Several polynomials associated with the harmonic numbers

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Abstract

We develop polynomials in $z \in \mathbb{C}$ for which some generalized harmonic numbers are special cases at $z = 0$. By using the Riordan array method, we explore interesting relationships between these polynomials, the generalized Stirling polynomials, the Bernoulli polynomials, the Cauchy polynomials and the Nörlund polynomials.

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1. Introduction

Some well-known classical numbers have been generalized to the polynomials in $z$ for which the numbers are the special case at $z = 0$. For example, the Bernoulli numbers $B_n$ are generalized to the Bernoulli polynomials of degree $n$, $B_n(z)$, defined by

$$\frac{t e^{zt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(z) \frac{t^n}{n!}, \quad z \in \mathbb{C},$$

where $B_n(0) = B_n$. As a result, numerous interesting applications and relationships to the other polynomials can be found in many literatures (e.g. see [5,7,8,11,15]).

As well as the Bernoulli numbers, the harmonic numbers $H_n = \sum_{k=1}^{n} 1/k$ frequently arise in combinatorial problems or in representations for well-known special functions, e.g. gamma functions and the Riemann and Hurwitz zeta functions (see [1,3,4,10]). These numbers have been generalized by several authors. One of them is the generalized harmonic numbers $H_n^{(r)}$ of rank $r$ (see [6,12]) defined by

$$H_n^{(r)} = \sum_{1 \leq n_0 + n_1 + \cdots + n_r \leq n} \frac{1}{n_0! n_1! \cdots n_r!}.$$

When $r = 0$, they reduce to the ordinary harmonic numbers.

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In the present paper, we are mainly interested in the polynomials in \( z \in \mathbb{C} \) for which the numbers \( H_n^{(r)} \) are the special case at \( z = 0 \). The concept of Riordan arrays introduced by Shapiro et al. [13] is extensively used throughout this paper.

The paper is organized in the following way. In Section 2, we briefly introduce the concept of a Riordan array method. In Section 3, we develop the harmonic polynomials \( H_n^{(r)}(z) \) of degree \( n - r \) in \( z \in \mathbb{C} \). In Section 4, we explore some relationships between the harmonic polynomials, the generalized Stirling polynomials and the Bernoulli polynomials. As a result, we establish that the multiple gamma function may be expressed by means of the integration of the harmonic polynomials together with the derivatives of the Hurwitz zeta function. In Section 5, we show that the harmonic polynomials are closely related to the Cauchy numbers. As a result, we develop the Cauchy polynomials in \( z \) for which the Cauchy numbers are the special case at \( z = 0 \), and then we obtain a relationship between the harmonic polynomials and the Cauchy polynomials. In Section 6, we observe that the Cauchy polynomials of both types may be related to the Nörlund polynomials.

2. Riordan array method

Throughout this paper, we let \( N_0 = \{0, 1, 2, \ldots\} \). A Riordan array [13] is an infinite, lower triangular array \( \{d_{n,k}\}_{n,k \in N_0} \) defined by a pair of generating functions \( g(t) \) and \( f(t) \) such that

\[
d_{n,k} = [t^n] g(t)(tf(t))^k,
\]

where \( g(0) \neq 0, f(0) \neq 0 \) and \( [t^n] \) is the coefficient operator. As usual, we denote it by \( \mathcal{R}(d_{n,k}) = (g(t), f(t)) \). The Riordan group is the set of all Riordan arrays with the operation being matrix multiplication. In terms of the generating functions this works out as

\[
\mathcal{R}(d_{n,k}) = \mathcal{R}(g(t), f(t))(h(t), \ell(t)) = \mathcal{R}(g(t)h(tf(t)), f(t)\ell(tf(t))).
\] (3)

It is easy to show that \( I = (1, 1) \) is the identity element of the Riordan group.

A common example of a Riordan array is the infinite Pascal matrix \( P = \mathcal{R} \left( \binom{n}{k} \right) = \left( \frac{1}{1-x}, \frac{1}{1-x} \right) \) for which we have

\[
\binom{n}{k} = [x^n] \frac{1}{1-x} \left( \frac{x}{1-x} \right)^k.
\]

The sums involving the rows of a Riordan array together with coefficients of some generating function can be performed by operating a suitable transformation on a generating function. In fact, the summation property (SP) [9] for a Riordan array \( \mathcal{R}(d_{n,k}) \) is

\[
\sum_{k=0}^{n} d_{n,k} h_k = [t^n] g(t) h(tf(t)),
\] (4)

where \( h(t) = \sum_{k \geq 0} h_k t^k \). The SP can help us to find a closed form for the generating function of many combinatorial sums. This Riordan array method has been extensively used in [14].

