Small complete caps in $\text{PG}(r,q)$, $r \geq 3$

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Abstract

A very difficult problem for complete caps in $\text{PG}(r,q)$ is to determine their minimum size. The results on this topic are still scarce and in this paper we survey the best results now known. Furthermore, we construct new interesting sporadic examples of complete caps in $\text{PG}(3,q)$ and in $\text{PG}(4,q)$ such that their size are smaller than the currently known. As a consequence, we get that the Pellegrino’s conjecture on the minimal size of a complete $k$-cap in $\text{PG}(3,q)$, $q$ odd, is in general false.

1. Introduction

A $k$-cap in the $r$-dimensional projective space $\text{PG}(r,q)$ over the Galois field $\text{GF}(q)$, is a set of $k$ points, no three of which are collinear. A $k$-cap $K$ is said to be complete if it is not contained in a $(k+1)$-cap of the same projective space. For a detailed description of the most important properties of these geometric structures, we would like to refer the reader to [1, 7, 8, 12, 14]. If $r = 2$, then a $k$-cap is also called a $k$-arc. For this particular concept, detailed surveys are in [10, 15].

The main problem about $k$-caps, posed by Segre [12], is to determine the values of $k$ for which there exists a complete $k$-cap. The cardinality of the largest cap in $\text{PG}(r,q)$ is denoted by $m_2(r,q)$, while the size of the smallest complete cap in $\text{PG}(r,q)$ is denoted by $n_2(r,q)$. The determination of the cardinalities of the largest and the smallest complete caps in $\text{PG}(r,q)$ is a very important question for the close relationship between the theory of the complete $k$-caps and the coding theory and the mathematical statistics.

In particular, we would like to remind that in [16, 2, 7, 14] this relationship was presented.

Much work has been done on finding complete $k$-caps with the maximum possible number of points. On the other hand, not much progress has been made on determining...
the minimum possible size of a complete cap in PG(r, q) and the cardinalities of the known complete k-caps are still relatively high with respect to the theoretically minimal values.

The aim of this paper is to survey all the actually known constructions which give smaller complete k-caps in PG(r, q), with r ≥ 3, and to give, with computer assistance, interesting examples of small complete k-caps in PG(3, q), with q ∈ {4, 7, 11, 17}, and in PG(4, 4) which have a cardinality which is smaller than the one of all known complete caps in the corresponding projective spaces.

We conclude with some tables which give the best results now known about the size of the smallest complete caps in PG(r, q), with r ≥ 3.

2. Small complete caps in PG(r, q)

General methods were developed for constructing small complete k-caps in PG(r, q). In our context, the following list shows the most significant among them. First of all, we recall that the Lunelli–Sce bound on the minimal cardinality k of the complete k-caps of PG(r, q) essentially says (cf. [14]):

(2.1) If K is a complete k-cap of PG(r, q), then \( k > \sqrt{2q^{r-1}/2} \).

For \( r = 3 \) and q even, this bound is surprisingly sharp. Namely, B. Segre proved that:

(2.2) There exist complete \((3q + 2)\)-caps in PG(3, q), \( q = 2^h \) [12].

For \( r = 3 \) and \( q = 2 \) all complete caps are readily identified:
(2.3) In PG(3, 2), a complete k-cap K is one of the following:
   (i) \( k = 8 \) and K is the complement of a plane;
   (ii) \( k = 5 \) and K is an elliptic quadric [8].

Other interesting results about small complete caps are the following:
(2.4) In PG(3, 3), there exists a complete 8-cap [12].
(2.5) In PG(3, 4), there exist: (i) a complete 12-cap; (ii) a complete 14-cap [12].
(2.6) Let \( r \) be the remainder of the division \(((q - 3)/2) : 3\). In PG(3, q), \( q = 9 \) or q prime with \( q ≥ 5 \), there exist complete k-caps with \( k = (q^2 + rq + 6)/3 \) [3].
(2.7) Let \( q = 4t ± 1 \) and let \( m \) be the maximum integer such that \( \binom{m}{2} < t \). In PG(3, q), there exist complete k-caps with \( k = (m + 1)(q + 1) + 2 \) [11].
(2.8) Let \( q = 4t ± 1 \). In PG(3, q), there exist complete k-caps with \( k = 4 + [(q - 1)/2]^2 \) [11].
(2.9) Suppose \( r ≥ 3 \). In PG(r, 4), there exist complete k-caps with \( k = 2^{r+1} - 2 \) [12].
(2.10) In PG(4, q), \( q > 2 \) even, there exist complete k-caps with \( k = 2q^2 + q + 5 \) [17].
(2.11) In PG(4, q), \( q ≥ 3 \) odd, there exist complete k-caps with \( k = 2q^2 + 1 \) [17].
(2.12) In PG(4, q), \( q ≥ 4 \) even, there exist complete k-caps with \( k = 2q^2 + q \) [17].
(2.13) In PG(4, 2), there exist complete k-caps with \( k = 9 \) [4].
(2.14) In PG(5, 2), there exist complete k-caps with \( k = 13 \) [4].
(2.15) In PG(6, 2), there exist complete k-caps with \( k = 21 \) [4].
(2.16) In PG(7, 2), there exist complete k-caps with k = 29 [4].

