On the Long-Run Distribution of Capital in the Ramsey Model

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We study issues related to the long-run distribution of wealth in two variants of Ramsey's model of optimal capital accumulation. First we show that, in the case where a government levies a progressive income tax, there exist infinitely many stationary equilibria in which all households own positive capital stocks. Moreover, it is demonstrated that non-stationary equilibria can exhibit complicated dynamics. Then we discuss the case where households exercise market power on the capital market and we show that this may also lead to equilibria in which all households own positive amounts of capital. Journal of Economic Literature Classification Numbers: E62, O41. © 2001 Elsevier Science (USA)

1. INTRODUCTION

The model of optimal capital accumulation developed by Ramsey [6] is one of the most popular models in macroeconomics. It forms the core of many models of economic growth and it is extensively used for analyzing the effects of government policy. In the majority of applications of this model it is assumed that households have time-additive utility functions and identical time-preference rates. It is well known that, in the case where households have time-additive utility functions but different time-preference rates, only the most patient household owns a positive stock of capital in the long run. This property has already been conjectured by Ramsey and it has been formally proved by Becker [1] and Becker and Foias [2] for the case in which households face borrowing constraints and by Bewley [4] for the case without borrowing constraints. There are, however, variants of the Ramsey model in which the long-run distribution of wealth can be non-degenerate. For example, Lucas and Stokey [5] have shown that in the case where preferences are described by recursive utility

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functionals satisfying certain assumptions, there exist stationary equilibria in which all households own positive amounts of capital. Sarte [7] has identified progressive taxation as another reason for the existence of stationary equilibria with a non-degenerate wealth distribution.

In the present paper we add a few observations to the above results. We start with the case where a government levies a progressive income tax, which implies that rich households face a higher marginal tax rate than poor households. As shown by Sarte [7], this implies that there exist equilibria in which all households have positive long-run wealth levels. We show that there can actually exist infinitely many stationary equilibria with this property, provided that the marginal tax rate is constant on non-trivial intervals (as it is the case in many real-world economies). Within this framework we also address the issue of complicated equilibrium dynamics. It is known that the existence of a non-negativity constraint on wealth (i.e., a borrowing constraint) in the Ramsey model can be the cause of complicated dynamics, indeterminacy, and sunspot equilibria; see Becker and Foias [2, 3] and Sorger [8, 9]. The present paper demonstrates that the model with progressive taxation can explain complicated dynamics even if the borrowing constraint is replaced by the assumption that a household’s present value of consumption must not exceed its present value of income (no-Ponzi game condition). We do this by demonstrating that flip bifurcations and indeterminacy can occur at a stationary equilibrium with positive wealth for all households. In contrast to the above mentioned result on the existence of a continuum of stationary equilibria, this result is shown for an after-tax income function with non-zero second derivative, i.e., for a non-constant and smooth marginal tax rate.

To explain the second variant of the Ramsey model discussed in this paper let us start from the standard case with time-additive utility functions but without taxation. If the Ramsey conjecture were correct and households would have different time-preference rates, then only the most patient household would own a positive capital stock in the long run. In such a case it would be unrealistic to assume that this household takes the interest rate as given. Instead, the household would realize that it has market power, so that the capital market would be more accurately described as a monopoly. This suggests to model the capital market not as a competitive market but rather as an oligopolistic market. It turns out that in this setting, too, the long-run wealth distribution is not necessarily degenerate. Impatient households may hold positive stocks of capital, although it is still the case that more patient households own higher capital stocks than less patient households. The intuition for this result is quite different from the one for the case of progressive taxation. Whereas progressive taxation leads to different interest rates for rich and poor households, the interest rate is the same for all households in the model.
with an oligopolistic capital market. However, because of their market power, rich households charge such a high interest rate that poor households still find it optimal to save.

The following section formulates the Ramsey model with taxation. Section 3 considers the case of progressive taxation and shows that there can exist infinitely many stationary competitive equilibria in which all households own positive capital stocks. We also present an example in which periodic and indeterminate equilibria exist even if the borrowing constraint is not binding for any household. Section 4 considers the model without taxation but abandons price taking behavior. We start by studying a version of the model in which households have market power on the capital market but behave as price takers on all other markets. We then show that the results carry over to a situation where households realize that they can influence the wage rate as well.

2. MODEL FORMULATION

In this section we formulate a version of the Ramsey [6] model that allows for heterogeneous households and a government. A similar model has been used by Sarte [7]. The difference between the present model and the one in [7] is that we restrict wealth to be non-negative and that we assume that tax revenue is distributed back to households by transfer payments.

Time is measured in discrete periods \( t \in I = \{0, 1, 2, \ldots \} \). There exists a continuum of identical firms which use capital and labor to produce a homogeneous output good. Output can be either consumed or set aside to form capital for the next period. The set of firms is assumed to have measure 1. At the beginning of period \( t \), every firm hires labor \( L_t \) and capital \( K_t \) from the households. Since firms are identical and the set of firms has unit measure, \( L_t \) and \( K_t \) can also be interpreted as the aggregate demands for labor and capital in period \( t \). The common technology of all firms is described by a neoclassical production function \( F: \mathbb{R}_+^2 \rightarrow \mathbb{R} \). Thus, output in period \( t \) is given by \( Y_t = F(L_t, K_t) \) (again, this is both the output of an individual firm and aggregate output). In every period \( t \in I \), firms solve the profit maximization problem

\[
\begin{align*}
\text{maximize} & \quad Y_t - w_t L_t - r_t K_t \\
\text{subject to} & \quad Y_t = F(L_t, K_t), \\
& \quad L_t \geq 0, \quad K_t \geq 0,
\end{align*}
\]

