_n) = \mathbb{E}_{x, y_1, \ldots, y_n} \{ u(x, \{ y_i \}_{i=1}^n, \{ s_i(y_i) \}_{i=1}^n) \}.\]

\[v(s_1^\circ, \ldots, s_n^\circ): \text{value of the team.}\]
Problem TO is a functional optimization (infinite-dimensional programming) problem:

- a functional has to be maximized, in order to find optimal strategies that are functions of $d_i$ real-valued variables, $i = 1, \ldots, n$.

Example of a functional problem:

- strategies = routing functions in packet-switching telecommunication networks
- variables = lengths of the packet queues in the nodes.
- objective functional = sum of individual objectives, each associated with a node and dependent only on its strategies and the ones of its neighbors in the network.

In applications of interest, often one has a large number of variables.
Suboptimal solutions to team optimization problems with stochastic information structure

Closed-form optimal versus approximate solutions to team optimization problems

- **Optimal solutions in closed form** to team optimization problems available only under quite **strong assumptions**
  - on the team utility
  - on the way in which each DM's information is influenced by the state of the world and, for dynamic teams, by the decisions taken by other DMs.

- **Most results** about closed-form optimal solutions hold under:
  - **LQG hypotheses** (linear information structure, concave quadratic team utility, and Gaussian random variables) and
  - for dynamic teams, **partially nested information**: each DM $i$ can reconstruct all the information known to the DMs that affect its own information, and their information is not affected by the decision of DM $i$ (**causality condition**).

- **Search for suboptimal strategies** that are
  - **accurate**
  - **easily implementable** (i.e., dependent on a reasonably small number of tunable parameters).
Main contributions of this work

- Studying **existence** and **smoothness properties** of an optimal $n$-tuple of strategies for Problem TO.
  - $\rightarrow$ **structural properties** of the (unknown) optimal strategies help in narrowing the search for good suboptimal strategies.

- Exploiting such structural properties to study **methods of approximate solution**.
  - $\rightarrow$ **estimates of the accuracy of suboptimal solutions**
    - expressed in terms of linear combinations of **simple computational units** (Extended Ritz Method, ERIM).
    - containing **parameters to be optimized via nonlinear programming (NLP) algorithms**.
Recall Problem TO:
(static team optimization with stochastic information)

Given

- joint probability density $\rho(x, y_1, \ldots, y_n)$
- team utility function $u(x, y_1, \ldots, y_n, a_1, \ldots, a_n)$

find

$$(s_1^\circ, \ldots, s_n^\circ) \in \arg\max_{s_i \in \mathcal{M}(Y_i, A_i), i=1,\ldots,n} v(s_1, \ldots, s_n),$$

where

$\mathcal{M}(Y_i, A_i) =$ set of bounded and measurable functions from $Y_i$ to $A_i$,

$$v(s_1, \ldots, s_n) = \mathbb{E}_{x, y_1, \ldots, y_n} \left\{ u(x, \{y_i\}_{i=1}^n, \{s_i(y_i)\}_{i=1}^n) \right\}.$$

$v(s_1^\circ, \ldots, s_n^\circ):$ value of the team.
Assumptions (I)

- Smoothness of the team utility and of the probability density
- Some form of concavity for the team utility function
- Interiority of optimal decisions
Assumption A1 (smoothness)

The set $X$ of the states of the world is compact, the sets $Y_1, \ldots, Y_n$ and $A_1, \ldots, A_n$ are compact, convex, and with nonempty interiors. For a positive integer $m \geq 2$, the team utility function $u(\cdot)$ is of class $C^m$ on an open set containing $X \times \prod_{i=1}^n Y_i \times \prod_{i=1}^n A_i$ and $\rho$ is a (strictly positive) probability density on $X \times \prod_{i=1}^n Y_i$, which can be extended to a (strictly positive) function of class $C^m$ on an open set containing $X \times \prod_{i=1}^n Y_i$. 
Assumption A2 (separate strong concavity)

There exists $\tau > 0$ such that the team utility function is separately strongly concave with constant $\tau$ in each of the decision variables $a_1, \ldots, a_n$. 
A concave function $f$ defined on a convex set $X$ is strongly concave with constant $\tau > 0$ iff for all $z, w \in X$ and every supergradient $\alpha_z$ of $f$ at $z$ one has

$$f(w) - f(z) \leq \alpha_z \cdot (w - z) - \tau \|w - z\|^2$$

(supergradient: $\alpha_z$ such that $f(w) - f(z) \leq \alpha_z \cdot (w - z)$).

