SYMBOLIC MODELS FOR NONLINEAR CONTROL SYSTEMS WITHOUT STABILITY ASSUMPTIONS

M AJID ZAMANI 1 , GIORDANO POLA 2 , MANUEL MAZO JR. 3 , AND PAULO TABUADA 1

ABSTRACT. Finite-state models of control systems were proposed by several researchers as a convenient mechanism to synthesize controllers enforcing complex specifications. Most techniques for the construction of such symbolic models have two main drawbacks: either they can only be applied to restrictive classes of systems, or they require the exact computation of reachable sets. In this paper, we propose a new abstraction technique that is applicable to any smooth control system as long as we are only interested in its behavior in a compact set. Moreover, the exact computation of reachable sets is not required.

1. INTRODUCTION

In the past years several different abstraction techniques have been developed to assist in the synthesis of controllers enforcing complex specifications. This paper is concerned with symbolic abstractions resulting from replacing aggregates or collections of states of a control system by symbols. When a symbolic abstraction with a finite number of states or symbols is available, the synthesis of the controllers can be reduced to a fixed-point computation over the finite-state abstraction [1]. Moreover, by leveraging computational tools developed for discrete-event systems [2, 3] and games on automata [4, 5, 6], one can synthesize controllers satisfying specifications difficult to enforce with conventional control design methods. Examples of such specification classes include logic specifications expressed in linear temporal logic or automata on infinite strings.

The quest for symbolic abstractions has a long history including results on timed automata [7], rectangular hybrid automata [8], and o-minimal hybrid systems [9, 10]. Early results for classes of control systems were based on dynamical consistency properties [11], natural invariants of the control system [12], l-complete approximations [13], and quantized inputs and states [14, 15]. Recent results include work on piecewise-affine and multi-affine systems [16, 17], abstractions based on an elegant use of convexity of reachable sets for sufficiently small time [18], and the use of incremental input-to-state stability [19, 20, 21, 22].

Our results improve upon most of the existing techniques in two directions: i) by being applicable to larger classes of control systems; ii) by not requiring the exact computation of reachable sets which is an hard task in general. In the first direction, our technique improves upon the results in [15, 16, 17] by being applicable to systems not restricted to nonholonomic chained-form, piecewise-affine, and multi-affine systems, respectively, and upon the results in [19, 20, 21, 22] by not requiring any stability or stabilizability assumptions. In the second direction, our technique improves upon the results in [13, 14] by not requiring the exact computation of reachable sets. In [18] a different abstraction technique is proposed that is also applicable to a wide class of control systems and does not require the exact computation of reachable sets. Such technique is based on convexity of reachable sets which requires very small sampling times. In contrast, our technique imposes no restrictions on the sampling time.

In this paper, we show that symbolic models exist if the control systems satisfy an incremental forward completeness assumption which is an incremental version of forward completeness. The main contribution of this paper is to establish that:

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For every nonlinear control system satisfying the incremental forward completeness assumption, one can construct a symbolic model that is alternatingly approximately simulated by the control system and that approximately simulates the control system.

Furthermore, since a weaker version of incremental forward completeness holds on compacts for every smooth control system, a symbolic model describing the behavior on compacts can always be constructed.

These relationships are weaker than the approximate bisimulation relationships, established in [19, 20, 21, 22], in the sense that they are only sufficient but not necessary to guarantee that any controller synthesized for the symbolic model can be refined to a controller enforcing the desired specifications on the original control system. In other words, any controller synthesized for the abstraction can be converted into a controller enforcing the specification on the original control system. However, failing to find a controller enforcing the specification on the symbolic model does not prevent the existence of a controller for the original control system. Hence, control designers are confronted with the choice between the following two alternatives when using approximate abstractions:

1. design a controller rendering the original control system incrementally input-to-state stable and then apply the existing abstraction techniques in [19, 20, 21, 22];
2. or construct an abstraction using the results presented in this paper.

Since most of the existing controller design techniques provide controllers enforcing stability rather than incremental stability, the second alternative provides a concrete approach to symbolic control design for unstable control systems.

2. Control Systems and Incremental Forward Completeness

2.1. Notation. The identity map on a set $A$ is denoted by $1_A$. If $A$ is a subset of $B$ we denote by $ι_A : A → B$ or simply by $ι$ the natural inclusion map taking any $a ∈ A$ to $ι(a) = a ∈ B$. The symbols $N$, $Z$, $R$, $R^+$ and $R^n_{+}$ denote the set of natural, integer, real, positive, and nonnegative real numbers, respectively. The symbol $I_n$ denotes the identity matrix on $R^n$. Given a vector $x ∈ R^n$, we denote by $x_i$ the $i$-th element of $x$, by $∥x∥$ the infinity norm of $x$, and by $∥x∥_2$ the Euclidean norm of $x$; we recall that $∥x∥ = max(|x_1|, |x_2|, \ldots, |x_n|)$, and $∥x∥_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, where $|x_i|$ denotes the absolute value of $x_i$. Given a matrix $M = \{m_{ij}\} ∈ R^{n×m}$, the infinity norm of $M$ is $∥M∥ = max_{1≤i≤m} \sum_{j=1}^{n} |m_{ij}|$. The closed ball centered at $x ∈ R^n$ with radius $ε$ is defined by $B_ε(x) = \{y ∈ R^n \mid ∥x − y∥ ≤ ε\}$. For any $A ⊆ R^n$ and $μ ∈ R^+$ define $[A]_μ = \{a ∈ A \mid a_i = k_iμ, k_i ∈ Z, i = 1, \ldots, n\}$. The set $[A]_μ$ will be used as an approximation of the set $A$ with precision $μ$. Geometrically, for any $μ ∈ R^+$ and $λ ≥ μ$ the collection of sets $\{B_λ(p)\}_{p∈[A]_μ}$ is a covering of $A$, i.e. $A ⊆ \bigcup_{p∈[A]_μ} B_λ(p)$. In the special case where $A = R^n$, the set $\bigcup_{p∈[A]_μ} B_λ(p)$ remains a cover even if we reduce $λ$ so as to satisfy $λ ≥ \frac{μ}{2}$. A function $d : R^n × R^n → R^n_+$ is a metric on $R^n$ if for any $x, y, z ∈ R^n$, the following three conditions are satisfied: i) $d(x, y) = 0$ if and only if $x = y$; ii) $d(x, y) = d(y, x)$; and iii) $d(x, z) ≤ d(x, y) + d(y, z)$. Given a measurable function $f : R^n_+ → R^n$, the (essential) supremum (sup norm) of $f$ is denoted by $∥f∥_∞$; we recall that $∥f∥_∞ = (\text{ess sup} \{∥f(t)∥, t ≥ 0\})$. A continuous function $γ : R^n_+ → R^n_+$, is said to belong to class $K$ if it is strictly increasing and $γ(0) = 0$; $γ$ is said to belong to class $K_∞$ if $γ ∈ K$ and $γ(r) → ∞$ as $r → ∞$.