where \( w_t \) is the wage rate and \( r_t \) the rental rate of capital.
There are $H$ infinitely lived households in the economy. The preferences of household $h \in \{1, 2, \ldots, H\}$ are characterized by the pair $(u^h, \beta^h)$, where $u^h : \mathbb{R}_+ \rightarrow \mathbb{R}$ is the instantaneous utility function and $\beta^h \in (0, 1)$ is the discount factor. Every household is endowed with one unit of labor per period which it inelastically supplies to the firms. We denote by $k^h_t$ the capital stock held by household $h$ at the beginning of period $t$. In particular, $k^h_0 \geq 0$ is the initial capital endowment. Household $h$’s factor income in period $t$ consists of labor income $w_t$ and rental income $r_t k^h_t$. This income is taxed and we denote by $g(w_t + r_t k^h_t)$ the after-tax income of household $h$ in period $t$. The function $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ is called the after-tax income function. In addition to factor income, each household receives a transfer payment $s_t$ from the government. If we denote by $c^h_t$ the consumption of household $h$ in period $t$, then the utility maximization problem of household $h$ can be stated as

$$
\text{maximize } \sum_{t=0}^{\infty} (\beta^h)^t u^h(c^h_t)
$$
subject to

$$
c^h_t + k^h_{t+1} - k^h_t = g(w_t + r_t k^h_t) + s_t, \quad t \in I,
$$

$$
c^h_t \geq 0, \quad k^h_t \geq 0, \quad t \in I.
$$

The objective functional is the discounted stream of utility derived from consumption over the entire lifetime. The budget constraint for period $t$ requires that the sum of after-tax income and transfers received during that period is equal to the sum of consumption and saving. The constraint $k^h_t \geq 0$ states that households must have non-negative wealth at each point in time. In other words, they are not allowed to finance present consumption by borrowing against future income. This is a form of a capital market imperfection. Capital markets would be perfect if one would replace the infinitely many budget constraints $k^h_t \geq 0, \quad t \in I$, by a single intertemporal budget constraint, i.e., by the assumption that the present value of lifetime consumption must not exceed the present value of lifetime earnings. Such an intertemporal budget constraint is known as a no-Ponzi game condition.

The economy $\mathcal{E}$ is specified by the fundamentals $\mathcal{F} = (F, g, \{(u^h, \beta^h) | h = 1, 2, \ldots, H\})$ and the vector of initial capital endowments $k_0 = (k^1_0, k^2_0, \ldots, k^H_0)$. We assume that $\mathcal{F}$ and $k_0$ have the following properties.

---

2 Note that $g(x) = x - \tau(x)$, where $\tau(x)$ is the income tax to be paid by an individual with income $x$. The function $\tau : \mathbb{R}_+ \rightarrow \mathbb{R}$ is called the tax function.

3 For simplicity, we assume that capital does not depreciate. It would be possible to include positive capital depreciation in the model.
(A1) The function \( f: \mathbb{R}_+ \times \mathbb{R} \) defined by \( f(K) = F(H, K) \) is continuous, twice continuously differentiable on \((0, +\infty)\), and satisfies \( f(0) = 0 \), \( f'(K) > 0 \), and \( f''(K) < 0 \) for all \( K > 0 \) as well as \( \lim_{K \to 0} f'(K) = +\infty \) and \( \lim_{K \to +\infty} f''(K) = 0 \).

(A2) For each \( h = 1, 2, \ldots, H \), the function \( u^h: \mathbb{R}_+ \times \mathbb{R} \) is continuous, twice continuously differentiable on \((0, +\infty)\), and satisfies \( (u^h)^'(c) > 0 \) and \( (u^h)^''(c) < 0 \) for all \( c > 0 \) as well as \( \lim_{c \to 0} (u^h)^'(c) = +\infty \).

(A3) Households are ordered according to increasing impatience, i.e., \( 1 > \beta^1 > \beta^2 > \cdots > \beta^H > 0 \).

(A4) The function \( g: \mathbb{R}_+ \times \mathbb{R} \) is continuous, twice continuously differentiable on \((0, +\infty)\), and satisfies \( g(0) = 0 \), \( 0 < g'(z) \leq 1 \), and \( g''(z) \leq 0 \) for all \( z > 0 \).

(A5) The aggregate initial endowment with capital is positive, i.e., \( \sum_{h=1}^{H} k^h_0 > 0 \).

Assumptions (A1) and (A2) are standard smoothness and convexity assumptions. The condition \( f(0) = 0 \) says that capital is necessary for production. Assumption (A3) is only made to simplify the exposition. As long as different households have different discount factors this condition can always be satisfied by relabelling the households. Assumption (A4) implies that the after-tax income function is increasing and concave and that \( g(z) \leq z \) holds for all income levels \( z \). The corresponding tax function \( \tau \) defined by \( \tau(z) = z - g(z) \) (see footnote 2) is therefore non-negative, non-decreasing, and convex. Concavity of \( g \) is usually referred to as marginal rate progressivity. Assumption (A4) implies also that \( g(x)/x \) is decreasing, which means that the tax system exhibits average rate progressivity. Assumption (A5) will guarantee the existence of non-trivial equilibria. If it were not satisfied then \( f(0) = 0 \) would imply that the only equilibrium for \( \bar{e} \) is the one where nothing is produced and nothing is consumed.

3. COMPETITIVE EQUILIBRIA

In this section we discuss the Ramsey model under the assumption that all agents take prices as given. A sequence

\[
E = (w_t, r_t, L_t, K_t, Y_t, s_t, \{(c^h_t, k^h_t) \mid h = 1, 2, \ldots, H\})_{t \in I}
\]

is called a competitive equilibrium from \( k_0 \) if the following conditions are satisfied.
(E1) For each $t \in I$, the pair $(L_t, K_t)$ solves the firms' profit maximization problem (1) given the factor prices $w_t$ and $r_t$.