Sometimes, it is very useful to use exponential generating functions instead of ordinary generating functions when we apply a Riordan array method. We call the resulting array an exponential Riordan array or simply \( E \)-Riordan array, and we denote it by \( \mathcal{R}(d_{n,k}^E) = (g(t), f(t))_E \), where \( d_{n,k}^E = [t^n/n!] g(t)(tf(t))^k/k! \). It is easy to show that

\[
(g(t), f(t))_E = D^{-1}(g(t), f(t)) D,
\] (5)

where \( D = \text{diag}(1, \frac{1}{1!}, \frac{1}{2!}, \ldots) \) is the infinite diagonal matrix. In this case, the SP associated with an \( E \)-Riordan array \( (g(t), f(t))_E \) may be applied as follows:

\[
\sum_{k=0}^{n} d_{n,k}^E h_k = [t^n/n!] g(t) h_E(tf(t)),
\]

where \( h_E(t) = \sum_{k \geq 0} h_k t^k/k! \).
3. Harmonic polynomials of degree \( n - r \)

In this section, we develop the polynomials generalizing the harmonic numbers \( H_n \) with the generating function \( \frac{1}{1 - t} \ln \frac{1}{1 - t} \).

For \( n, r \in \mathbb{N}_0 \), let us define the polynomials \( H^{(r)}_n(z) \) in \( z \in \mathbb{C} \) of degree \( n - r \) by

\[
\frac{[- \ln((1 - t))^{1+r}]}{t(1 - t)^{1-z}} = \sum_{n=0}^{\infty} H^{(r)}_n(z) t^n.
\]  

It is easy to show that the polynomials \( H^{(r)}_n(z) \) may be expressed in terms of the generalized harmonic numbers \( H^{(r)}_n \) of rank \( r \) given by (2) as

\[
H^{(r)}_n(z) = \sum_{k=0}^{n-r} (-1)^k H^{(r+k)}_{n+1} \frac{z^k}{k!}.
\]

We call \( H^{(r)}_n(z) \) the harmonic polynomials of degree \( n - r \). For \( r = 0 \), the harmonic polynomials have been studied in [2].

First, we observe that the generalized harmonic numbers \( H^{(r)}_{n+1} \) \((n, r \in \mathbb{N}_0)\) are special cases of the polynomials \( H^{(r)}_n(z) \) at \( z = 0 \), i.e. \( H^{(r)}_{n+1} = H^{(r)}_n(0) \).

Throughout this paper, \([z]^n\) denotes the Pochhammer symbol defined in terms of the gamma function \( \Gamma(z) \) by

\[
[z]^n = \prod_{k=1}^{n} (z + k - 1) = \frac{\Gamma(z+n)}{\Gamma(z)} = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] z^k \quad (z \in \mathbb{C}),
\]

where \([z]^0 = 1\) and \( \left[ \begin{array}{c} n \\ k \end{array} \right] \) denotes the Stirling cycle numbers.

Then we obtain basic properties for the harmonic polynomials analogous to those of the Bernoulli polynomials. The proofs are all easy exercises:

(i) \( H^{(r)}_n(z + 1) = H^{(r)}_n(z) - H^{(r)}_{n-1}(z) \) for all \( n \geq 1, r \geq 0 \);

(ii) \( \frac{d}{dz} H^{(r)}_n(z) = -H^{(r+1)}_{n-1}(z) \) for all \( n \geq 0, r \geq 0 \);

(iii) \( \int_0^1 H^{(r)}_n(z) \, dz = H^{(r-1)}_{n-1} \) for all \( n \geq 1, r \geq 1 \);

(iv) \( (n!/r!) \int_0^1 H^{(r)}_n(z) \, dz = \left[ \begin{array}{c} n+1 \\ r+1 \end{array} \right] \) for all \( n \geq 0, r \geq 0 \).

By setting \( r = 1 \), it follows from (iii) that the ordinary harmonic numbers \( H_n \) may be obtained by

\[
H_n = \int_0^1 H^{(1)}_n(z) \, dz.
\]

The concept of Riordan arrays is extensively used throughout this paper. In order to provide an appropriate setting for our purpose, let us define the harmonic Riordan array \( \mathcal{H}(z) \) by

\[
\mathcal{H}(z) = \left( \begin{array}{cccc} -\ln(1-t) & -\ln(1-t) \\ \frac{1}{t(1-t)^{1-z}} & \frac{1}{t} \end{array} \right).
\]