2.2. Control Systems. The class of control systems that we consider in this paper is formalized in the following definition.

**Definition 2.1.** A control system is a quadruple $Σ = (R^n, U, \mathcal{U}, f)$, where:

- $R^n$ is the state space;
- $U ⊆ R^n$ is the input set which is compact and convex;
- $\mathcal{U}$ is the set of all piecewise continuous functions of time from intervals of the form $[a, b] ⊆ R$ to $U$ with $a < 0$ and $b > 0$;
A curve $\xi : [a, b] \to \mathbb{R}^n$ is said to be a \textit{trajectory} of $\Sigma$ if there exists $v \in \mathcal{U}$ satisfying:

$$(2.1) \quad \xi(t) = f(\xi(t), v(t)),$$

for all $t \in [a, b]$. We also write $\xi_{xv}(\tau)$ to denote the point reached at time $\tau$ under the input $v$ from initial condition $x = \xi(xv)(0)$; this point is uniquely determined, since the assumptions on $f$ ensure existence and uniqueness of trajectories [24]. Although we have defined trajectories over open domains, we shall refer to trajectories $\xi_{xv} : [0, \tau] \to \mathbb{R}^n$ and input curves $v : [0, \tau] \to \mathcal{U}$ defined on closed domains $[0, \tau]$, $\tau \in \mathbb{R}^+$, with the understanding of the existence of a trajectory $\xi'_{xv} : [a, b] \to \mathbb{R}^n$ and input curve $v' : [a, b] \to \mathcal{U}$ such that $\xi_{xv} = \xi'_{xv} |_{[0, \tau]}$ and $v = v' |_{[0, \tau]}$.

A control system $\Sigma$ is said to be forward complete if every trajectory is defined on an interval of the form $[a, \infty]$. Sufficient and necessary conditions for a system to be forward complete can be found in [24]. A control system $\Sigma$ is said to be smooth if $f$ is an infinitely differentiable function of its arguments.

### 2.3. Incremental Forward Completeness

The results presented in this paper require a certain assumption that we introduce in this section.

**Definition 2.2.** A control system $\Sigma$ is incrementally forward complete ($\delta$-FC) if there exist continuous functions $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ and $\gamma : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that for every $s \in \mathbb{R}^+_0$, the functions $\beta(\cdot, s)$ and $\gamma(\cdot, s)$ belong to class $\mathcal{K}_\infty$, and for any $x, x' \in \mathbb{R}^n$, any $\tau \in \mathbb{R}^+$, and any $v, v' \in \mathcal{U}$, where $v, v' : [0, \tau] \to \mathcal{U}$, the following condition is satisfied for all $t \in [0, \tau]$:

$$(2.2) \quad \|\xi_{x, v}(t) - \xi_{x', v'}(t)\| \leq \beta(\|x - x'\|, t) + \gamma(\|v - v'\|_\infty, t).$$

Incremental forward completeness requires the distance between two arbitrary trajectories to be bounded by the sum of two terms capturing the mismatch between the initial conditions and the mismatch between the inputs as shown in (2.2). As an example, for a linear control system:

$$\xi = A\xi + Bu, \quad \xi(t) \in \mathbb{R}^n, \quad v(t) \in \mathcal{U} \subseteq \mathbb{R}^m,$$

the functions $\beta$ and $\gamma$ can be chosen as:

$$(2.3) \quad \beta(r, t) = \|e^{At}\| r; \quad \gamma(r, t) = \left(\int_0^t \|e^{As}B\| \, ds\right) r,$$

where $\|e^{At}\|$ denotes the infinity norm of $e^{At}$. Whenever the origin is an equilibrium point for $\Sigma$, the choice $x' = 0$, $v' = 0$ results in the estimate $\|\xi_{x, v}(t)\| \leq \beta(\|x\|, t) + \gamma(\|v\|_\infty, t)$ which is shown in [24] to be equivalent to forward completeness of $\Sigma$ when it holds for all $t \in \mathbb{R}^+_0$. Hence, the systems satisfying (2.2) are termed incrementally forward complete. Descriptions of $\delta$-FC in terms of Lyapunov-like functions and expansion metrics are reported in Section 5.

### 3. Symbolic Models and Approximate Equivalence Notions

#### 3.1. Systems and Control Systems

We use systems to describe both control systems as well as their symbolic models. A more detailed exposition of the notion of system that we now introduce can be found in [1].

**Definition 3.1.** [1] A system $S$ is a quintuple $S = (X, U, \rightarrow, Y, H)$ consisting of:

1. We note that $\delta$-FC implies uniform continuity of the map $\phi_t : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$ defined by $\phi_t(x, v) = \xi_{xv}(t)$ for any fixed $t \in \mathbb{R}^+_0$. Here, uniform continuity is understood with respect to the topology induced by the infinity norm on $\mathbb{R}^n$, the sup norm on $\mathcal{U}$, and the product topology on $\mathbb{R}^n \times \mathcal{U}$.
• A set of states $X$;
• A set of inputs $U$;
• A transition relation $\rightarrow \subseteq X \times U \times X$;
• An output set $Y$;
• An output function $H : X \rightarrow Y$.

System $S$ is said to be:
• metric, if the output set $Y$ is equipped with a metric $d : Y \times Y \rightarrow \mathbb{R}^+_0$;
• countable, if $X$ is a countable set;
• finite, if $X$ is a finite set.

A transition $(x, u, x') \in \rightarrow$ is denoted by $x \xrightarrow{u} x'$. For a transition $x \xrightarrow{u} x'$, state $x'$ is called a $u$-successor, or simply successor, of state $x$. Since $\rightarrow \subseteq X \times U \times X$ is a relation, for any given state and input $u \in U$ there may be: no $u$-successors, one $u$-successor, or many $u$-successors. We denote the set of $u$-successors of a state $x$ by $\text{Post}_u(x)$ and by $U(x)$ the set of inputs $u \in U$ for which $\text{Post}_u(x)$ is nonempty. A system is deterministic if for any state $x \in X$ and any input $u$, there exists at most one $u$-successor (there may be none). A system is called nondeterministic if it is not deterministic. Hence, for a nondeterministic system it is possible for a state to have two (or possibly more) distinct $u$-successors.