(E2) For each $h=1, 2, \ldots, H$, the sequence $(c^h_t, k^h_t)_{t \in I}$ solves household $h$'s utility maximization problem (2) given the sequence of prices $(w_t, r_t)_{t \in I}$, the sequence of transfers $(s_t)_{t \in I}$, and the initial endowment $k^h_0$.

(E3) The factor markets clear in every period, i.e., $L_t = H$ and $K_t = \sum_{h=1}^{H} k^h_t$ for all $t \in I$.

(E4) The output market clears in every period, i.e., $\sum_{h=1}^{H} (c^h_t + k^h_{t+1} - k^h_t) = F(L_t, K_t)$ for all $t \in I$.

(E5) The government's budget is balanced in every period, i.e., $\sum_{h=1}^{H} y^h_t = H s_t$ for all $t \in I$, where $y^h_t = w_t + r_t k^h_t - g(w_t + r_t k^h_t)$ is the tax payment of household $h$ in period $t$.

We say that $E$ is a competitive equilibrium for $\mathcal{E}$ if there exists a vector of initial endowments $k_0$ satisfying (A5) such that $E$ is a competitive equilibrium from $k_0$. A competitive equilibrium for $\mathcal{E}$ is called stationary if it is a constant sequence.

It will be convenient to summarize necessary and sufficient equilibrium conditions for the model under consideration. This is done in the following lemma.

**Lemma 1.** Let $\mathcal{E}$ be an economy satisfying (A1)–(A4). If the sequence

$$(w_t, r_t, L_t, K_t, Y_t, s_t, \{(c^h_t, k^h_t) | h = 1, 2, \ldots, H\})_{t \in I}$$

is a competitive equilibrium, then the following conditions hold for all $t \in I$ and all $h = 1, 2, \ldots, H$:

$$Y_t = f(K_t), \quad L_t = H, \quad K_t = \sum_{j=1}^{H} k^j_t, \quad k^h_t \geq 0, \quad c^h_t \geq 0, \quad (3)$$

$$r_t = f''(K_t), \quad w_t = \frac{1}{H} \left[ f(K_t) - f''(K_t) K_t \right], \quad (4)$$

$$k^h_{t+1} = k^h_t - c^h_t + \frac{1}{H} \left[ f(K_t) + H g(w_t + r_t k^h_t) - \sum_{j=1}^{H} g(w_t + r_t k^j_t) \right], \quad (5)$$

$$\frac{(u'^r(c^h_t))}{\beta^h(u'^r(c^{h+1}_{t+1}))} \geq 1 + g'(w_{t+1} + r_{t+1} k^h_{t+1}) r_{t+1} \quad \text{with equality if } k^h_{t+1} > 0, \quad (6)$$

$$s_t = \frac{1}{H} \left[ f(K_t) - \sum_{j=1}^{H} g(w_t + r_t k^j_t) \right]. \quad (7)$$
Conversely, if the sequence \((w_t, r_t, L_t, K_t, Y_t, s_t, \{(c^h_t, k^h_t) | h = 1, 2, ..., H\})_{t \in T}\) satisfies (3)–(7) as well as (A5) and the transversality condition
\[
\lim_{t \to +\infty} (\beta^h)' (\alpha^h)' (\alpha^h)^h k^h_t = 0
\]
for all \(h = 1, 2, ..., H\), then it is a competitive equilibrium for \(E\).

**Proof. Necessity.** The conditions in (3) follow immediately from the definition of a competitive equilibrium. Conditions (4) are the first order optimality conditions for the firms’ profit maximization problem, and condition (6) is the first order optimality condition for household \(h\)’s utility maximization problem. Condition (7) is derived from the government’s budget constraint, the equations in (4), and the fact that \(\sum_{j=1}^H (w_t + r_t k^j_t) = w_t H + r_t K_t\). Condition (5) follows from (7) and (2).

**Sufficiency.** Conditions (3), (4), and (7) imply (E5), and conditions (5) and (7) imply the budget constraint in (2). Because of the assumed concavity of all functions, the first order optimality conditions (4) and (6) together with the respective feasibility conditions and the transversality condition are sufficient for (E1) and (E2) to hold. Condition (E3) holds because of (3), and (E4) follows by summing (5) over all households.

For the case without a government, Ramsey [6] conjectured that in the long run only the most patient household owns a positive stock of capital. A formal proof of this conjecture was given by Becker [1] and Becker and Foias [2]. They showed that, in every economy \(E\) satisfying (A1)–(A3) and \(g(z) = z\) for all \(z \geq 0\), there exists a unique stationary competitive equilibrium for \(E\).\(^4\) In this stationary competitive equilibrium household 1 owns the entire capital stock, whereas the other households consume their wages and do not save anything. Sarte [7] considered the Ramsey model with an after-tax income function satisfying \(g^*(z) < 0\) for all \(z > 0\). He showed that there exists a unique stationary competitive equilibrium as well, but that the wealth distribution is non-degenerate. The stationary wealth levels of the households are decreasing with respect to the discount factor. This means that the most patient household owns the largest capital stock, and the most impatient household owns the smallest capital stock. Sarte [7] simulated his model in order to study how the stationary income distribution is related to the degree of heterogeneity of discount factors.

\(^4\)The result can be generalized to the case of a constant tax rate \(\tau \in [0, 1]\), i.e., to the case \(g(z) = (1 - \tau) z\).
The following example shows that stationary equilibria for $\mathcal{E}$ are not uniquely determined if the after-tax income function contains flat segments. This is an important observation because many countries employ after-tax income functions with piecewise constant marginal tax rates. The possibility of multiple stationary equilibria for $\mathcal{E}$ is related to an analogous result for the standard Ramsey model without taxation and identical households: only the aggregate capital stock in a stationary competitive equilibrium is uniquely determined, but the distribution of wealth among the households is indeterminate (it depends on the vector of initial capital endowments $k_0$).