For a function of class $C^2$, necessary and sufficient condition for its strong concavity $\tau$:

$$\sup_X \lambda_{\text{max}}(\nabla^2 f(z)) \leq -2\tau$$

($\lambda_{\text{max}}(\nabla^2 f(z))$: maximum eigenvalue of the Hessian $\nabla^2 f(z)$).
Let $X_1, \ldots, X_n$ be convex sets, $\tau > 0$, and $S \subseteq \{1, \ldots, n\}$. A function $f$ defined on $\prod_{i=1}^{n} X_i$ is separately strongly concave with constant $\tau > 0$ in the arguments $x_i$ for $i \in S$ iff, for any fixed choice of all the arguments except one of them of the form $x_{\hat{i}}$ with $\hat{i}$ in $S$, the resulting function is strongly concave with constant $\tau$ in the only remaining argument $x_{\hat{i}}$. 
Assumption A3 (interiority)

For every n-tuple \( \{s_1, \ldots, s_n\} \) of strategies and every
\((y_1, \ldots, y_n) \in \prod_{i=1}^n Y_i\), the sets

\[
\text{argmax}_{a_1 \in A_1} E_x, y_2, \ldots, y_n \mid y_1 \{u(x, \{y_i\}_{i=1}^n, a_1, \{s_i(y_i)\}_{i=2}^n)\},
\]

\[
\text{argmax}_{a_2 \in A_2} E_x, y_1, y_3, \ldots, y_n \mid y_2 \{u(x, \{y_i\}_{i=1}^n, s_1(y_1), a_2, \{s_i(y_i)\}_{i=3}^n)\},
\]

\[
\ldots
\]

\[
\text{argmax}_{a_n \in A_n} E_x, y_1, \ldots, y_{n-1} \mid y_n \{u(x, \{y_i\}_{i=1}^n, \{s_i(y_i)\}_{i=1}^{n-1}, a_n)\}
\]

have nonempty intersections with the interiors of \( A_1, A_2, \ldots, A_n \), respectively.
Lemma 1

Let Assumptions A1 and A2 hold. Then

\[
\max \left\{ v(s_1, \ldots, s_n) \mid s_i \in \mathcal{M}(Y_i, A_i), \ i = 1, \ldots, n \right\}
= \max \left\{ v(s_1, \ldots, s_n) \mid s_i \in \mathcal{C}(Y_i, A_i), \ i = 1, \ldots, n \right\}.
\]

\(\mathcal{C}(Y_i, A_i)\): set of continuous functions from \(Y_i\) to \(A_i\).
Theorem 1

Let Assumptions A1, A2, and A3 hold. Then

Problem TO admits $C^{m-2}$ optimal strategies $(s_1^o, \ldots, s_n^o)$ with partial derivatives that are Lipschitz up to the order $m - 2$. 

Useful consequences of Theorem 1

- Numerical computation of a multi-dimensional integral, i.e., the expected value of the team utility.
- Numerical computations of suboptimal solutions.
Evaluation of

\[ \nu(s_1, \ldots, s_n) = \mathbb{E}_{x, y_1, \ldots, y_n} \{ u(x, \{y_i\}_{i=1}^n, \{s_i(y_i)\}_{i=1}^n) \} . \]

- When \( \rho(\cdot), u(\cdot) \) and \( s_1(\cdot), \ldots, s_n(\cdot) \) are smooth enough, this can be approximated by quasi-Monte Carlo methods.