The results in this Section and in Section 4 rely on additional assumptions on $U$ and $\mathcal{U}$ that we now describe. Such assumptions are not required for the results in Sections 2 and 5. We restrict attention to control systems $\Sigma = (\mathbb{R}^n, U, \mathcal{U}, f)$ with input sets $U$ of the form $U = \prod_{i=1}^m [a_i, b_i] \subseteq \mathbb{R}^m$ with $b_i > a_i$. For such input sets we define the constant $\tilde{\mu}$ by $\tilde{\mu} = \min\{b_i - a_i, \ldots, |b_m - a_m|\}$. We further restrict attention to trajectories generated by piece-wise constant input curves, by requiring $\mathcal{U}$ to contain only constant curves of duration $\tau \in \mathbb{R}^+$:

$$\mathcal{U} = \{v : [0, \tau] \rightarrow U \mid v(t) = v(0), t \in [0, \tau]\}.$$ 

Given a control system $\Sigma = (\mathbb{R}^n, U, \mathcal{U}, f)$, and a time discretization parameter $\tau \in \mathbb{R}^+$, consider the system $\Sigma_{\tau}(\Sigma) = (X_{\tau}, U_{\tau}, \xrightarrow{\tau}, Y_{\tau}, H_{\tau})$ consisting of:

• $X_{\tau} = \mathbb{R}^n$;
• $U_{\tau} = U$;
• $x_{\tau} \xrightarrow{u_{\tau}} x'_{\tau}$ if there exists a trajectory $\xi_{x_{\tau}u_{\tau}} : [0, \tau] \rightarrow \mathbb{R}^n$ of $\Sigma$ satisfying $\xi_{x_{\tau}u_{\tau}}(\tau) = x'_{\tau}$;
• $Y_{\tau} = \mathbb{R}^n$;
• $H_{\tau} = 1_{\mathbb{R}^n}$.

The above system can be thought of as the time discretization of the control system $\Sigma$.

### 3.2. System relations

We first consider approximate simulation relations, introduced in [25], that are useful when analyzing or synthesizing controllers for deterministic systems.

**Definition 3.2.** Let $S_a = (X_a, U_a, \xrightarrow{a}, Y_a, H_a)$ and $S_b = (X_b, U_b, \xrightarrow{b}, Y_b, H_b)$ be metric systems with the same output sets $Y_a = Y_b$ and metric $d$, and consider a precision $\varepsilon \in \mathbb{R}^+$. A relation $R \subseteq X_a \times X_b$ is said to be an $\varepsilon$-approximate simulation relation from $S_a$ to $S_b$, if the following three conditions are satisfied:

(i) for every $x_a \in X_a$, there exists $x_b \in X_b$ with $(x_a, x_b) \in R$;
(ii) for every $(x_a, x_b) \in R$ we have $d(H_a(x_a), H_b(x_b)) \leq \varepsilon$;
(iii) for every $(x_a, x_b) \in R$ we have that:

$$x_a \xrightarrow{a} x'_a \text{ in } S_a \text{ implies the existence of } x_b \xrightarrow{b} x'_b \text{ in } S_b \text{ satisfying } (x'_a, x'_b) \in R.$$ 

System $S_a$ is $\varepsilon$-approximately simulated by $S_b$ or $S_b$ $\varepsilon$-approximately simulates $S_a$, denoted by $S_a \preceq_S S_b$, if there exists an $\varepsilon$-approximate simulation relation from $S_a$ to $S_b$. 
For nondeterministic systems we need to consider relationships that explicitly capture the adversarial nature of nondeterminism. The notion of alternating approximate simulation relation is shown in [20] to be appropriate to this effect.

**Definition 3.3.** Let $S_a$ and $S_b$ be metric systems with the same output sets $Y_a = Y_b$ and metric $d$, and consider a precision $\varepsilon \in \mathbb{R}^+$. A relation $R \subseteq X_a \times X_b$ is said to be an $\varepsilon$-approximate alternating simulation relation from $S_a$ to $S_b$ if the following three conditions are satisfied:

(i) for every $x_a \in X_a$, there exists $x_b \in X_b$ with $(x_a, x_b) \in R$;
(ii) for every $(x_a, x_b) \in R$ we have $d(H_a(x_a), H_b(x_b)) \leq \varepsilon$;
(iii) for every $(x_a, x_b) \in R$ and for every $u_a \in U_a(x_a)$ there exists $u_b \in U_b(x_b)$ such that for every $x'_b \in \text{Post}_{u_b}(x_b)$ there exists $x'_a \in \text{Post}_{u_a}(x_a)$ satisfying $(x'_a, x'_b) \in R$.

System $S_a$ is alternatingly $\varepsilon$-approximately simulated by $S_b$ or $S_b$ alternatingly $\varepsilon$-approximately simulates $S_a$, denoted by $S_a \preceq_{\varepsilon AS} S_b$, if there exists an alternating $\varepsilon$-approximate simulation relation from $S_a$ to $S_b$.

It is readily seen from the above definitions that the notions of approximate simulation and of alternating approximate simulation coincide when the systems involved are deterministic.

The importance of the preceding notions lies in enabling the transfer of controllers designed for a symbolic model to controllers acting on the original control system. More details about these notions and how the refinement of controllers can be performed are reported in [1].

4. **Symbolic Models for $\delta$-FC Control Systems**

This section contains the main contribution of the paper. We show that the time discretization of a $\delta$-FC control system, suitably restricted to a compact set, admits a finite abstraction.

We consider a $\delta$-FC control system $\Sigma = (\mathbb{R}^n, U, \mathcal{U}, f)$, and a quadruple $q = (\tau, \eta, \mu, \theta)$ of quantization parameters, where $\tau \in \mathbb{R}^+$ is the time quantization, $\eta \in \mathbb{R}^+$ is the state space quantization, $\mu \in \mathbb{R}^+$ is the input space quantization, and $\theta \in \mathbb{R}^+$ is a design parameter. Given $\Sigma$ and $q$, define the system:

$$
S_q(\Sigma) = (X_q, U_q, \longrightarrow^q, Y_q, H_q),
$$

consisting of:

- $X_q = [\mathbb{R}^n]_\eta$;
- $U_q = [\mathbb{R}]_\mu$;
- $x_q \xrightarrow{u_q} x'_q$ if $\|\xi_{x_qu_q}(\tau) - x'_q\| \leq \beta(\theta, \tau) + \gamma(\mu, \tau) + \frac{\eta}{2}$;
- $Y_q = \mathbb{R}^n$;
- $H_q = \iota : X_q \mapsto Y_q$.

where $\beta$ and $\gamma$ are the functions appearing in [22]. In the definition of the transition relation, and in the remainder of the paper, we abuse notation by identifying $u_q$ with the constant input curve with domain $[0, \tau]$ and value $u_q$.

The transition relation of $S_q(\Sigma)$ is well defined in the sense that for every $x_q \in X_q$ and every $u_q \in U_q$ there always exists $x'_q \in X_q$ such that $x_q \xrightarrow{u_q} x'_q$. This can be seen by noting that by definition of $X_q$, for any $x \in \mathbb{R}^n$ there always exists a state $x'_q \in X_q$ such that $\|x - x'_q\| \leq \eta/2$. Hence, for $x = \xi_{x_qu_q}(\tau)$ there always exists a state $x'_q \in X_q$ satisfying $\|\xi_{x_qu_q}(\tau) - x'_q\| \leq \frac{\eta}{2} \leq \beta(\theta, \tau) + \gamma(\mu, \tau) + \frac{\eta}{2}$.

We can now state the main result of the paper which relates $\delta$-FC to existence of symbolic models.