**Example 1.** Consider an economy $\mathcal{E}$ with $H = 2$ households which have discount factors $\beta^1 = 9/10$ and $\beta^2 = 4/5$, respectively. Let the utility functions $u^1$ and $u^2$ be arbitrary functions satisfying (A2). Moreover, assume that the technology is given by the Cobb–Douglas production function $F(L, K) = \frac{LK}{2}$, and let the after-tax income function satisfy
\[
g(z) = \begin{cases} 
z & \text{if } z \in [0, 1-\varepsilon), \\
\left(\frac{5+4z}{9}\right) & \text{if } z \in (1+\varepsilon, +\infty),
\end{cases}
\]
where $\varepsilon$ is a positive number less than $1/2$. It is obvious that one can extend $g$ to the entire domain $\mathbb{R}_+$ in such a way that (A4) is satisfied. Let $\kappa$ be any number from the interval $[0, 2-4\varepsilon)$. Substitution into the conditions of Lemma 1 shows that the economy has the following stationary competitive equilibrium:
\[
\begin{align*}
\bar{w} &= 1/2, & \bar{r} = 1/4, & \bar{L} = 2, & \bar{K} = 4, & \bar{Y} = 2, & \bar{s} = \left(10-5\kappa\right)/72, \\
\bar{c}_1 &= \left(98-13\kappa\right)/72, & \bar{c}_2 &= \left(46+13\kappa\right)/72, & \bar{k}_1 &= 4-\kappa, & \bar{k}_2 &= \kappa.
\end{align*}
\]
It follows that there exists a one-dimensional manifold (parametrized by $\kappa$) of stationary competitive equilibria for $\mathcal{E}$.

We have already mentioned that Sarte [7] proved that the wealth levels of households in a stationary competitive equilibrium are ordered in the same way as the discount factors, i.e., the less patient a household is the smaller is its long-run capital stock. The same result is also true in the present model and we state it here for the sake of completeness.

**Lemma 2.** Let $\mathcal{E}$ be an economy satisfying (A1)–(A4). Let $k^1, k^2, \ldots, k^H$ be the capital stocks of households 1, 2, ..., $H$ in a stationary competitive
equilibrium. Then it follows that $k^h \geq k^j$ whenever $h < j$, and $k^h > k^j$ whenever $k^h > 0$ and $h < j$.

**Proof.** Assume that the capital stock $k^h$ is equal to 0. Then it follows from condition (6) that $1/\beta^h \geq 1 + g(w) r$, where $w$ and $r$ denote the wage rate and the interest rate in the stationary competitive equilibrium. For all $j > h$ we therefore obtain from (A3) and (A4) that $1/\beta^j > 1/\beta^h \geq 1 + g(w) r \geq 1 + g(w + rk^j) r$. Using condition (6) again, we see that this implies $k^h = 0$. We can therefore conclude that there exists $\bar{h} \in \{1, 2, \ldots, H\}$ such that $k^h = 0$ for all $h > \bar{h}$ and $k^h > 0$ for all $h \leq \bar{h}$. It remains to show that $k^\bar{h} > k^j$ for all $h < j < \bar{h}$. From (6) we obtain $g(w + rk^h) = (1/\beta^h - 1)/r < (1/\beta^j - 1)/r = g(w + rk^j)$. Together with concavity of $g$ this implies $k^h > k^j$.

The dynamic properties of competitive equilibria in the model without taxation were discussed in Becker and Foias [2, 3] and Sorger [8, 9]. In particular, it was shown in these papers that competitive equilibria may exhibit very complicated dynamics like cycles of arbitrary long periods, chaos, and non-uniqueness. Moreover, it was demonstrated that there may exist sunspot equilibria with expectations-driven stochastic fluctuations. All these properties were proved for (non-stationary) competitive equilibria in which only the most patient household owns capital (i.e., the non-negativity constraint on wealth is binding for all households except household 1). We shall now show that competitive equilibria with similar dynamic properties can occur in the model with taxation even if the borrowing constraint $k^h \leq 0$ is not active for any household. This means that the equilibrium conditions remain to be satisfied if the borrowing constraints $k^h \leq 0$ are replaced by a single intertemporal budget constraint (no-Ponzi game condition). We restrict ourselves to presenting examples of periodic equilibria and indeterminate equilibria.

The existence of periodic and indeterminate equilibria will be proved by a local analysis of the equilibrium conditions around a stationary competitive equilibrium. Let us denote by $k = (k^1, k^2, \ldots, k^H)$ the capital stocks in the stationary equilibrium. We will show that a flip bifurcation can occur which creates period-two cycles, and that the stationary competitive equilibrium can have a stable manifold, the dimension of which exceeds the dimension of the state space. The latter property implies the existence of a continuum of non-stationary competitive equilibria from $k$ converging to the stationary equilibrium.