- For the optimal team strategies \( s_1^\circ(\cdot), \ldots, s_n^\circ(\cdot) \) and \( m \geq \sum_{i=0}^n d_i + 2 \), Theorem 1 provides a degree of smoothness sufficient to apply Koksma-Hlawka’s Inequality (upper bound on the approximate integration error for quasi-Monte Carlo methods).
Numerical computation of the expected value (II)

Additional simplifications when the team utility function has a particular structure

- network team optimization problems: the team utility function $u$ is the sum of individual utility functions $u_i$, (Rantzer 2008) each dependent only on a small group of DMs.
Theorem 2

Let the assumptions of Theorem 1 hold for scalar-valued strategies \((l_i = 1, i = 1, \ldots, n)\) and an odd integer \(m > \max_{i=1}^n \{d_i\} + 1\). Then there exists an \(n\)-tuple of strategies \((\tilde{s}_1^k, \ldots, \tilde{s}_n^k)\) of the form

\[
\tilde{s}_i^k(y_i) = \Prj_{A_i} \left( \sum_{j=1}^k c_{ij} e^{-\frac{\|y_i - t_{ij}\|^2}{\delta_{ij}}} \right),
\]

(projections of linear combinations of \(k\) Gaussians), such that

\[
\nu(s_1^o, \ldots, s_n^o) - \nu(\tilde{s}_1^k, \ldots, \tilde{s}_n^k) \leq \frac{C}{\sqrt{k}},
\]

where \(c_{ij} \in \mathbb{R}, \ t_{ij} \in \mathbb{R}^{d_i}, \ \delta_{ij} > 0\) are to be optimized via a NLP algorithm, \(C > 0\) is a constant \((\text{independent of } k)\), and \(\Prj_{A_i}\) is the projection onto the set \(A_i\).

- Similar estimates for vector-valued strategies and sigmoidal and sinusoidal computational units.
Centralization versus decentralization: a comparison

Using $k$ computational units for each DM: upper bounds of approximate optimization of order $\frac{1}{\sqrt{k}}$, for a degree of smoothness $m$ in the problem formulation that grows linearly in $\max_{i=1}^n \{d_i\}$ (i.e., the dimension of the largest information vector).

More decentralization $\rightarrow$ lower degree of smoothness required.
- Centralized case (one single-member team with information vector of size $\sum_{i=1}^n d_i$): $m > \sum_{i=1}^n d_i + 1$;
- Decentralized case: $m > \max_{i=1}^n \{d_i\} + 1$.

More decentralization $\rightarrow$ (usually strictly) smaller value of the team.
Example: Optimal production in a multidivisional firm (I)

- **Two autonomous divisions** producing **two different goods** in quantities $a_1 \in [0, a_{1,\text{max}}]$ and $a_2 \in [0, a_{2,\text{max}}]$, respectively.

- **Goods**: sold in two **markets** at **prices** $\xi \in [\xi_{\text{min}}, \xi_{\text{max}}]$ and $\zeta \in [\zeta_{\text{min}}, \zeta_{\text{max}}]$.

- $\xi$ and $\zeta$: not known exactly before choosing $a_1$ and $a_2$.

- The first division has a “**private estimate**” $y_1$ of $\xi$ and the second division has a “**private estimate**” $y_2$ of $\zeta$. 
Example: Optimal production in a multidivisional firm (II)

- **Total cost** of producing the amounts $a_1$ and $a_2$:

  \[ c(a_1, a_2) = \frac{1}{2}c_{11}a_1^2 + c_{12}a_1a_2 + \frac{1}{2}c_{22}a_2^2. \]

- $c_{1,2} \neq 0$: interaction term.

- **Team utility function**:

  \[ u(\xi, \zeta, s_1(y_1), s_2(y_2)) = \]

  \[ \xi s_1(y_1) + \zeta s_2(y_2) - \frac{1}{2}c_{11}s_1(y_1)^2 - c_{12}s_1(y_1)s_2(y_2) - \frac{1}{2}c_{22}s_2(y_2)^2. \]
Example: Optimal production in a multidivisional firm (III)

Checking the assumptions:

- **Assumption A1**: choose a sufficiently smooth joint probability density for \( x = (\xi, \zeta), y_1, y_2 \) (the required smoothness of the team utility function \( u \) holds automatically since it is quadratic).

- **Assumption A2**: choose \( c_{11}, c_{12} \neq 0 \) and \( c_{22} \) such that the cost function is positive definite (i.e., \( c_{11} > 0 \) and \( c_{11}c_{22} - c_{12}^2 > 0 \)).

- **Assumption A3**: satisfied with suitable values of the parameters (i.e., \( \{a_i, \max\}, \xi_{\min}, \xi_{\max}, \zeta_{\min}, \zeta_{\max}, \{c_{ij}\} \)).