**Theorem 4.1.** Let $\Sigma$ be a $\delta$-FC control system. For any desired precision $\varepsilon \in \mathbb{R}^+$, and any quadruple $q = (\tau, \eta, \mu, \theta)$ of quantization parameters satisfying $\mu \leq \hat{\mu}$ and $\eta \leq 2\varepsilon \leq 2\theta$, we have $S_q(\Sigma) \preceq_{\varepsilon AS} S_\tau(\Sigma) \preceq_S S_q(\Sigma)$. 
Proof of Theorem 4.1. We start by proving $S_q(\Sigma) \subseteq S_q(\Sigma)$. Consider the relation $R \subseteq X_\tau \times X_q$ defined by $(x_\tau, x_q) \in R$ if and only if $\|H_\tau(x_\tau) - H_q(x_q)\| = \|x_\tau - x_q\| \leq \varepsilon$. Since $X_\tau \subseteq \bigcup_{p \in [r_n]} B_p(\mu)$, for every $x_\tau \in X_\tau$ there exists $x_q \in X_q$ such that:

$$
\|x_\tau - x_q\| \leq \eta_2 \leq \varepsilon.
$$

Hence, $(x_\tau, x_q) \in R$ and condition (i) in Definition 3.2 is satisfied. Now consider any $(x_\tau, x_q) \in R$. Condition (ii) in Definition 3.2 is satisfied by the definition of $R$. Let us now show that condition (iii) in Definition 3.2 holds.

Consider any $v_\tau \in U_\tau$. Choose an input $u_q \in U_q$ satisfying:

$$
\|v_\tau - u_q\|_{\infty} = \|v_\tau(0) - u_q(0)\| \leq \mu.
$$

Note that the existence of such $u_q$ is guaranteed by the special form of $U$, described in Section 3, and by the inequality $\mu \leq \tilde{\mu}$ which guarantees that $U \subseteq \bigcup_{p \in [r_n]} B_p(p)$. Consider the unique transition $x_\tau \xrightarrow{v_\tau} x'_\tau = \xi_{x_\tau, v_\tau}(\tau)$ in $S_\tau(\Sigma)$. It follows from the $\delta$-FC assumption that the distance between $x'_\tau$ and $\xi_{x_\tau u_q}(\tau)$ is bounded as:

$$
\|x'_\tau - \xi_{x_\tau u_q}(\tau)\| \leq \beta(\varepsilon, \tau) + \gamma(\mu, \tau).
$$

Since $X_\tau \subseteq \bigcup_{p \in [r_n]} B_\tau(p)$, there exists $x'_q \in X_q$ such that:

$$
\|x'_\tau - x'_q\| \leq \eta_2
$$

Using the inequalities $\varepsilon \leq \theta$, (4.4), and (4.5), we obtain:

$$
\|\xi_{x_\tau u_q}(\tau) - x'_q\| = \|\xi_{x_\tau u_q}(\tau) - x'_\tau + x'_\tau - x'_q\| \leq \|\xi_{x_\tau u_q}(\tau) - x'_\tau\| + \|x'_\tau - x'_q\| \\
 \leq \beta(\varepsilon, \tau) + \gamma(\mu, \tau) + \eta_2 \leq \beta(\theta, \tau) + \gamma(\mu, \tau) + \frac{\eta_2}{\frac{\eta_2}{2}},
$$

which implies the existence of $x_q \xrightarrow{u_q} x'_q$ in $S_q(\Sigma)$ by definition of $S_q(\Sigma)$. Therefore, from inequality (4.5) and $\frac{\eta_2}{2} \leq \varepsilon$, we conclude $(x'_\tau, x'_q) \in R$ and condition (iii) in Definition 3.2 holds.

Now we prove $S_q(\Sigma) \subseteq S_\tau(\Sigma)$. Consider the relation $R \subseteq X_\tau \times X_q$. For every $x_q \in X_q$, by choosing $x_\tau = x_q$, we have $(x_\tau, x_q) \in R$ and condition (i) in Definition 3.3 is satisfied. Now consider any $(x_\tau, x_q) \in R$. Condition (ii) in Definition 3.3 is satisfied by the definition of $R$. Let us now show that condition (iii) in Definition 3.3 holds. Consider any $u_q \in U_q$. Choose the input $v_\tau = u_q$ and consider the unique $x'_\tau = \xi_{x_\tau, v_\tau}(\tau) \in \text{Post}_{v_\tau}(x_\tau)$ in $S_\tau(\Sigma)$. From the $\delta$-FC assumption, the distance between $x'_\tau$ and $\xi_{x_\tau u_q}(\tau)$ is bounded as:

$$
\|x'_\tau - \xi_{x_\tau u_q}(\tau)\| \leq \beta(\varepsilon, \tau).
$$

Since $X_\tau \subseteq \bigcup_{p \in [r_n]} B_\tau(p)$, there exists $x'_q \in X_q$ such that:

$$
\|x'_\tau - x'_q\| \leq \frac{\eta_2}{2}
$$

Using the inequalities $\varepsilon \leq \theta$, (4.6), and (4.7), we obtain:

$$
\|\xi_{x_\tau u_q}(\tau) - x'_q\| = \|\xi_{x_\tau u_q}(\tau) - x'_\tau + x'_\tau - x'_q\| \\
 \leq \|\xi_{x_\tau u_q}(\tau) - x'_\tau\| + \|x'_\tau - x'_q\| \leq \beta(\varepsilon, \tau) + \frac{\eta_2}{2} \\
 \leq \beta(\theta, \tau) + \gamma(\mu, \tau) + \frac{\eta_2}{2},
$$

which implies the existence of $x_q \xrightarrow{u_q} x'_q$ in $S_q(\Sigma)$ by definition of $S_q(\Sigma)$. Therefore, from inequality (4.7) and $\frac{\eta_2}{2} \leq \varepsilon$, we can conclude that $(x'_\tau, x'_q) \in R$ and condition (iii) in Definition 3.2 holds.
The symbolic model $S_q(\Sigma)$ has a countably infinite state set. In order to construct a finite symbolic model we note that in practical applications the physical variables are restricted to a compact set. Velocities, temperatures, pressures, and other physical quantities cannot become arbitrarily large without violating the operational envelop defined by the control problem being solved. By making use of this fact, we can directly compute a finite abstraction $S_{qD}(\Sigma)$ of $S_r(\Sigma)$ capturing the behavior of $S_r(\Sigma)$ within a given compact set $D \subset \mathbb{R}^n$ describing the valid range for the physical variables. The system $S_{qD} = (X_{qD}, U_{qD}, \overrightarrow{a_{qD}}, Y_{qD}, H_{qD})$ is defined by:

- $X_{qD} = [D]_\eta$;
- $U_{qD} = [\mu]$;
- $x_{qD} \xrightarrow{u_{qD}} x'_q$ if every $x'_q \in [\mathbb{R}^n]_\eta$ satisfying $\|\xi_{x_{qD}u_{qD}}(\tau) - x'_q\| \leq \beta(\theta, \tau) + \gamma(\mu, \tau) + \frac{\eta}{2}$ belongs to $X_{qD}$;
- $Y_{qD} = \mathbb{R}^n$;
- $H_{qD} = \iota : X_{qD} \leftrightarrow Y_{qD}$.