In order to demonstrate the above properties we have to derive the linearization of the equilibrium dynamics around a stationary competitive equilibrium. As in Example 1 we consider an economy with $H = 2$ households. Let us assume that $(w, r, L, K, Y, s, c^1, c^2, k^1, k^2)$ is a stationary competitive equilibrium for $E$ with $k^1 > 0$ and $k^2 > 0$. According to Lemma 1
the dynamics of competitive equilibria close to the stationary one are described by the four difference equations

\[ \begin{align*}
&k_{1+1} = k_1 - c_1' + [f(K_t) + g(w_t + r_t k_1) - g(w_t + r_t k_2)]/2, \\
&k_{2+1} = k_2 - c_2' + [f(K_t) + g(w_t + r_t k_2) - g(w_t + r_t k_1)]/2, \\
&(u')' (c_1') = \beta'(u') (c_1') \left[ 1 + g((w_{t+1} + r_{t+1} k_{1+1}) r_{t+1}) \right], \\
&(u')' (c_2') = \beta''(u') (c_2') \left[ 1 + g((w_{t+1} + r_{t+1} k_{2+1}) r_{t+1}) \right],
\end{align*} \]

(8)

where \( K_t = k_1^2 + k_2^2, \ r_t = f'(K_t), \) and \( w_t = [f(K_t) - f'(K_t) K_t]/2. \) The Jacobian matrix of this system evaluated at the steady state \((k_1, k_2, c_1, c_2)\) is given by

\[
J = \begin{pmatrix}
\frac{dk_{1+1}}{dk_1} & \frac{dk_{1+1}}{dc_1} & \frac{dk_{2+1}}{dc_1} & \frac{dk_{2+1}}{dc_2} \\
\frac{dk_{2+1}}{dk_1} & \frac{dk_{2+1}}{dc_1} & \frac{dk_{2+1}}{dc_2} & \frac{dk_{2+1}}{dc_2} \\
\frac{dc_{1+1}}{dk_1} & \frac{dc_{1+1}}{dc_1} & \frac{dc_{2+1}}{dc_1} & \frac{dc_{2+1}}{dc_2} \\
\frac{dc_{2+1}}{dk_1} & \frac{dc_{2+1}}{dc_1} & \frac{dc_{2+1}}{dc_1} & \frac{dc_{2+1}}{dc_2}
\end{pmatrix},
\]

where, for \( i, j \in \{1, 2\} \) with \( i \neq j, \)

\[
\begin{align*}
\frac{dk_{1+1}}{dk_1} &= \frac{1}{2} \left[ 1 + \frac{1}{\beta} + f'(K) + \left( \frac{1}{\beta} + \frac{1}{\beta} - 2 \right) \frac{(k_1' - k_2') f''(K)}{2 f'(K)} \right], \\
\frac{dk_{1+1}}{dk_2} &= \frac{1}{2} \left[ 1 - \frac{1}{\beta} + f'(K) + \left( \frac{1}{\beta} + \frac{1}{\beta} - 2 \right) \frac{(k_1' - k_2') f''(K)}{2 f'(K)} \right], \\
\frac{dk_{1+1}}{dc_1} &= -1, \\
\frac{dk_{1+1}}{dc_2} &= 0, \\
\frac{dc_{1+1}}{dk_1} &= - \beta \frac{(u')' (c')}{(u')'' (c')} \left\{ \gamma f''(K) [1 + f'(K)] + \gamma'' f'(K) \frac{dk_{1+1}}{dk_1} \right\}, \\
&\quad + \gamma'' f'(K) [1 + f'(K)] \frac{(k_1' - k_2') f''(K)}{2}, \\
\frac{dc_{1+1}}{dk_2} &= - \beta \frac{(u')' (c')}{(u')'' (c')} \left\{ \gamma f''(K) [1 + f'(K)] + \gamma'' f'(K) \frac{dk_{1+1}}{dk_1} \right\}, \\
&\quad + \gamma'' f'(K) [1 + f'(K)] \frac{(k_1' - k_2') f''(K)}{2},
\end{align*}
\]
\[
\frac{dc_{i+1}}{dc_i} = 1 + \beta_i \left( \frac{(u')^n(c')^i}{(u')^n(c')^i} \right) \left( \gamma_i f''(K) + \gamma_i f'(K) \left[ f'(K) + \frac{(k^i - k_i) f''(K)}{2} \right] \right),
\]

\[
\frac{dc_{j+1}}{dc_j} = \beta_j \left( \frac{(u')^n(c')}{(u')^n(c')} \right) f''(K) \left[ \gamma_j f'(K) \frac{k^j - k_j}{2} \right].
\]

Here, \( K = k^1 + k^2 \), \( r = f'(K) \), \( w = [f(K) - f'(K) K]/2 \), \( \gamma'_i = g'(w + r k^i) \), and \( \gamma''_i = g''(w + r k^i) \). The following example shows that there exist economies with periodic or indeterminate competitive equilibria.

**Example 2.** Consider an economy \& with \( H = 2 \) households which have discount factors \( \beta^1 = 3/5 \) and \( \beta^2 = 4/7 \), respectively. Moreover, assume that the technology satisfies \( f(11) = 13 \) and \( f'(11) = 1 \), and that the after-tax income function satisfies \( g(2) = 7/4 \), \( g(11) = 8 \), \( g'(2) = 3/4 \), and \( g'(11) = 2/3 \). For the moment, the utility functions \( u^i \) and \( u^j \) are only required to satisfy (A2). Substitution into the conditions of Lemma 1 shows that the economy has the following stationary competitive equilibrium:

\[
w_t = 1, \quad r_t = 1, \quad L_t = 2, \quad K_t = 11, \quad Y_t = 13, \quad s_t = 13/8, \quad c^1_t = 77/8, \quad c^2_t = 27/8, \quad k^1_t = 10, \quad k^2_t = 1.
\]