Then the array \( \mathcal{H}(z) \) is of the form

\[
\mathcal{H}(z) = \left[ \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{3}{2} - z & 2 - z & 1 & 0 \\ \frac{11}{6} - 2z + \frac{1}{2}z^2 & \frac{35}{12} - \frac{5}{2}z + \frac{1}{2}z^2 & \frac{5}{2} - z & 1 \end{array} \right]
\]
We see that the generic elements of \( \mathcal{H}(z) = [h_{n,r}(z)]_{n,r \in \mathbb{N}_0} \) are harmonic polynomials \( H_n^{(r)}(z) \) of degree \( n - r \) by observing that
\[
h_{n,r}(z) = \left[ t^n \right] \left( -\frac{\ln(1-t)}{t(1-t)^{1-z}} \right) (-\ln(1-t))^r = [t^n] \left( \frac{-\ln(1-t)}{t(1-t)^{1-z}} \right)^{1+r} = H_n^{(r)}(z).
\]

By setting \( z = 0 \) and 1, the harmonic Riordan array \( \mathcal{H}(z) \) is simplified to the generalized harmonic number array and the associated Stirling cycle number array, respectively, i.e.
\[
\mathcal{H}(0) = \mathcal{R}(H_{n+1}^{(r)}) \quad \text{and} \quad \mathcal{H}(1) = \mathcal{R} \left( \frac{(k+1)!}{(n+1)!} \left[ \frac{n+1}{k+1} \right] \right).
\]

**Theorem 3.1.** The harmonic polynomials \( H_{n-1}^{(r)}(z) \) may be expressed by means of the Stirling cycle numbers:
\[
H_{n-1}^{(r)}(z) = \frac{r!}{n!} \sum_{k=r}^{n} \binom{n}{k} \binom{k}{r} (1-z)^{k-r} \quad (n, r \geq 1).
\]

**Proof.** Let us observe that
\[
\mathcal{R} \left( \frac{(k+1)!}{(n+1)!} \left[ \frac{n+1}{k+1} \right] \right) = \left( \frac{-\ln(1-t)}{t}, \frac{-\ln(1-t)}{t} \right).
\]

By applying the SP for the Riordan array (10) together with \( h(t) = t^r e^{(1-z)t} = \sum_{k=r}^{\infty} ((1-z)^k - (k-r)!t^k, \) we obtain
\[
\sum_{k=r}^{n} \left( \frac{(k+1)!}{(n+1)!} \left[ \frac{n+1}{k+1} \right] \right) (1-z)^{k-r} \frac{(k-r)!}{(k-r)!} = [t^n] \left( \frac{-\ln(1-t)}{t} \right) e^{(1-z)t} \bigg|_{t = -\ln(1-t)}.
\]

Simplifying above equation gives us that
\[
\frac{(r+1)!}{(n+1)!} \sum_{k=r}^{n} \binom{n+1}{k+1} \binom{k+1}{r+1} (1-z)^{k-r} = [t^n] \left( \frac{-\ln(1-t)}{t(1-t)^{1-z}} \right)^{1+r}
\]
\[
= H_n^{(r)}(z).
\]

This leads to (9) and the proof is completed. \( \square \)

Since \( H_{n+1}^{(r)} = H_{n+1}^{(r)}(0) \), by setting \( z = 0 \) the polynomials \( H_{n-1}^{(r)}(z) \) given by (9) are simplified to the generalized harmonic numbers \( H_n^{(r-1)} \):
\[
H_n^{(r-1)} = \frac{r!}{n!} \sum_{k=r}^{n} \binom{n}{k} \binom{k}{r} (1-z)^{k-r} \quad (n, r \geq 1).
\]

**4. Connection with the multiple gamma function**

The multiple gamma function \( \Gamma_n(z) \) is defined as a generalization of the Euler gamma function \( \Gamma(z) \) by the recurrence-functional equation (see [1,3]):
\[
\Gamma_1(z) = \Gamma(z), \quad \Gamma_{n+1}(z + 1) = \frac{\Gamma_{n+1}(z)}{\Gamma_n(z)}, \quad z \in \mathbb{C},
\]
where \( \Gamma_1(1) = 1 \). It is known that the multiple gamma function is closely related to the Hurwitz zeta function \( \zeta(s, z) \) defined by the series
\[
\zeta(s, z) = \sum_{k=0}^{\infty} \frac{1}{(k+z)^s}, \quad \Re(s) > 0,
\]
as a generalization of the Riemann zeta function \( \zeta(s) = \zeta(s, 1) \). In fact, Adamchik [1] proved that the multiple gamma function \( \Gamma_{n}(z) \) may be expressed by means of the derivatives of the Hurwitz zeta function:

\[
\ln \Gamma_{n}(z) = \frac{(-1)^{n}}{(n-1)!} \sum_{k=0}^{n-1} P_{n,k}(z) (\zeta'(-k) - \zeta'(-k, z)),
\]

where \( P_{n,k}(z) \) are the generalized Stirling polynomials of the first kind defined by

\[
P_{n,k}(z) = \sum_{j=k+1}^{n} (-z)^{j-k-1} \binom{n}{j} \right\}
\]

with the alternative representation:

\[
P_{n,k}(z) = \frac{(-1)^{k}}{k!} \left( \frac{\epsilon^{n-1} \ln(1-t)^{k}}{\epsilon^{n-1} (1-t)^{1-z}} \right)_{t=0}.
\]