Note that $S_{qD}(\Sigma)$ is a finite system since $D$ is a compact set. Moreover, the relation $R \subseteq X_{qD} \times X_{qD}$ defined by $(x_{qD}, x_q) \in R$ if $x_{qD} = x_q$ is a 0-approximate alternating simulation relation from $S_{qD}(\Sigma)$ to $S_q(\Sigma)$. By combining $S_{qD}(\Sigma) \subseteq^0_{AS} S_q(\Sigma)$ with $S_q(\Sigma) \subseteq^0_{AS} S_r(\Sigma)$ we conclude $S_{qD}(\Sigma) \subseteq^0_{AS} S_r(\Sigma)$. Hence, any controller synthesized for the finite model $S_{qD}(\Sigma)$ can be refined to a controller enforcing the same specification on $S_r(\Sigma)$. Moreover, in order to compute $S_{qD}(\Sigma)$ we only need inequality \((5.1)\) in the definition of $\delta$-FC to hold for times $t$ for which $\xi(t) \in D$. As will be discussed in Section 5 this weaker version of inequality \((5.1)\) holds for any smooth control system on a compact set $D$.

We refer the interested readers to Appendix I for a numerical example showing the effectiveness of the proposed results. In the example, a controller is synthesized for an inverted pendulum subject to a schedulability constraint defined by a finite system.

5. Descriptions of Incremental Forward Completeness

This section contains the description of $\delta$-FC in terms of Lyapunov-like functions and expansion metrics. We start by introducing the following definition which was inspired by the notion of incremental input-to-state stability ($\delta$-ISS) Lyapunov function presented in [26].

**Definition 5.1.** Consider a control system $\Sigma$ and a smooth function $V : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$. Function $V$ is called a $\delta$-FC Lyapunov function for $\Sigma$, if there exist $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, $\sigma$, and $\kappa \in \mathbb{R}$ such that:

(i) for any $x, x' \in \mathbb{R}^n$

\[\underline{\alpha}(\|x - x'\|) \leq V(x, x') \leq \overline{\alpha}(\|x - x'\|);\]

(ii) for any $x, x' \in \mathbb{R}^n$ and for any $u, u' \in U$

\[\frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial x'} f(x', u') \leq \kappa V(x, x') + \sigma(\|u - u'\|)\]

The following theorem describes $\delta$-FC in terms of the existence of a $\delta$-FC Lyapunov function.

**Theorem 5.2.** A control system $\Sigma$ is $\delta$-FC if it admits a $\delta$-FC Lyapunov function. Moreover, the functions $\beta$ and $\gamma$ in \((5.1)\) are given by:

\[\beta(r, t) = \alpha^{-1}\left(2e^{\kappa t} \overline{\alpha}(r)\right), \quad \gamma(r, t) = \alpha^{-1}\left(2e^{\kappa t} \frac{1}{\kappa} \sigma(r)\right).\]

The proof of the preceding result is reported in Appendix II and was inspired by the work in [24].

\[\text{It is shown in [1] that the composition of two alternating simulation relations is still an alternating simulation relation.}\]

\[\text{Condition (ii) of Definition 3.1 can be replaced by } \frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial x'} f(x', u') \leq \rho(\|x - x'\|) + \sigma(\|u - u'\|), \text{ where } \rho \text{ is a } \mathcal{K}_\infty \text{ function. It is known that there is no loss of generality in considering } \rho(\|x - x'\|) = \kappa V(x, x'), \text{ by appropriately modifying the } \delta\text{-FC Lyapunov function } V \text{ (see Lemma 11 in [27]).}\]
Remark 5.3. It can be easily checked that the quadratic Lyapunov function \( V(x, x') = \| x - x' \|^2 \) satisfies conditions (i) and (ii) in Definition 5.1 if we restrict \( x \) and \( x' \) to range in a compact subset of \( \mathbb{R}^n \). The arguments in the proof of Theorem 5.2 show that any such control system satisfies inequality (2.2), in the definition of \( \delta \)-FC, for all \( t \) for which the trajectories \( \xi \) remain within the aforementioned compact set. This weaker version of inequality (2.2) is sufficient for the construction of the finite symbolic model \( S_qD(\Sigma) \) introduced in Section 4.

In addition to Lyapunov-like functions, the \( \delta \)-FC condition can be described by resorting to expansion metrics. The notion of contraction metric was introduced in control theory in [28] as a tool for the study of stability properties of nonlinear systems. The interested reader may also wish to consult [29] where it is shown that contraction metrics were investigated more than 40 years before being used in control theory. In this paper we consider systems that are not necessarily stable and thus consider also expansion metrics.

The variational system associated with a smooth control system \( \Sigma \), when we have variations of the state and input, is given by the differential equation:

\[
\frac{d}{dt}(\delta \xi) = \frac{\partial f}{\partial x} \bigg|_{x=\xi} \delta \xi + \frac{\partial f}{\partial u} \bigg|_{x=\xi} \delta \nu
\]

where \( \delta \xi \) and \( \delta \nu \) are variations\(^4\) of the state and input, respectively.

A Riemannian metric \( G : \mathbb{R}^n \to \mathbb{R}^{n \times n} \) on \( \mathbb{R}^n \) is a smooth map such that, for any \( x \in \mathbb{R}^n \), \( G(x) \) is a symmetric positive definite matrix [30]. For any \( x \in \mathbb{R}^n \) and smooth functions \( Z, W : \mathbb{R}^n \to \mathbb{R}^n \), one can define the scalar function \( \langle Z, W \rangle_G \) as \( Z^T(x)G(x)W(x) \). We will still use the notation \( \langle Z, W \rangle_G \) to denote \( Z^TGW \) even if \( G \) does not represent any Riemannian metric.

In the following, we introduce the notion of exponential expansion which was inspired by the notion of exponential contraction in [31] [28]. Note that in [31] [28], the notion of exponential contraction does not capture the variations on the input. In contrast, the notion of exponential expansion does contain variations of the state and input.

Definition 5.4. Let \( \Sigma = (\mathbb{R}^n, U, U, f) \) be a smooth control system on the manifold \( \mathbb{R}^n \) equipped with a Riemannian metric \( G \). Control system \( \Sigma \) is said to be an exponential expansion with respect to a metric \( G \), if there exists some \( \lambda \in \mathbb{R} \) and \( \alpha \in \mathbb{R}^n \) such that:

\[
\langle Z, Z \rangle_F + 2 \left( \frac{\partial f}{\partial u} \right)^T G(x) + G(x) \frac{\partial f}{\partial u} + \frac{\partial G}{\partial f} f(x, u), \text{any } Z \in \mathbb{R}^n, \text{ and any } W \in \mathbb{R}^m, \text{ or equivalently:}
\]

\[
Z^T \left( \left( \frac{\partial f}{\partial u} \right)^T G(x) + G(x) \frac{\partial f}{\partial u} + \frac{\partial G}{\partial f} f(x, u) \right) Z + 2W^T \left( \frac{\partial f}{\partial u} \right)^T G(x)Z \leq \lambda Z^T G(x)Z + \alpha (Z^T G(x)Z)^{\frac{1}{2}} (W^T W)^{\frac{1}{2}},
\]

where the constant \( \lambda \) is called expansion rate.