We shall now show that periodic competitive equilibria and indeterminacy can exist when the curvature of the after-tax income function is close to the critical value \( \bar{\gamma} = -3052035/1451864 \approx -2.1021 \). The Jacobian matrix \( J \) depends on the first order derivatives of \( u^i \) and \( u^j \) and on the second order derivatives of \( u^i \), \( u^j \), \( f \), and \( g \), which we have not yet specified. Let the utility functions \( u^1 \) and \( u^2 \) be such that \( (u')^i(77/8) = (u')^j(27/8) = 1 \), \( (u')^i(77/8) = -2 \), and \( (u')^j(27/8) = -2648775/1270381 \approx -2.085 \). Furthermore, assume that \( f''(11) = -5 \), \( g''(2) = \bar{\gamma} \), and \( g''(11) = -4 \). Using these values, one can show that the Jacobian matrix evaluated at the steady state has the two negative eigenvalues \( \sigma_1 = -5 \) and \( \sigma_2 = -1 \) as well as a pair of complex conjugate eigenvalues with absolute values \( \sqrt{615}/30 \approx 0.8266 \). We choose \( g''(2) \) as the bifurcation parameter. As \( g''(2) \) changes from \(-2\) to \(-2.2\), the eigenvalue \( \sigma_2 \) moves from \(-0.8251\) to \(-1.2774\) passing through \(-1\) at the critical value \( g''(2) = \bar{\gamma} \). On the other hand, none of the other three eigenvalues crosses the unit circle as \( g''(2) \) changes from \(-2\) to \(-2.2\). This allows us to draw two conclusions. First, as \( g''(2) \) changes from \(-2\) to \(-2.2\) a flip bifurcation occurs at the critical value \( g''(2) = \bar{\gamma} \), which implies that, for an open interval of values of \( g''(2) \) close to \( \bar{\gamma} \), there exist periodic competitive equilibria of period 2. The second observation is that,
for values of \( g''(2) \) close to but larger than \( \bar{g} \), the Jacobian matrix \( J \) has three eigenvalues with absolute values smaller than 1. Thus, the stable manifold of the stationary competitive equilibrium is three-dimensional. If the two initial conditions \( k_1^0 \) and \( k_2^0 \) are given by the stationary values 10 and 1, there exists a one-dimensional manifold of initial conditions \( c_1^0 \) and \( c_2^0 \) such that trajectories of (8) starting at these values qualify as competitive equilibria from (10, 1) and converge to the stationary equilibrium. This implies that the stationary competitive equilibrium is indeterminate.

4. NASH EQUILIBRIA

In this section we consider the case without any government activities; that is, we assume \( g(z) = z \) for all \( z \geq 0 \) and \( s_t = 0 \) for all \( t \in I \). In the previous section we mentioned that in this situation the Ramsey model makes a rather unrealistic prediction about the long-run wealth distribution. In particular, it follows from the results by Becker [1] that in every stationary competitive equilibrium only the most patient household owns a positive capital stock. But if this is actually the case, the assumption of price taking behavior of households becomes untenable. Household 1, who is the sole owner of all capital, would definitely realize that it has market power in the capital market and would exercise its market power to its own advantage. Thus, the model makes a prediction which invalidates one of its most important assumptions. One way out of this dilemma is to assume that for each \( h = 1, 2, \ldots, H \), there exists a large group of households which have the common discount factor \( \beta^h \). In that case, the competitive assumption still makes sense even if only households with discount factor \( \beta^1 \) own positive amounts of capital. In this section, however, we take the assumption of households having mutually different discount factors seriously. Thus, we have to abandon the assumption of price taking behavior in the capital market. Instead we postulate that all households know the market demand function for capital and that they play a Nash equilibrium in the capital market. As for the labor market and the output market we maintain the assumption of price taking behavior.\(^5\)

Since all households supply labor inelastically, aggregate labor supply in each period is equal to \( H \) and the wage rate adjusts to clear the labor market. This means that aggregate labor demand in each period is also equal to \( H \). The inverse demand function for capital is therefore given by

\(^5\) A variant of the model in which price taking behavior in the labor market is also abandoned will be discussed briefly at the end of the section.
Households know this demand function. Thus, household \( h \) solves the problem

\[
\begin{align*}
\text{maximize} & \quad \sum_{t=0}^{\infty} \left( \beta^t \right) u^t(c_t^h) \\
\text{subject to} & \quad c_t^h + k_{t+1}^h - k_t^h = w_t + f'(k_t^h + k_{t+1}^h + \cdots + k_H^h) k_t^h, \quad t \in I, \\
& \quad c_t^h \geq 0, \quad k_t^h \geq 0, \quad t \in I.
\end{align*}
\]

(9)

A sequence

\[ E = (w_t, r_t, L_t, K_t, Y_t, \{(c_t^h, k_t^h) \mid h = 1, 2, \ldots, H\}) \]

is called a **Nash equilibrium from** \( k_0 \) if conditions (E1), (E3), and (E4) from Section 3 as well as the following condition are satisfied.

(E2') For each \( h = 1, 2, \ldots, H \), the sequence \( (c_t^h, k_t^h)_{t \in I} \) solves household \( h \)'s utility maximization problem (9) given the sequence of wages \( (w_t)_{t \in I} \), the initial capital stock \( k_0^h \), and the \( H-1 \) sequences of capital stocks \( (k_j^t)_{t \in I}, j \neq h \).

As in the case of competitive equilibria, we call \( E \) a **Nash equilibrium for** \( \mathcal{E} \), if there exists a vector of initial capital stocks \( k_0 \) satisfying (A5) such that \( E \) is a Nash equilibrium from \( k_0 \). Moreover, we call a Nash equilibrium for \( \mathcal{E} \) **stationary** if it is a constant sequence. The following lemma can be proved in exactly the same way as Lemma 1.