(2.17) In PG(8, 2), there exist complete k-caps with k = 45 [4].

(2.18) In PG(r, 2), for \( r \geq 9 \), there exist complete k-caps with \( k = 15 \times 2^{m-3} - 3 \), when \( r = 2m - 1 \), and \( k = 23 \times 2^{m-3} - 3 \), when \( r = 2m \) [4].

In this article, some of the previously mentioned results are improved by proving the following (enclosed in brackets the previous smaller size in the corresponding space is reported):

**Theorem 1.** (a) There exist complete 10-caps in PG(3, 4) \{12, see (2.5)};
(b) there exist complete 21-caps in PG(3, 7) \{22, see (2.8)};
(c) there exist complete 36-caps in PG(3, 11) \{38, see (2.7)};
(d) there exist complete 67-caps in PG(3, 17) \{74, see (2.7)};
(e) there exist complete 29-caps in PG(4, 4) \{30, see (2.9)}.

We are however not yet able to present a description of the construction of these complete k-caps which could be used to construct k-caps of this type for other values of q.

**Remark 1.** In [11], Pellegrino conjectures that the cardinality k of the complete k-caps of (2.7) is the minimum that can be theoretically reached in PG(3, q) for q ≥ 5 and q odd. From our Theorem we see that this conjecture is in general false.

**Remark 2.** It is well known that the parity check matrices of linear binary codes with length n, minimum distance d = 4 and covering radius R = 2 are equivalent to complete n-caps in projective geometry (see [4]). Using coding theory and an exhaustive computer search, in [4] it is proved that the 9-cap of (2.13) and the 13-cap of (2.14) have the minimum length among complete k-caps in PG(4, 2) and in PG(5, 2), respectively.

**Remark 3.** The surprising fact is how few exact values of the constant \( n_2(r, q) \), the minimum possible size of a complete k-cap in PG(r, q), are known. From our Remark 2 and from (2.3) we see that all the known exact values of \( n_2(r, q) \) are the following:

\[
\begin{align*}
(2.19) \quad n_2(3, 2) &= 5; \\
(2.20) \quad n_2(4, 2) &= 9; \\
(2.21) \quad n_2(5, 2) &= 13.
\end{align*}
\]

3. Proof of the theorem

It will be sufficient to give an example of a complete 10-cap in PG(3, 4), a complete 21-cap in PG(3, 7), a complete 36-cap in PG(3, 11), a complete 67-cap in PG(3, 17)
and an example of a complete 29-cap in PG(4, 4). These examples were also checked on completeness using a computer.

The points constituting the complete caps in the respective spaces PG(r, q) are denoted with homogeneous coordinates \((x_0, x_1, \ldots, x_r)\), in accordance with the terminology used in the preceding sections.

An example of a complete 21-cap in PG(3, 7):

\[(0, 1, 0, 0), (1, 1, 0, 0), (0, 0, 1, 0), (1, 0, 1, 0), (4, 0, 0, 1), (1, 1, 0, 1), (5, 4, 0, 1)\]
\[(6, 0, 1, 1), (3, 1, 1, 1), (0, 4, 1, 1), (5, 0, 2, 1), (2, 1, 2, 1), (6, 4, 2, 1), (3, 5, 3, 1)\]
\[(5, 6, 3, 1), (3, 2, 4, 1), (2, 3, 4, 1), (0, 5, 5, 1), (2, 6, 5, 1), (0, 3, 6, 1), (1, 2, 6, 1)\]