Note that the inequality (5.3) or (5.4) implies:

\[
\frac{d}{dt} \langle \delta \xi, \delta \xi \rangle_G \leq \lambda \langle \delta \xi, \delta \xi \rangle_G + \alpha \langle \delta \xi, \delta \xi \rangle_G^{\frac{1}{2}} \langle \delta \nu, \delta \nu \rangle_{I_m}^{\frac{1}{2}},
\]

where \( \delta \xi \) and \( \delta \nu \) are variations of a state and an input trajectory of the control system \( \Sigma \). The following result describes \( \delta \)-FC in terms of the existence of an expansion metric.

\(^4\)The variations \( \delta \xi \) and \( \delta \nu \) can be formally defined by considering a family of trajectories \( \xi_{x,u}(t, \epsilon) \) and inputs \( u(t, \epsilon) \) parametrized by \( \epsilon \in \mathbb{R} \). The variations of the state and input are then \( \delta \xi = \frac{\partial \xi}{\partial \epsilon} \) and \( \delta \nu = \frac{\partial u}{\partial \epsilon} \).
Theorem 5.5. If $\Sigma = (\mathbb{R}^n, U, \mathcal{U}, f)$ is an exponential expansion with respect to a Riemannian metric $G$ on $\mathbb{R}^n$, satisfying $\omega_\|y\|_2^2 \leq y^T G(x) y \leq \overline{\omega}_\|y\|_2^2$ for any $x, y \in \mathbb{R}^n$ and $\omega, \overline{\omega} \in \mathbb{R}^+$, then $\Sigma$ is a $\delta$-FC control system. Moreover, the functions $\beta$ and $\gamma$ in (2.2) are given by:

$$
(5.6) \quad \beta(r, t) = \sqrt{\frac{\sqrt{n}}{\omega}} e^{\frac{\lambda}{2} t} r, \quad \gamma(r, t) = \sqrt{\frac{m}{\omega}} e^{\frac{\lambda}{2} t - 1} r,
$$

where $n$ and $m$ denote the dimension of the state and input space, respectively.

The proof of the preceding result can be found in Appendix II and was inspired by the work in [31].

6. Discussion

In this paper we showed that any smooth control system, suitably restricted to a compact subset of states, admits a finite symbolic model. Our results improve upon the existing work by being applicable to a large class of control systems and by not requiring the exact computation of reachable sets or the convexity of reachable sets. The symbolic models constructed according to the results presented in this paper can be used to synthesize controllers enforcing complex specifications given in several different formalisms such as temporal logics or automata on infinite strings. The synthesis of such controllers is well understood and can be performed using simple fixed-point computations as described in [1]. The current limitation of this design methodology is the size of the computed abstractions. The authors are currently investigating several different techniques to address this limitation such as integrating the design of controllers with the construction of symbolic models [32]. Efforts by other researchers include the use of non-uniform quantization [33].

Acknowledgements

The authors would like to thank C. Manolescu for enlightening discussions.

References

Example 7.1. We illustrate the results in Theorem 4.1 on an inverted pendulum shown in Figure 1. We have the following model for the pendulum:

\[
\dot{x}_1 = x_2, \\
\dot{x}_2 = \frac{g}{l} \sin(x_1) - \frac{h}{ml} x_2 + \frac{1}{m} \cos(x_1) u,
\]

where \(x_1\) is the angular position, \(x_2\) is the angular velocity of the point mass, \(u\) is the applied force (control input), \(g = 9.8\) is gravity’s acceleration, \(l = 0.5\) is the length of the rod, \(m = 0.5\) is the mass, and \(h = 2\) is the rotational friction coefficient. All constants and variables are expressed in the International System. The eigenvalues of the linearized system around the equilibrium point \((0, 0)\) are \(\lambda_1 = 1.1433\) and \(\lambda_2 = -17.1433\) showing that the original nonlinear system is unstable at \((0, 0)\). Hence, the results in [31, 19, 20, 21, 35] do not apply to this system. We assume that \(u \in U = [-6, 6]\). We work on the subset \(X = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [-1, 1]\) of the
Inverted pendulum mounted on a cart.

Table 1. The constants $a_{hijk}$ defining the $\delta$-FC Lyapunov function $V$ for the inverted pendulum.

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State space of $\Sigma$. In order to construct a symbolic abstraction for the preceding model, we need to find functions $\beta$ and $\gamma$ satisfying the incremental forward completeness property in (2.2). We use SOS programming [36] to find a $\delta$-FC Lyapunov function of the form $V((x_1, x_2), (x'_1, x'_2)) = \sum a_{hijk} x_h^2 x_{i}^2 x_{j}^2 x_{k}^2$, $0 \leq h + i + j + k \leq 4$, for the inverted pendulum described above. The resulting $V$ is defined by the constants $a_{hijk}$ provided in Table 1. Function $V$ satisfies conditions (i) and (ii) in Definition 5.1 with $\mathcal{K}_{\infty}$ functions: $\alpha(r) = 0.5r^2$, $\sigma(r) = 3r^2$, $\sigma(r) = 5r^2$ and positive constant $\kappa = 2$. Using the results from Theorem 5.2, the functions $\beta$ and $\gamma$ are given by $\beta(r, t) = 2\sqrt{3}\varepsilon r$ and $\gamma(r, t) = 2\sqrt{3}\varepsilon^2 r - 1$. Our objective is to design a controller forcing the trajectories of the system to reach the target set $W = [-0.25, 0.25] \times [-1, 1]$ and to remain indefinitely inside $W$. We furthermore assume that the controller is implemented on a microprocessor that is executing other tasks in addition to the control task. We consider a periodic schedule with epochs of three time slots where the first two time slots are allocated to the control tasks and the third time slot to another task. The expression time slot refers to a time interval of the form $[kT, (k + 1)T]$ with $k \in \mathbb{N}$ and where $T$ is the time quantization parameter. Therefore, the microprocessor schedule is given by:

$|aau|aau|aau|aau|aau|aau|aau|aau|aau|aau|aau|aau|aau|aau|aau|aau|aau|aau|aau|aau|$
Figure 2. $S_{sc}$ describing the schedulability constraints. The lower part of the states are labeled with the outputs $a$ and $u$ denoting availability and unavailability of the microprocessor, respectively.