Here \( d_i = 1, i = 1, 2 \), but the example can be easily extended to a multidimensional setting.
Example: Optimal production in a multidivisional firm (IV)

- $X_1 = X_2 = Y_1 = Y_2 = [2, 10],$
- $A_1 = A_2 = [0, 12],$
- $c_{11} = c_{22} = 1, \ c_{12} = 0.15.$
- **Suboptimal strategies** with various numbers $k$ of computational units of various kinds:
  - sinusoidal
  - sigmoidal
  - Gaussian
Example: Optimal production in a multidivisional firm

\[ \tilde{s}^{(k)}_1(y_1) \]

\[ \tilde{s}^{(k)}_2(y_2) \]
Example: Optimal production in a multidivisional firm (VI)

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<th>Basis functions</th>
<th>Expected net profit $v(s_1, s_2)$</th>
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<td>Gaussian</td>
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Developments

- **dynamic team optimization problems**: several dynamic team optimization problems are equivalent to static ones (Witsenhausen, 1988).

- **other approximation schemes**.

- **network team optimization problems**: stochastic versions of the congestion, routing, and bandwidth allocation problems described in (Mansour, 2006)
Thanks for the attention
Extra slides
In network team optimization problems the team utility function $u$ is the sum of individual utility functions $u_i$, (Rantzer 2008) each dependent only on a small group of DMs.

E.g.: $n$ individual utility functions $u_i$, each associated with the DM $i$ and dependent only on the decisions of the DM $i$ and its neighbors $\mathcal{N}_i$ in the network.

$\implies$ Imposing the the interiority assumption A3 made in Theorem 1 may be ”easier”.

Network team optimization problems to which Theorem 1 may apply: stochastic versions of the congestion, routing, and bandwidth allocation problems considered in (Mansour, 2006):

- motivation: they are stated in terms of smooth and strictly concave utility functions.
Given any \( n \)-tuple of strategies \((s_1, \ldots, s_n)\), the \( n \)-tuple

\[
\hat{s}_1(y_1) := \arg\max_{a_1 \in A_1} \mathbb{E}_{x, y_2, \ldots, y_n | y_1} \left\{ u(x, \{y_i\}_{i=1}^n, a_1, \{\hat{s}_i(y_i)\}_{i=2}^n) \right\},
\]

\[
\hat{s}_2(y_2) := \arg\max_{a_2 \in A_2} \mathbb{E}_{x, y_1, y_3, \ldots, y_n | y_2} \left\{ u(x, \{y_i\}_{i=1}^n, \hat{s}_1(y_1), a_2, s_i(y_i)_{i=3}^n) \right\},
\]

\[\vdots\]

\[
\hat{s}_n(y_n) := \arg\max_{a_n \in A_n} \mathbb{E}_{x, y_1, \ldots, y_{n-1} | y_n} \left\{ u(x, \{y_i\}_{i=1}^n, \{\hat{s}_i(y_i)\}_{i=1}^{n-1}, a_n) \right\},
\]

is at least as good as \((s_1, \ldots, s_n)\).

For network team optimization problems, these formulas have a simpler form: for each \( i \), only the individual utility functions of the DMs interacting with the DM \( i \) have to be taken into account.

\[\rightarrow\] Number of integration variables in the evaluation of the conditional expectations smaller than in other team optimization problems.
Let $\mathcal{T}s := (\hat{s}_1, \ldots, \hat{s}_n)$. Under suitable assumptions $\mathcal{T}$ is a contraction operator and for any initial smooth $s^0 := (s^0_1, \ldots, s^0_n)$, the sequence $\{s^i\}$ s.t. $s^{i+1} = \mathcal{T}s^i$ converges to a (unique) optimal solution of Problem TO.

$\rightarrow$ rates of convergence.

For two DMs, under the assumptions of Theorem 1, $\mathcal{T}$ is a contraction operator if $\frac{\beta_{1,2}}{2\tau} < 1$, where $\beta_{1,2} > 0$ (see the next slide) measures the interaction between the two DMs.

- Similar estimates can be obtained for more than two DMs.

For network team optimization problems, most interaction terms $\beta_{i,j}$ are equal to 0, so it is easier to obtain estimates than for general team optimization problems.
Uniqueness of the optimal team strategies

- $\rho_i$: marginal probability density of $y_i$.