**Lemma 3.** Let \( \mathcal{E} \) be an economy satisfying (A1)–(A3) and \( g(z) = z \) for all \( z \geq 0 \). If the sequence

\[ (w_t, r_t, L_t, K_t, Y_t, \{(c_t^h, k_t^h) \mid h = 1, 2, \ldots, H\}) \]

is a Nash equilibrium, then conditions (3)–(4) as well as the following two conditions hold for all \( t \in I \) and all \( h = 1, 2, \ldots, H \):

\[
k_{t+1}^h = k_t^h - c_t^h + \frac{1}{H} [f(K_t) - f'(K_t) K_t] + f'(K_t) k_t^h, \tag{10}
\]

\[
\frac{(u^h)'(c_t^h)}{u^h(c_{t+1}^h)} \geq 1 + f'(K_{t+1}) + f''(K_{t+1}) k_{t+1}^h \quad \text{with equality if} \quad k_{t+1}^h > 0. \tag{11}
\]
Conversely, assume that the function $k^h_t \mapsto f'(k^h_t + k^j_t + \cdots + k^H_t) k^h_t$ is concave for all $h = 1, 2, \ldots, H$ and all $t \in I$. If the sequence $(w_t, r_t, L_t, K_t, Y_t, \{(c^h_t, k^h_t) | h = 1, 2, \ldots, H\})_{t \in I}$ satisfies (3)–(4), (10)–(11), (A5), and the transversality condition
\[
\lim_{t \to +\infty} (\beta^h)'(u^h)'(c^h_t) k^h_t = 0
\]
for all $h = 1, 2, \ldots, H$, then it is a Nash equilibrium for $E$.

We shall now demonstrate that in the case of households acting strategically, Ramsey’s conjecture about the degenerate long-run distribution of wealth need not hold. As in the previous section we restrict ourselves to the presentation of an example with $H = 2$ households.

**Example 3.** Consider an economy $E$ with $H = 2$ households which have discount factors $\beta^1 = 32/37$ and $\beta^2 = 32/39$, respectively. Moreover, assume that the technology is given by the Cobb–Douglas production function $F(L, K) = \sqrt{LK}/2$ and that the utility functions are arbitrary functions satisfying (A2). First of all note that the function $k^h_t \mapsto f'(k^h_t + k^j_t) k^h_t = k^h_t / (2 \sqrt{k^h_t + k^j_t})$ is strictly concave for every $k^j > 0$. Therefore, conditions (3)–(4), (10)–(11), and the transversality condition are sufficient equilibrium conditions. We shall now prove that the economy has a unique stationary Nash equilibrium given by

\[
w_t = 1/2, \quad r_t = 1/4, \quad L_t = 2, \quad K_t = 4, \quad Y_t = 2,
\]
\[
c^1_t = 5/4, \quad c^2_t = 3/4, \quad k^1_t = 3, \quad k^2_t = 1.
\]

The fact that this is indeed a Nash equilibrium is easily verified by substitution into the sufficient conditions of Lemma 3. To prove uniqueness we use the necessary conditions. We have to distinguish two cases, namely that both households own positive capital stocks and that only household 1 owns a positive capital stock (the case that only household 2 owns a positive capital stock cannot occur; see Lemma 4 below). If $k^1_t = k^1 > 0$ and $k^2_t = k^2 > 0$ holds in a stationary Nash equilibrium, then Eq. (11) implies that

\[
\frac{37}{32} = 1 + \frac{1}{2} (k^1 + k^2)^{-1/2} \frac{k^1}{4} (k^1 + k^2)^{-3/2},
\]
\[
\frac{39}{32} = 1 + \frac{1}{2} (k^1 + k^2)^{-1/2} \frac{k^2}{4} (k^1 + k^2)^{-3/2}.
\]
Adding these two equations and defining \( K = k^1 + k^2 \) we obtain
\[
\frac{19}{8} = 2 + 1/\sqrt{K} - 1/(4 \sqrt{K}),
\]
which implies \( k^1 + k^2 = K = 4 \). Substituting this back into the above two equations it follows that \( k^1 = 3 \) and \( k^2 = 1 \). Now let us consider the case where only household 1 owns a positive stock of capital. In this case Eq. (11) implies that
\[
\frac{37}{32} = 1 + \frac{1}{2 \sqrt{k^1}} - \frac{1}{4 \sqrt{k^1}}, \quad \frac{39}{32} > 1 + \frac{1}{2 \sqrt{k^1}}.
\]
The first condition yields \( \sqrt{k^1} = 8/5 \). Substituting this into the second condition leads to a contradiction. Thus, there cannot be a stationary Nash equilibrium in which only household 1 owns capital.

In the above example there exists a unique stationary Nash equilibrium and it has the property that all households own positive capital stocks. The following lemma proves that in every stationary Nash equilibrium, patient households are richer than impatient ones.

**Lemma 4.** Let \( \mathcal{E} \) be an economy satisfying (A1)–(A3) and \( g(z) = z \) for all \( z \geq 0 \). Let \( k^1, k^2, \ldots, k^H \) be the capital stocks of households 1, 2, ..., \( H \) in a stationary Nash equilibrium. Then it follows that \( k^h \geq k^j \) whenever \( h < j \), and \( k^h > k^j \) whenever \( k^h > 0 \) and \( h < j \).

**Proof.** Assume that the capital stock \( k^h \) is equal to 0. Then it follows from (11) that \( 1/\beta^h \geq 1 + f'(K) \), where \( K \) denotes the aggregate capital stock in the stationary Nash equilibrium. For all \( j > h \) we therefore obtain from (A1) and (A3) that \( 1/\beta^j > 1/\beta^h \geq 1 + f'(K) \geq 1 + f'(K) + f''(K) k^j \). Using condition (11) again, we see that this implies \( k^j = 0 \). We can therefore conclude that there exists \( \bar{h} \in \{1, 2, \ldots, H\} \) such that \( k^h = 0 \) for all \( h > \bar{h} \) and \( k^h > 0 \) for all \( h \leq \bar{h} \). It remains to show that \( k^h > k^j \) for \( h < j \leq \bar{h} \). From (11) we obtain \( f''(K) k^h = 1/\beta^h - f'(K) < 1/\beta^j - f'(K) = f''(K) k^j \). Together with \( f''(K) < 0 \) this implies \( k^h > k^j \).