In this section, we explore some connections between the harmonic polynomials \( H_{n}^{(r)}(z) \), the generalized Stirling polynomials \( P_{n,r}(z) \) and the Bernoulli polynomials \( B_{n}(z) \).

First, it follows from (6) and (12) that the harmonic polynomials \( H_{n}^{(r)}(z) \) are closely related to the Stirling polynomials \( P_{n,r}(z) \) by

\[
H_{n}^{(r)}(z) = \frac{(r+1)!}{(n+1)!} P_{n+2,r+1}(z).
\]

By setting \( z = 0 \) and \( 1 \), it follows from (9) and (13) that the polynomials \( P_{n+1,r}(z) \) are simplified to the Stirling cycle numbers, respectively (also see [1]):

\[
P_{n+1,r}(0) = \binom{n+1}{r+1} \quad \text{and} \quad P_{n+1,r}(1) = \binom{n}{r}.
\]

Now, let us define the infinite lower triangular matrices \( P(z) \) and \( B(z) \) by

\[
P(z) = [P_{n,k}(z)]_{n,k \in \mathbb{N}_{0}} \quad \text{and} \quad B(z) = \left[ \binom{n}{k} B_{n-k}(z) \right]_{n,k \in \mathbb{N}_{0}}.
\]

It is easy to show that \( P(z) \) and \( B(z) \) may be expressed by the exponential Riordan arrays, respectively:

\[
P(z) = \left( \frac{1}{(1-t)^{1-z}}, -\frac{\ln(1-t)}{t} \right)_{E} \quad \text{and} \quad B(z) = \left( \frac{te^{zt}}{e^{t} - 1}, 1 \right)_{E}.
\]

We note that \( B(z) \) is the Bernoulli matrix defined in [15].

Very interestingly, by performing the matrix multiplication (3) of exponential Riordan arrays \( P(z) \) and \( B(x) \) given by (16) we can obtain

\[
P(z)B(x) = \left( \frac{-\ln(1-t)}{t(1-t)^{1-z}}, -\frac{\ln(1-t)}{t} \right)_{E}
\]

for any independent complex variables \( z \) and \( x \). Hence we have the following theorem from (5) and (7).

**Theorem 4.1.** The harmonic Riordan array \( \mathcal{H}(z - x + 1) \) may be factorized by means of the generalized Stirling matrix \( P(z) \) and the Bernoulli matrix \( B(x) \) for any independent complex variables \( z \) and \( x \):

\[
D^{-1} \mathcal{H}(z - x + 1) D = P(z)B(x),
\]

where \( D = \text{diag} \left( \frac{1}{n!}, \frac{1}{(n+1)!}, \frac{1}{(n+2)!}, \ldots \right) \).
By setting \(z = x\) it follows from (8) and (17) that the Stirling cycle numbers may be expressed by means of the generalized Stirling polynomials \(P_{n,k}(z)\) together with the Bernoulli polynomials \(B_n(z)\) for any complex number \(z\):

\[
\binom{n+1}{k+1} = \sum_{r=k}^{n} \binom{r}{k} P_{n+1,r}(z) B_{r-k}(z) \quad (n, k \geq 0).
\]

Further, by observing that \(B(x)D^{-1} = \left(d^k/dx^k \right) B_n(x)\) for \(n, k \in \mathbb{N}_0\), it follows from (17) that the Stirling polynomials \(H_r^{(r)}(z)\) may be expressed by means of the generalized Stirling polynomials together with the derivatives of the Bernoulli polynomials:

\[
H_{n}^{(r)}(z - x + 1) = \frac{1}{n!} \sum_{k=r}^{n} P_{n+1,k}(z) \frac{d^r}{dx^r} B_k(x).
\]

Finally, by integrating we observe that \(\int_{0}^{1} B(x) \, dx\) is the identity matrix \(I = (1, 1)\). Hence, it follows from (17) that the Stirling polynomials \(P_{n+1,k}(z)\) may be expressed by means of the integration of the harmonic polynomials:

\[
P_{n+1,k}(z) = \frac{n!}{k!} \int_{0}^{1} H_{n}^{(k)}(z - x + 1) \, dx. \quad (18)
\]

The following theorem is an immediate consequence of (11) and (18).