**Theorem 3**

Let Assumptions A1, A2, and A3 hold with $n = 2$, $u$ be a quadratic function with respect to $a_1$ and $a_2$, and let

$$
\beta_{1,2} := \sqrt{d_1} \sqrt{d_2} \max_{(x, y_1, y_2) \in X \times Y_1 \times Y_2} \max_{q=1, \ldots, q_{l_1}, r=1, \ldots, r_{l_2}} \left| \frac{\partial^2}{\partial a_{1,q} \partial a_{2,r}} u(x, y_1, y_2, a_1, a_2) \right|.
$$

If $\frac{\beta_{1,2}}{2\pi} < 1$, then $(s_1^0, s_2^0)$ given in Theorem 1 is the unique optimal pair of strategies in $\mathcal{L}_2(Y_1, \rho_{y_1}, \mathbb{R}^{l_1}) \times \mathcal{L}_2(Y_2, \rho_{y_2}, \mathbb{R}^{l_2})$ and also in $C^{m-2}(Y_1, A_1) \times C^{m-2}(Y_2, A_2)$.

The proof of Theorem 3 exploits Banach Contraction Mapping Theorem and a result on Fréchet derivatives from (Li & Basar, 1987).

Theorem 3 can be extended to the case of more than $n = 2$ DMs.
Theorem 1 can be applied to study

- the applicability of quasi-Monte Carlo methods;
- upper bounds on the approximation of team optimal strategies via various kinds of neural networks (w.r.t. the dimension $d_i$ and the number $k_i$ of computational units used by each neural network), e.g., Radial-Basis-Function (RBF) networks (see Theorem 2).

Theorem 3 can be applied to study

- algorithms for suboptimal strategies based on Banach Contraction Mapping Theorem.
(Idealized) algorithms for suboptimal strategies (I)

- The operator

\[ T' : \mathcal{L}_2(Y_1, \rho_{y_1}, \mathbb{R}^l_1) \times \mathcal{L}_2(Y_2, \rho_{y_2}, \mathbb{R}^l_2) \to \mathcal{L}_2(Y_1, \rho_{y_1}, \mathbb{R}^l_1) \times \mathcal{L}_2(Y_2, \rho_{y_2}, \mathbb{R}^l_2) \]

such that

\[ T'_1(s_1, s_2)(y_1) = \operatorname{argmax}_{a_1 \in A_1} \mathbb{E}_{x, y_2 | y_1} \{ u(x, y_1, y_2, a_1, s_2(y_2)) \} \quad \forall y_1 \in Y_1, \]

\[ T'_2(s_1, s_2)(y_2) = \operatorname{argmax}_{a_2 \in A_2} \mathbb{E}_{x, y_1 | y_2} \{ u(x, y_1, y_2, T'_1(s_1, s_2)(y_1), a_2) \} \quad \forall y_2 \in Y_2 \]

is a contraction operator with constant \( \frac{\beta^2_{1,2}}{4\tau^2} < 1 \).

- Rate of convergence \( \frac{\beta^2_{1,2}}{4\tau^2} \) to the optimal pair of strategies.
For certain instances of Problem TO: the parameters of the neural networks may be found by the greedy algorithm developed in (Zhang, 2003) for finding sparse solutions to the problem of maximizing a strongly concave functional over the convex hull of some set of basis functions.
Model of store-and-forward packet-switching networks

Directed graph with a set $\mathcal{N}$ of $|\mathcal{N}| = n$ nodes and a set $\mathcal{L}$ of links.
At each node $i \in \mathcal{N}$: stochastic messages arrive from outside and there is a queue in which messages are stored for the destination.

$\mathcal{P}(i), \mathcal{S}(i)$: sets of upstream and downstream neighbors of node $i$.

The messages are transmitted and their flows are modified synchronously at instants $0, 1, \ldots, T - 1$.

$p_{ij}$: delay in transmitting, propagating, and processing a message (rounded off to a multiple of the sample period).

Very large number of data packets $\Rightarrow$ traffic flows described by continuous variables.
Store-and-forward packet-switching network with link delays and capacities
Optimal routing problem
as a functional optimization problem

- $n$ individual utility functions $u_i$, each associated with the DM $i$ and dependent only on the decisions of the DM $i$ and its neighbors $\mathcal{N}_i$ in the network.

- The team utility function $u$ is the sum of individual utility functions $u_i$ (Rantzer 2008) each dependent only on a small group of DMs.