Figure 3. Upper and central panels: evolution of $x_1$ and $x_2$ with initial condition $(0.7, 0)$. Lower panel: input signal.

where $a$ denotes a slot available for the control task and $u$ denotes a slot allotted to a different task. The symbol $|$ separates each epoch of three time slots. The schedulability constraint on the microprocessor can be represented by the finite system $S_{sc}$ in Figure 2. When $S_{sc}$ is in state $q_1$ or $q_2$ the microprocessor computes the control input for the inverted pendulum. On the other hand, when $S_{sc}$ is in state $q_3$, the microprocessor computes another task. Although we can easily consider more complex schedules, the constraints described by $S_{sc}$ in Figure 2 already illustrate the computational constraints imposed by implementing control laws on shared microprocessors.

For a precision $\varepsilon = 0.01$, we construct a symbolic model $S_{q}(\Sigma)$ by choosing $\theta = 0.01$, $\eta = 0.02$, $\tau = 0.5$, and $\mu = 0.4$ so that assumptions of Theorem 4.1 are satisfied. The computation of the abstraction $S_{q}(\Sigma)$ was performed in the tool \texttt{Pessoa} \cite{pessoa}. A controller enforcing the specification is found by performing simple fixed-point computations on $S_{q}(\Sigma)$ using standard algorithms from game theory \cite{game_theory}. We solved a reachability game and a safety game, both implemented in Pessoa, to reach and stay indefinitely in the target set, respectively. In Figure 3 we show the closed-loop trajectory stemming from the initial condition $(0.7, 0)$ and the evolution of the input signal. The domain of the synthesized controller for different initial states of the finite system modeling the schedulability of the microprocessor is shown in Figure 4.

\texttt{Pessoa} can be freely downloaded from \url{http://www.cyphylab.ee.ucla.edu/pessoa}. All the files necessary to recreate this example are also available on Pessoa’s website.
Figure 4. Domain of the controller forcing the inverted pendulum to reach and remain in $[-0.25, 0.25] \times [-1, 1]$ under the constraint described by $S_{sc}$ in Figure 2. The different figures in (a), (b), and (c) correspond to $S_{sc}$ initialized from states $q_1$, $q_2$, or $q_3$, respectively.
8. Appendix II

Proof of Theorem 5.2. The proof is inspired by the work in [24]. By using property (i) in Definition 5.1, we obtain:

\[
\|\xi_{x,v}(t) - \xi_{x',v'}(t)\| \leq \alpha^{-1}\left(V(\xi_{x,v}(t), \xi_{x',v'}(t))\right).
\]

By using the property (ii) and the comparison lemma \[38\], one gets:

\[
V(\xi_{x,v}(t), \xi_{x',v'}(t)) \leq e^{\kappa t}V(\xi_{x,v}(0), \xi_{x',v'}(0)) + e^{\kappa t} \ast \sigma(\|v(t) - v'(t)\|)
\]

where * denotes the convolution integral. By combining inequalities (8.1) and (8.2), one gets:

\[
\|\xi_{x,v}(t) - \xi_{x',v'}(t)\| \leq \alpha^{-1}\left(e^{\kappa t}V(x, x') + e^{\kappa t} \ast \sigma(\|v - v'\|)\right)
\]

where \(\gamma(\rho, \phi) = \alpha^{-1}(\rho + \phi)\), \(\rho = e^{\kappa t}V(x, x')\), and \(\phi = \frac{e^{\kappa t} - 1}{\kappa} \sigma(\|v - v'\|)\). Since \(\gamma\) is nondecreasing in each variable, we have:

\[
\|\xi_{x,v}(t) - \xi_{x',v'}(t)\| \leq h\left(e^{\kappa t}V(x, x')\right) + h\left(\frac{e^{\kappa t} - 1}{\kappa} \sigma(\|v - v'\|)\right)
\]

where \(h(r) = \gamma(r, r) = \alpha^{-1}(2r)\) and \(h : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+\) is a \(K_\infty\) function. Moreover, using \(V(x, x') \leq \overline{\alpha}(\|x - x'\|)\), one obtains:

\[
\|\xi_{x,v}(t) - \xi_{x',v'}(t)\| \leq \alpha^{-1}\left(2e^{\kappa t}\overline{\alpha}(\|x - x'\|)\right) + \alpha^{-1}\left(2\frac{e^{\kappa t} - 1}{\kappa} \sigma(\|v - v'\|)\right).
\]

Therefore, by defining functions \(\beta\) and \(\gamma\) as

\[
\beta(\|x - x'\|, t) = \alpha^{-1}(2e^{\kappa t}\overline{\alpha}(\|x - x'\|))
\]

\[
\gamma(\|v - v'\|, t) = \alpha^{-1}\left(2\frac{e^{\kappa t} - 1}{\kappa} \sigma(\|v - v'\|)\right),
\]

the condition (2.2) is satisfied. Hence, the system \(\Sigma\) is \(\delta\)-FC.

Proof of Theorem 5.3. The proof of Theorem 5.3 requires the following preliminary result.

Lemma 8.1. The Riemannian manifold \(\mathbb{R}^n\) equipped with a Riemannian metric \(G\), satisfying \[\|x\|_2^2 \leq y^T G(x) y\] for any \(x, y \in \mathbb{R}^n\) and for some positive constant \(\omega\), is complete as a metric space, with respect to the Riemannian distance determined by \(G\).

Proof. The proof was suggested to us by C. Manolescu. First, for each pair of points \(x, y \in \mathbb{R}^n\) we define the path space:

\[\Omega(x, y) = \{\chi : [0, 1] \rightarrow \mathbb{R}^n \mid \chi\text{ is piecewise smooth, } \chi(0) = x, \text{ and } \chi(1) = y\}.
\]

Recall that a function \(\chi : [a, b] \rightarrow \mathbb{R}^n\) is piecewise smooth if \(\chi\) is continuous and if there exists a partition \(a = a_1 < a_2 < \cdots < a_k = b\) of \([a, b]\) such that \(\chi|_{[a_i, a_{i+1}]}\) is smooth for \(i = 1, \cdots, k - 1\). We can then define the Riemannian distance \(d_G(x, y)\) between points \(x, y \in \mathbb{R}^n\) as

\[
d_G(x, y) = \inf_{\chi \in \Omega(x, y)} \int_0^1 \sqrt{\left(\frac{d\chi(s)}{ds}\right)^T G(\chi(s)) \frac{d\chi(s)}{ds}} ds.
\]

\[6e^{\kappa t} \ast \sigma(\|v(t) - v'(t)\|) = \int_0^t e^{\kappa(t - \tau)} \sigma(\|v(\tau) - v'(\tau)\|) d\tau.
\]

\[7\]This condition is nothing more than uniform positive definitness of \(G\).
It follows immediately that $d_G$ is a metric on $\mathbb{R}^n$. The Riemannian manifold $\mathbb{R}^n$ is a complete metric space, equipped with the metric $d_G$, if every Cauchy sequence of points in $\mathbb{R}^n$ has a limit in $\mathbb{R}^n$. Assume $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}^n$, equipped with the metric $d_G$. By using the assumption on $G$, we have

$$d_G(x_n, x_m) = \inf_{\chi \in \Omega(x_n, x_m)} \int_0^1 \sqrt{\frac{d\chi(s)}{dt} G(\chi(s)) \frac{d\chi(s)}{dt}} ds \geq \sqrt{\omega} \inf_{\chi \in \Omega(x_n, x_m)} \int_0^1 \sqrt{\left(\frac{d\chi(s)}{dt}\right)^T G(\chi(s)) \frac{d\chi(s)}{dt}} ds = \sqrt{\omega} \|x_n - x_m\|_2,$$