**Theorem 4.2.** The multiple gamma function \(\Gamma_n(z)\) may be expressed by means of the integration of the harmonic polynomials together with the derivatives of the Hurwitz zeta function:

\[
\ln \Gamma_n(z) = \sum_{k=0}^{n-1} \frac{(-1)^{n}}{k!} \left( \int_{0}^{1} H_{n-1}^{(k)}(z - x + 1) \, dx \right) \left( \zeta''(k) - \zeta''(k, z) \right).
\]

**5. Connection with the Cauchy numbers**

The Cauchy numbers of the first type and the second type [5] are defined by

\[
c_n^{(1)} := \int_{0}^{1} x^n \, dx \quad \text{and} \quad c_n^{(2)} := \int_{0}^{1} \lfloor x \rfloor_n \, dx,
\]

respectively, where \((x)_n = x(x - 1) \cdots (x - n + 1) = (-1)^n [-x]_n\).

Recently, the Cauchy numbers of both types have been extensively studied by Merlini et al. [9] by using the Riordan array method.

In this section, we show that the harmonic polynomials are closely related to the Cauchy numbers.

**Theorem 5.1.** The generalized binomial coefficients [5] with repetition, \(\begin{pmatrix} x \\ n \end{pmatrix} = \lfloor x \rfloor_n / n!\) may be expressed by means of the harmonic polynomials:

\[
\begin{pmatrix} x - z + 1 \\ n \end{pmatrix} = \sum_{k=0}^{n} \frac{1}{(k+1)!} H_{n}^{(k)}(z - x + 1). \quad (19)
\]

**Proof.** By applying the SP for the Riordan array

\[
\mathcal{R}(H_{n}^{(k)}(z - x + 1)) = \left( \frac{-\ln(1-t)}{t(1-t)^{x-z}}, \frac{-\ln(1-t)}{t} \right)
\]
together with \( h(t) := (e^t - 1)/t = \sum_{k \geq 0} 1/(k + 1)! \), we obtain
\[
\sum_{k=0}^n \frac{1}{(k+1)!} H_n^{(k)}(z-x+1) = \left[ t^n \right] \frac{-\ln(1-t)}{t(1-t)^{x-z}} \left( \frac{e^t - 1}{t} \right)_{t=-\ln(1-t)} = \left[ t^n \right] \frac{1}{(1-t)^{1-z+x}} = \frac{[x - z + 1]_n}{n!}
\]
which completes the proof. \( \square \)

Formula (19) suggests that the harmonic polynomials may be used to get a generalization of the Cauchy numbers in connection with the Stirling polynomials. In fact, it follows from (18) and (19) that
\[
\int_0^1 [x-z+1]_n \, dx = \sum_{k=0}^n \frac{1}{k+1} P_{n+1,k}(z). \tag{20}
\]
By setting \( z = 1 \), from (14) we obtain the explicit formulas for the Cauchy numbers of both types (also see [5,9]):
\[
c_n^{(1)} = \int_0^1 (x)_n \, dx = \sum_{k=0}^n (-1)^{n-k} \left[ \begin{array}{c} n \\ k \end{array} \right]
\]
and
\[
c_n^{(2)} = \int_0^1 [x]_n \, dx = \sum_{k=0}^n \frac{1}{k+1} \left[ \begin{array}{c} n \\ k \end{array} \right].
\]

Now, we observe that the right-hand side of (20) may be expressed by a polynomial in \( z + 1 \) of degree \( n \). More generally, let us consider the \textit{Cauchy polynomials of degree} \( n \) \textit{of the second type} \( \Phi_n^{(2)}(z) \) defined by
\[
\Phi_n^{(2)}(z) = \int_0^1 [x-z]_n \, dx.
\]
The first few polynomials are
\[
\Phi_0^{(2)}(z) = 1,
\]
\[
\Phi_1^{(2)}(z) = \frac{1}{2} - z,
\]
\[
\Phi_2^{(2)}(z) = \frac{5}{6} - 2z + z^2,
\]
\[
\Phi_3^{(2)}(z) = \frac{9}{4} - 6z + \frac{9}{2}z^2 - z^3.
\]