An example of a complete 36-cap in PG(3, 11):

\[(1, 0, 10, 1), (0, 1, 3, 6), (1, 7, 5, 1), (9, 1, 2, 1), (7, 1, 2, 2), (1, 9, 2, 9)\]
\[(2, 1, 3, 3), (10, 1, 4, 3), (6, 1, 4, 4), (2, 1, 5, 4), (8, 1, 5, 5), (0, 1, 7, 1)\]
\[(8, 1, 6, 6), (2, 1, 7, 6), (6, 1, 7, 7), (1, 4, 7, 8), (2, 1, 8, 8), (5, 1, 9, 8)\]
\[(7, 1, 9, 9), (1, 6, 4, 3), (10, 1, 10, 10), (1, 0, 1, 0), (10, 0, 0, 1), (0, 0, 1, 10)\]
\[(0, 1, 2, 4), (1, 3, 2, 0), (1, 9, 9, 8), (1, 9, 4, 3), (1, 0, 5, 2), (1, 8, 8, 9)\]
\[(0, 1, 1, 0), (1, 0, 4, 10), (1, 10, 1, 0), (1, 7, 2, 0), (1, 1, 6, 0), (1, 8, 5, 3)\]

An example of a complete 67-cap in PG(3, 17):

\[(0, 0, 16, 1), (0, 1, 0, 0), (16, 1, 1, 1), (15, 1, 2, 1), (13, 1, 2, 2), (11, 1, 3, 2)\]
\[(8, 1, 3, 3), (5, 1, 4, 3), (1, 1, 4, 4), (14, 1, 5, 4), (9, 1, 5, 5), (4, 1, 6, 5)\]
\[(15, 1, 6, 6), (9, 1, 7, 6), (2, 1, 7, 7), (12, 1, 8, 7), (4, 1, 8, 8), (13, 1, 9, 8)\]
\[(4, 1, 9, 9), (12, 1, 10, 9), (2, 1, 10, 10), (9, 1, 11, 10), (15, 1, 11, 11), (4, 1, 12, 11)\]
\[(9, 1, 12, 12), (14, 1, 13, 12), (1, 1, 13, 13), (5, 1, 14, 13), (8, 1, 14, 14), (11, 1, 15, 14)\]
\[(13, 1, 15, 15), (15, 1, 16, 15), (16, 1, 16, 16), (1, 0, 1, 0), (16, 0, 0, 1), (1, 0, 16, 1)\]
\[(0, 0, 1, 0), (0, 1, 1, 2), (1, 0, 3, 4), (1, 1, 0, 1), (1, 2, 0, 4), (1, 3, 0, 6)\]
\[(1, 4, 0, 3), (1, 5, 0, 8), (1, 6, 1, 5), (1, 7, 0, 7), (1, 8, 1, 6), (1, 9, 1, 3)\]
\[(1, 10, 1, 11), (1, 11, 1, 9), (1, 12, 0, 2), (1, 13, 3, 6), (1, 14, 4, 1), (1, 15, 4, 13)\]
\[(1, 16, 2, 11), (0, 1, 1, 6), (1, 0, 12, 13), (1, 1, 4, 16), (1, 3, 3, 15), (1, 4, 13, 2)\]
\[(1, 5, 5, 13), (1, 7, 0, 12), (1, 9, 4, 8), (1, 10, 2, 8), (1, 12, 15, 7), (1, 15, 11, 2)\]
\[(1, 7, 8, 2)\]
Finally, let the elements of the field \( \text{GF}(4) \) be denoted by \( 0, 1, y, z \). The operations \textit{addition} and \textit{multiplication} of this field are given by the following operation tables:

\[
\begin{array}{c|cccc}
+ & 0 & 1 & y & z \\
0 & 0 & 1 & y & z \\
1 & 1 & 0 & z & y \\
y & y & z & 0 & 1 \\
z & z & y & 1 & 0
\end{array}
\quad
\begin{array}{c|cccc}
\times & 0 & 1 & y & z \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & y & z \\
y & y & 0 & y & z \\
z & z & 0 & 1 & y
\end{array}
\]

An example of a complete 10-cap in PG(3,4):

\[(1,0,0,0),(1,0,1,0),(0,0,0,1),(y,0,z,1),(0,1,y,1),
(y,1,z,1),(0,y,1,1),(1,y,1,1),(0,z,z,1),(1,z,1,1).\]