So far in this section we have assumed that households have market power on the capital market, but that they act as price takers on all other markets. This was motivated by the fact that the original Ramsey model predicts that, in the long run, only the most patient household supplies capital whereas it assumes that all households supply labor. One might
argue, however, that households should realize that they can affect the wage rate because the latter is determined as a residual from the profit condition. In the remainder of this section we briefly describe how the above analysis has to be changed in order to capture this aspect. As we shall see, this version of the model also allows for the possibility of all households holding positive capital in a stationary equilibrium.

If household \( h \) realizes that the rental rate of capital is \( r_t = f'(K_t) \) and the wage rate is \( w_t = \frac{1}{H}[f(K_t) - f'(K_t) K_t] \), then the relevant optimization problem is

\[
\max_{c_t^h, k_t^h} \sum_{t=0}^{\infty} \beta^t u^h(c_t^h)
\]

subject to

\[
\begin{align*}
& c_t^h + k_{t+1}^h - k_t^h = \frac{1}{H}[f(K_t) - f'(K_t) K_t] + f'(K_t) k_t^h, \\
& c_t^h \geq 0, \quad k_t^h \geq 0, \quad t \in I.
\end{align*}
\]

(12)

where, as usual, \( K_t \) stands for \( K_t = \sum_{j=1}^{H} k_j^t \).

In this context, a sequence \( E = (w_t, r_t, L_t, K_t, Y_t, \{(c_t^h, k_t^h) | h=1, 2, ..., H\})_{t \in I} \) is a Nash equilibrium from \( k_0 \) if conditions (E1), (E3), and (E4) from Section 3 as well as the following condition are satisfied.

\((E2'')\) For each \( h = 1, 2, ..., H \), the sequence \( (c_t^h, k_t^h)_{t \in I} \) solves household \( h \)'s utility maximization problem (12) given the initial capital stock \( k_0^h \), and the \( H-1 \) sequences of capital stocks \( (k_j^t)_{t \in I}, j \neq h \).

As before \( E \) is called a Nash equilibrium for \( \mathcal{E} \), if there exists a vector of initial capital stocks \( k_0 \) satisfying (A5) such that \( E \) is a Nash equilibrium from \( k_0 \), and \( \mathcal{E} \) is called a stationary Nash equilibrium if it is a constant sequence. Lemma 3 has to be replaced by the following result. Note that the only difference to Lemma 3 is in the Euler equation and the concavity requirement.

**Lemma 5.** Let \( \mathcal{E} \) be an economy satisfying (A1)–(A3) and \( g(z) = z \) for all \( z \geq 0 \). If the sequence

\[
(w_t, r_t, L_t, K_t, Y_t, \{(c_t^h, k_t^h) | h=1, 2, ..., H\})_{t \in I}
\]

is an economy satisfying (A1)–(A3) and \( g(z) = z \) for all \( z \geq 0 \). If the sequence

\[
(w_t, r_t, L_t, K_t, Y_t, \{(c_t^h, k_t^h) | h=1, 2, ..., H\})_{t \in I}
\]

4 This version of the model has been suggested by an anonymous referee.

7 Note that the variable \( K_t \) is used in (12) merely to simplify the notation. Household \( h \) does not take \( K_t \) as given but realizes that it can influence \( K_t = \sum_{j=1}^{H} k_j^t \) via \( k_t^h \).
is a Nash equilibrium, then conditions (3)–(4), (10), and
\[
\frac{(u^h)'(c^h_t)}{\beta'(u^h)'(c^h_t)} \geq 1 + f'(K_{t+1}) + f''(K_{t+1}) \left[ k^h_{t+1} - \frac{K_{t+1}}{H} \right]
\]
with equality if \( k^h_{t+1} > 0 \) hold for all \( t \in I \) and all \( h = 1, 2, \ldots, H \). Conversely, assume that the function
\[
k^h_t \mapsto \frac{1}{H} \left[ f \left( \sum_{j=1}^H k^j_t \right) - f' \left( \sum_{j=1}^H k^j_t \right) \sum_{j=1}^H k^j_t \right] + f'' \left( \sum_{j=1}^H k^j_t \right) k^h_t
\]
is concave for all \( h = 1, 2, \ldots, H \) and all \( t \in I \). If the sequence \((w_t, r_t, L_t, K_t, Y_t, \{(c^h_t, k^h_t) \mid h = 1, 2, \ldots, H\})_{t \in I}\) satisfies (3)–(4), (10), (13), (A5), and the transversality condition
\[
\lim_{t \to +\infty} (\beta^h)'(u^h)'(c^h_t) k^h_t = 0
\]
for all \( h = 1, 2, \ldots, H \), then it is a Nash equilibrium for \( \delta \).

The following example shows that the result on the possibility of positive capital stocks for impatient households carries over to the present case.

**Example 3 (continued).** Using arguments identical to those employed before, one can show that under the present definitions, the economy has a unique stationary Nash equilibrium given by

\[
w_t = 2/3, \ r_t = 3/16, \ L_t = 2, \ K_t = 64/9, \ Y_t = 8/3,
\]
\[
c^1_t = 16/9, \ c^2_t = 8/9, \ k^1_t = 160/27, \ k^2_t = 32/27.
\]

Thus, both households own positive capital stocks.

Obvious modifications of the proof of Lemma 4 show that this lemma carries over to the present situation as well, that is, in a stationary Nash equilibrium it must hold that \( k^h \geq k^j \) whenever \( h < j \), and \( k^h > k^j \) whenever \( k^h > 0 \) and \( h < j \).

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