\textbf{Theorem 5.2.} The \textit{Cauchy polynomials of the second kind} \( \Phi_n^{(2)}(z) \) have the following exponential generating function:
\[
\frac{-t}{(1-t)^{1-z} \ln(1-t)} = \sum_{n=0}^{\infty} \Phi_n^{(2)}(z) \frac{t^n}{n!}. \tag{21}
\]
\textbf{Proof.} By applying the SP for the \( E \)-Riordan array \( P(z+1) = 1/(1-t)^{-z} \), \(-\ln(1-t)/t \) \( E \) given by (16) together with \( h_E(t) := \frac{e^t - 1}{t} = \sum_{k \geq 0} (1/(k+1)) t^k / k! \), we obtain
\[
\Phi_n^{(2)}(z) = \sum_{k=0}^n \frac{1}{k+1} P_{n+1,k}(z+1) = \left[ t^n \right] \frac{-t}{n! (1-t)^{1-z} \ln(1-t)} \left( \frac{e^t - 1}{t} \right)_{t=-\ln(1-t)}
\]
which completes the proof. \( \square \)
By a similar argument, if we note that
\[ \int_0^1 (x - z)_n \, dx = \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k+1} P_{n+1,k}(1 - z), \]
we may define the Cauchy polynomials of degree \( n \) of the first type \( \Phi^{(1)}_n(z) \) by
\[ \Phi^{(1)}_n(z) = \int_0^1 (x - z)_n \, dx. \]
The first few polynomials are
\[ \Phi^{(1)}_0(z) = 1, \quad \Phi^{(1)}_1(z) = \frac{1}{2} - z, \quad \Phi^{(1)}_2(z) = -\frac{1}{6} + z^2, \quad \Phi^{(1)}_3(z) = \frac{1}{4} - \frac{3}{2} z^2 - z^3. \]

Since
\[ \left[ (-1)^{n-k} P_{n+1,k}(1 - z) \right]_{n,k \in \mathbb{N}_0} = \left( \frac{1}{(1 + t)^2} \cdot \frac{\ln(1 + t)}{t} \right)_E, \] by applying the SP for the \( E \)-Riordan array (22) together with \( h_E(t) := (e^t - 1)/t \), we can establish the following exponential generating function for the Cauchy polynomials of the first type \( \Phi^{(1)}_n(z) \):
\[ \frac{t}{(1 + t)^2 \ln(1 + t)} = \sum_{n=0}^{\infty} \Phi^{(1)}_n(z) \frac{t^n}{n!}. \] (23)

Now, we observe an interesting connection between the Cauchy polynomials of both types. It follows from (21) and (23) that
\[ \Phi^{(1)}_n(z) = (-1)^n \Phi^{(2)}_n(1 - z). \] (24)
By setting \( z = 0 \) and \( 1 \), we obtain \( c^{(1)}_n = (-1)^n \Phi^{(2)}_n(1) \) and \( c^{(2)}_n = (-1)^n \Phi^{(1)}_n(1) \), respectively. Note that \( \Phi^{(1)}_n(1) \) has been studied in [9].

**Theorem 5.3.** The Cauchy polynomials of both types, \( \Phi^{(1)}_n(z) \) and \( \Phi^{(2)}_n(z) \), are connected in the following way:
\[ \Phi^{(2)}_n(z) = \sum_{k=0}^{n} L_{n,k} \Phi^{(1)}_k(z), \] (25)
where \( L_{n,k} \) are the signless Lah numbers given by \( L_{n,k} = (n!/k!) \binom{n-1}{k-1} \) for \( n, k \geq 1 \), \( L_{0,0} = 1 \) and \( L_{n,0} = 0 \) for \( n > 0 \).

**Proof.** First note that the Lah numbers array \( \{L_{n,k}\}_{n,k \in \mathbb{N}_0} \) may be expressed by the \( E \)-Riordan array \( L \):
\[ L = [L_{n,k}]_{n,k \in \mathbb{N}_0} = \left( 1, \frac{1}{1-t} \right)_E, \] (26)
where \( L_{0,0} = 1 \) and \( L_{n,0} = 0 \) for \( n > 0 \).
By applying the SP for the E-Riordan array $L$ together with $h_E(t) = t/(1+t)^2 \ln(1+t)$ for the Cauchy polynomials of the first type given by (23), we obtain

$$
\sum_{k=0}^{n} L_{n,k} \phi_k^{(1)}(z) = \left[ \frac{t^n}{n!} \left( \frac{t}{(1+t)^2 \ln(1+t)} \right)_{t=1-z} \right] = \left[ \frac{t^n}{n!} \frac{-t}{(1-t)^{1-z} \ln(1-t)} = \phi_n^{(2)}(z) \right]
$$