It is readily seen from the inequality (8.3) that the sequence $\{x_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in $\mathbb{R}^n$ with respect to the Euclidean metric. Since the Riemannian manifold $\mathbb{R}^n$ with respect to the Euclidean metric is a complete metric space, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a point, named $x^*$, in $\mathbb{R}^n$. By picking a convex compact subset $D \subset \mathbb{R}^n$, containing $x^*$, and using Lemma 8.18 in [30], we have: $\overline{\omega} \|y\|^2 \geq y^T G(x)y$ for any $y \in \mathbb{R}^n$, $x \in D$, and some positive constant $\overline{\omega}$. Since the sequence $\{x_n\}_{n=1}^{\infty}$ converges to $x^* \in D$, there exists some integer $N$ such that the sequence $\{x_n\}_{n=N}^{\infty}$ remains forever inside $D$. Hence, we have:

$$\sqrt{\omega} \|x_n - x^*\|_2 \leq d_G(x_n, x^*) \leq \sqrt{\omega} \|x_n - x^*\|_2,$$

for $n > N$. Therefore, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to $x^* \in \mathbb{R}^n$, equipped with the metric $d_G$. Therefore, $\mathbb{R}^n$ with respect to the metric $d_G$ is a complete metric space.

By using the assumption on the metric and Lemma 8.1 we know that $\mathbb{R}^n$ is a complete metric space with respect to the Riemannian distance determined by $G$. Using the Hopf-Rinow theorem [39], we conclude that $\mathbb{R}^n$ with respect to the metric $G$ is geodesically complete. The rest of proof is inspired by the proof of Theorem 2 in [31]. Consider two points $x$ and $x'$ in $\mathbb{R}^n$ and a geodesic $\chi : [0, 1] \rightarrow \mathbb{R}^n$ joining $x = \chi(0)$ and $x' = \chi(1)$. The geodesic distance between the points $x$ and $x'$ is given by:

$$d_G(x, x') = \int_0^1 \sqrt{\left(\frac{d\chi(s)}{dt}\right)^T G(\chi(s)) \frac{d\chi(s)}{dt}} ds.$$

Consider the straight line $\tilde{\chi}(s) = (1 - s)v(t) + sv'(t)$, for fixed $t \in \mathbb{R}_+^n$, fixed $v, v' \in \mathcal{U}$, and for any $s \in [0, 1]$. The curve $\tilde{\chi}$ is a geodesic, with respect to the Euclidean metric, on the subset $\mathcal{U} \subseteq \mathbb{R}^n$ joining $v(t) = \tilde{\chi}(0)$ and $v'(t) = \tilde{\chi}(1)$. Consider also the input curve $\upsilon_s$ defined by $\upsilon_s = \tilde{\chi}(s)$. Let $l(t)$ be the length of the curve $\xi(\upsilon_s, t)$ parametrized by $s$ and with respect to the metric $G$, i.e.:

$$l(t) = \int_0^1 \sqrt{\partial T G(\xi(\upsilon_s(t))) \partial T \xi} ds, \text{ s.t. } \delta \xi = \frac{\partial}{\partial s} \xi(\upsilon_s(t)).$$

In the rest of the proof, we drop the argument of the metric $G$ for simplicity. By taking the derivative of (8.6) with respect to time, we obtain:

$$\frac{d}{dt} l(t) = \int_0^1 \frac{\frac{d}{ds} (\delta T G \delta T)}{2 \sqrt{T G \delta T}} ds = \int_0^1 \sqrt{\frac{\partial T G + \frac{\partial G}{\partial T} \partial T + \frac{\partial T G}{\partial T} \partial T}{2 \sqrt{T G \delta T}} G \delta T} ds, \text{ s.t. } \delta \upsilon = \frac{\partial}{\partial s} \upsilon_s(t).$$

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8A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space $X$, equipped with a metric $d$, is a Cauchy sequence if $\lim_{n,m \to \infty} d(x_n, x_m) = 0$. 
Since $\Sigma$ is an exponential expansion with $\lambda$ and $\alpha$ the constants introduced in Definition 5.4, the following inequality holds:

\[
\frac{d}{dt} l(t) \leq \frac{\lambda}{2} l(t) + \frac{\alpha}{2} \int_0^1 \sqrt{\delta u^T \delta v} ds
\]

\begin{align*}
= \frac{\lambda}{2} l(t) + \frac{\alpha}{2} \|v(t) - v'(t)\|_2.
\end{align*}

Using the inequality $\|v(t)\|_2 \leq \sqrt{m} \|v(t)\|$, in which $m$ denotes the dimension of the input space, one gets:

\[
l(t) \leq e^{\frac{\lambda}{2} t} l(0) + \frac{\sqrt{m} \alpha}{2} e^\frac{\lambda}{2} t * \|v(t) - v'(t)\|
\]

\[
\leq e^{\frac{\lambda}{2} t} l(0) + \frac{\sqrt{m} \alpha}{\lambda} \left( e^{\frac{\lambda}{2} t} - 1 \right) \|v - v'\|_\infty,
\]

where $*$ denotes the convolution integral. From (8.5) and (8.6), it can be seen that $l(0) = d_G(x, x')$. However, for $t \in \mathbb{R}^+$, $l(t)$ is not necessarily the Riemannian distance, determined by $G$, because $\xi_{xv}(t)$ is not necessarily a geodesic. Nevertheless, $d_G(\xi_{xv}(t), \xi_{x'v'}(t)) \leq l(t)$, and one has:

\[
d_G(\xi_{xv}(t), \xi_{x'v'}(t)) \leq e^{\frac{\lambda}{2} t} d_G(x, x') + \frac{\sqrt{m} \alpha}{\lambda} \left( e^{\frac{\lambda}{2} t} - 1 \right) \|v - v'\|_\infty.
\]

Using the assumptions on the metric, it can be readily checked that:

\[
\sqrt{n} \|\xi_{xv}(t) - \xi_{x'v'}(t)\| \leq d_G(\xi_{xv}(t), \xi_{x'v'}(t)),
\]

\[
d_G(x, x') \leq \sqrt{\frac{\alpha}{\omega}} \|x - x'\|,
\]

where $n$ denotes the dimension of the state space. Hence, the condition (8.8) reduces to:

\[
\|\xi_{xv}(t) - \xi_{x'v'}(t)\| \leq \sqrt{\frac{\alpha}{\omega}} e^{\frac{\lambda}{2} t} \|x - x'\| + \sqrt{\frac{m \alpha}{\omega}} \left( e^{\frac{\lambda}{2} t} - 1 \right) \|v - v'\|_\infty,
\]

which is the $\delta$-FC condition in (2.2). \qed