An example of a complete 29-cap in PG(4,4):

\[(1,0,0,0,0),(1,0,1,0,0),(0,1,0,0,0),(y,1,z,0,0),(0,1,y,1,0),(y,1,1,0,0),
(0,1,y,0),(1,1,y,y,0),(0,1,z,z,0),(1,1,1,z,0),(0,1,0,1,z),
(0,1,1,0,1),(y,1,z,0,1),(0,1,1,0,y),(1,1,y,0,y),(1,0,0,1,1),(0,1,1,1,1),
(0,1,1,z,z),(1,1,y,1,1),(1,y,0,z,y),(1,z,0,y,y),(1,y,0,z,y),(0,0,1,y,y),
(0,1,0,y,y),(1,1,y,y,y),(1,0,0,1,z),(1,0,0,z,1),(1,0,1,1,1)\]

All necessary verifications have been performed by means of a computer, and so the proof of our theorem now ends. □

4. Cardinality of the known smaller complete caps

<table>
<thead>
<tr>
<th>( q )</th>
<th>( k )</th>
<th>Deducible from</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>8</td>
<td>(2.4)</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
<td>(2.6) and (2.8)</td>
</tr>
<tr>
<td>7</td>
<td>21</td>
<td>(b)</td>
</tr>
<tr>
<td>9</td>
<td>29</td>
<td>(2.6)</td>
</tr>
<tr>
<td>11</td>
<td>36</td>
<td>(c)</td>
</tr>
<tr>
<td>13</td>
<td>44</td>
<td>(2.7)</td>
</tr>
<tr>
<td>17</td>
<td>67</td>
<td>(d)</td>
</tr>
<tr>
<td>( \geq 19 )</td>
<td>((m+1)(q+1) + 2)</td>
<td>(2.7)</td>
</tr>
</tbody>
</table>
Table 2
Cardinality $k$ of smaller complete caps in $\text{PG}(3,q)$ for $q$ even

<table>
<thead>
<tr>
<th>$q$</th>
<th>$k$</th>
<th>Deducible from</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>5</td>
<td>(2.3)</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>(a)</td>
</tr>
<tr>
<td>$\geq 8$</td>
<td>$3q + 2$</td>
<td>(2.2)</td>
</tr>
</tbody>
</table>

Table 3
Cardinality $k$ of smaller complete caps in $\text{PG}(4,q)$ for $q$ odd

<table>
<thead>
<tr>
<th>$q$</th>
<th>$k$</th>
<th>Deducible from</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\geq 3$</td>
<td>$2q^2 + 1$</td>
<td>(2.11)</td>
</tr>
</tbody>
</table>

Table 4
Cardinality $k$ of smaller complete caps in $\text{PG}(4,q)$ for $q$ even

<table>
<thead>
<tr>
<th>$q$</th>
<th>$k$</th>
<th>Deducible from</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>9</td>
<td>(2.13)</td>
</tr>
<tr>
<td>4</td>
<td>29</td>
<td>(e)</td>
</tr>
<tr>
<td>$\geq 8$</td>
<td>$2q^2 + q$</td>
<td>(2.12)</td>
</tr>
</tbody>
</table>

Table 5
Cardinality $k$ of smaller complete caps in $\text{PG}(r,2)$ for $r \geq 5$

<table>
<thead>
<tr>
<th>$q$</th>
<th>$k$</th>
<th>Deducible from</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>13</td>
<td>(2.14)</td>
</tr>
<tr>
<td>6</td>
<td>21</td>
<td>(2.15)</td>
</tr>
<tr>
<td>7</td>
<td>29</td>
<td>(2.16)</td>
</tr>
<tr>
<td>8</td>
<td>45</td>
<td>(2.17)</td>
</tr>
<tr>
<td>$\geq 9$</td>
<td>$15 \times 2^{m-3} - 3$, for $r = 2m - 1$, or $23 \times 2^{m-3} - 3$, for $r = 2m$</td>
<td>(2.18)</td>
</tr>
</tbody>
</table>

Table 6
Cardinality $k$ of smaller complete caps in $\text{PG}(r,4)$ for $r \geq 4$

<table>
<thead>
<tr>
<th>$q$</th>
<th>$k$</th>
<th>Deducible from</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\geq 4$</td>
<td>$2^{r+1} - 2$</td>
<td>(2.9)</td>
</tr>
</tbody>
</table>

References