which completes the proof. □

By setting $z = 0$, it immediately follows from (25) that

$$
c_n^{(2)} = \sum_{k=0}^{n} \frac{n!}{k!} \binom{n-1}{k-1} c_k^{(1)}. 
$$

In order to obtain a relationship between the harmonic polynomials and the Cauchy polynomials of both types, we observe that their exponential generating functions are closely related to the reciprocal of the ordinary generating function:

$$
h(t, z) := - \ln(1-t) \frac{t}{(1-t)^{1-z} \ln(1-t)} = \sum_{n \geq 0} H_n(z) t^n \quad \text{(27)}
$$

for the harmonic polynomials $H_n^{(0)}(z)$ of degree $n$ where $H_n(z) = H_n^{(0)}(z)$. In fact, it follows from (21) and (23) that the exponential generating functions for $\psi_n^{(1)}(z)$ and $\psi_n^{(2)}(z)$ are the reciprocal inverses of $h(-t, 1+z)$ and $h(t, 2-z)$, respectively. Hence, by the aid of Wronski’s formula (see [5], p. 157) for determinant, we obtain interesting expressions with determinants of order $n$ for the Cauchy polynomials of degree $n$ in terms of the harmonic polynomials $H_k(z)$:

$$
\psi_n^{(1)}(z) = \int_0^1 (x - z)_n \, dx = n! \det C_n(1+z)
$$

and

$$
\psi_n^{(2)}(z) = \int_0^1 [x - z]_n \, dx = (-1)^n n! \det C_n(2-z),
$$

where

$$
C_n(z) = \begin{bmatrix}
H_1(z) & 1 & & O \\
H_2(z) & H_1(z) & & \\
& \ddots & \ddots & \\
& & H_n(z) & \cdots & H_2(z) & H_1(z)
\end{bmatrix}
$$

is the $n \times n$ matrix of the Hessenberg form.

By setting $z = 0$ we immediately obtain the determinantal expressions for the Cauchy numbers of both types:

$$
c_n^{(1)}(z) = \int_0^1 (x)_n \, dx = n! \det \begin{bmatrix}
\frac{1}{2} & 1 & & O \\
\frac{1}{3} & \frac{1}{2} & & \\
& \ddots & \ddots & \\
& & \frac{1}{n+1} & \cdots & 1
\end{bmatrix}
$$

and

$$
c_n^{(2)}(z) = \int_0^1 [x]_n \, dx = (-1)^n n! \det \begin{bmatrix}
\frac{1}{2} & 1 & & O \\
\frac{1}{3} & \frac{1}{2} & & \\
& \ddots & \ddots & \\
& & \frac{1}{n+1} & \cdots & 1
\end{bmatrix}
$$
6. Connection with the Nörlund polynomials

The Cauchy polynomials of both types may be related to the Nörlund polynomials $B_n^{(z)}$ and $b_n^{(z)}$ defined by
\[
\left( \frac{t}{e^t - 1} \right)^z = \sum_{n=0}^{\infty} B_n^{(z)} \frac{t^n}{n!}
\]
and
\[
\left( \frac{t}{\ln(1 + t)} \right)^z = \sum_{n=0}^{\infty} b_n^{(z)} t^n,
\]
respectively. We note that $B_n^{(0)}$ and $b_n := b_n^{(1)}$ are called the Nörlund numbers and the Bernoulli numbers of the second kind, respectively (see [8]). The first few values for the Nörlund numbers are
\[
B_0^{(0)} = 1, \quad B_1^{(1)} = -\frac{1}{2}, \quad B_2^{(2)} = \frac{5}{6}, \quad B_3^{(3)} = -\frac{9}{4}, \ldots
\]

Recently, Liu and Srivastava [8] obtained some explicit formulas and representations for both Nörlund polynomials. Also they observed the exponential generating function for the Nörlund numbers with no mention for a relationship to the Cauchy polynomials:
\[
\sum_{n=0}^{\infty} B_n^{(n)} \frac{t^n}{n!} = \frac{t}{(1 + t) \ln(1 + t)}.
\]

This suggests that Nörlund numbers and the Bernoulli numbers of the second kind may be closely related to the Cauchy polynomials of both types. In fact, it immediately follows from (21), (23) and (29) that
\[
B_n^{(n)} = (-1)^n c_n^{(2)} = \Phi_n^{(1)}(1)
\]
for each $n = 0, 1, 2, \ldots$ where $c_n^{(2)} = \Phi_n^{(2)}(0)$ is the $n$th Cauchy number of the second type.

**Theorem 6.1.** Let $b_n$ be the Bernoulli numbers of the second kind. Then
\[
\Phi_n^{(1)}(z) = n! \sum_{k=0}^{n} (-1)^k b_{n-k} \left[ z \atop k \right]
\]
where $\left[ z \atop k \right]$ are the generalized binomial coefficients given by (19).

**Proof.** Since $(t / \ln(1 + t), 1)_E = [(n! / k!)b_{n-k}]_{n,k \in \mathbb{N}_0}$ is the $E$-Riordan array associated with Bernoulli numbers of the second kind, it follows from the SP for $t / \ln(1 + t), 1)_E$ together with $h_E(t) = 1/(1 + t)^z = \sum_{n \geq 0} (-1)^n [z]_n / n!$ that
\[
\Phi_n^{(1)}(z) = \left[ \frac{t^n}{n!} \right] \left( \frac{t}{\ln(1 + t)} \right) \left( \frac{1}{(1 + t)^z} \right) = n! \sum_{k=0}^{n} (-1)^k b_{n-k} [z]_k / k!
\]
which completes the proof. □
Theorem 6.2. Let $b_n$ be the Bernoulli numbers of the second kind. Then

$$n! \sum_{k=0}^{n} (-1)^k b_{n-k} H_k(z) = (z-1)_n,$$

where $H_n(z)$ are the Harmonic polynomials of degree $n$ given by (27).

Proof. Applying the SP for the Riordan array $(t/\ln(1+t), -1) = [(-1)^k b_{n-k}]_{n,k \in \mathbb{N}_0}$ together with the generating function $h(t, z)$ given by (27), we obtain

$$\sum_{k=0}^{n} (-1)^k b_{n-k} H_k(z) = [t^n] \left( \frac{t}{\ln(1+t)} \right) \left. \left( \frac{-\ln(1-t)}{t(1-t)^{1-z}} \right) \right|_{t=-t} \bigg|_{t=-t}$$

$$= [t^n] \frac{1}{(1+t)^{1-z}} = \frac{(z-1)_n}{n!}$$

which completes the proof. \(\square\)

Finally, we obtain a relationship between the Bernoulli polynomials and the Cauchy polynomials of the second type. For the purpose we begin with the $E$-Riordan array given by

$$\left( \frac{t}{e^t-1}, e^t \right)_E = \left[ \sigma_{n,k} \binom{n}{k} \left( \frac{n-k+2}{2} \right)^{-1} \right]_{n,k \in \mathbb{N}_0} \quad (30)$$

where $\sigma_{n,k}$ is the (0,1)-valued function defined as $\sigma_{n,k} = 1$ if $n-k$ is even and 0 otherwise. We note that the matrix given by (30) may be considered as the $E$-Riordan array representing the generalized Bernoulli polynomials $B_n^{(2)}(z)$ with the exponential generating function $(t/(e^t-1))^{2}e^{zt}$ when $x = -2$ and $z = -1$.

Theorem 6.3. Let $B_n(z)$ be the Bernoulli polynomials given by (1). Then

$$\sum_{k=0}^{n} \sigma_{n,k} \binom{n}{k} \left( \frac{n-k+2}{2} \right)^{-1} B_k(z) = \sum_{k=0}^{n} (-1)^k S(n, k) \Phi_k^{(2)}(z), \quad (31)$$

where $S(n, k)$ is the Stirling number of the second kind.

Proof. Let us consider the $E$-Riordan array associated Stirling numbers of the second kind:

$$\left( 1, \frac{e^t - 1}{t} \right)_E = [S(n, k)]_{n,k \in \mathbb{N}_0} \quad (32)$$

Applying the SP twice for the $E$-Riordan arrays given by (30) and (32) yields

$$\sum_{k=0}^{n} \sigma_{n,k} \binom{n}{k} \left( \frac{n-k+2}{2} \right)^{-1} B_k(z) = \left[ \frac{t^n}{n!} \right] \left( \frac{t}{e^t-1} \right)^{-2} e^{-t} \left( \frac{t}{e^t-1} \right) e^{zt}$$

$$= \left[ \frac{t^n}{n!} \right] \left( \frac{e^t - 1}{t} \right) e^{(z-1)t}$$

$$= \left[ \frac{t^n}{n!} \right] \left( \frac{-t}{(1-t)^{1-z} \ln(1-t)} \right)_{t:=1-e^t}$$

$$= \sum_{k=0}^{n} (-1)^k S(n, k) \Phi_k^{(2)}(z)$$

which completes the proof. \(\square\)
7. Concluding remark

Throughout the paper, we saw that the Riordan array method using the SP is very powerful to get some explicit formulas or relationships between several polynomials and numbers. Of course, numerous results involving the polynomials and numbers considered in this paper can also be derived by using the Riordan array method. We believe that this method will be very useful in future works arising in enumerative combinatorics.

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References