Reconstruction of polyharmonic functions from samples

Gerhard Schmeisser

Mathematical Institute, University of Erlangen-Nuremberg, 91054 Erlangen, Germany

Received 18 February 2006; received in revised form 13 February 2007; accepted 21 February 2007

Communicated by Hans G Feichtinger

Available online 06 March 2007

Abstract

We study planar complex-valued functions that satisfy a certain Wirtinger differential equation of order $k$. Our considerations include entire functions ($k = 1$), harmonic functions ($k = 2$), biharmonic functions ($k = 4$), and polyharmonic functions ($k$ even) in general. Under the assumption of restricted exponential growth and square integrability along the real axis, we establish a sampling theorem that extends the classical sampling theorem of Whittaker–Kotel’nikov–Shannon and reduces to the latter when $k = 1$. Intermediate steps, which may be of independent interest, are representation theorems, uniqueness theorems, and the construction of fundamental functions for interpolation. We also consider supplements, variants, generalizations, and an algorithm.

© 2007 Elsevier Inc. All rights reserved.

MSC: 31A30; 94A20

Keywords: Polyharmonic functions; Wirtinger calculus; Representation theorems; Uniqueness theorems; Fundamental functions; Sampling; Reconstruction formulae

1. Introduction and motivation

Throughout this paper, we shall use the following notation. For $\sigma > 0$, we denote by $\mathcal{E}_\sigma$ the set of all functions $f$ which are entire, that is, analytic in the whole complex plane, and satisfy

$$\limsup_{r \to \infty} \frac{\log \max_{|z| = r} |f(z)|}{r} \leq \sigma.$$  \hfill (1)

We call $\mathcal{E}_\sigma$ the class of entire functions of exponential type $\sigma$. 

E-mail address: schmeisser@mi.uni-erlangen.de.
When we have a function \( f : \mathbb{C} \rightarrow \mathbb{C} \) and write \( f \in L^p(\mathbb{R}) \), we actually mean that the restriction of \( f \) to \( \mathbb{R} \) belongs to \( L^p(\mathbb{R}) \). The spaces \( \mathcal{B}^p_\sigma := \mathcal{E}_\sigma \cap L^p(\mathbb{R}) \) are called Bernstein classes; see [26, Definition 6.5]. As a consequence of the Paley–Wiener theorem [26, Ch. 7], a signal \( \phi : \mathbb{R} \rightarrow \mathbb{C} \) is bandlimited to \([-\sigma, \sigma]\) and belongs to \( L^2(\mathbb{R}) \) if and only if it is the restriction to \( \mathbb{R} \) of a function from \( \mathcal{B}^2_\sigma \). Therefore, the classical sampling theorem of Whittaker–Kotel’nikov–Shannon (see [36, p. 49]) may be extended from \( \mathbb{R} \) (the time axis) to \( \mathbb{C} \) and stated as follows.

**Theorem A.** Let \( f \in \mathcal{B}^2_\sigma \), where \( \sigma > 0 \). Then

\[
f(z) = \sum_{n=-\infty}^{\infty} f \left( \frac{n\pi}{\sigma} \right) \frac{\sin(\sigma z - n\pi)}{\sigma z - n\pi} \quad (z \in \mathbb{C}).
\]

(2)

The series converges absolutely and uniformly on every strip \( |\Im z| \leq K \) for any \( K > 0 \).

The hypotheses on \( f \) consist of three crucial requirements:

(i) the analyticity on \( \mathbb{C} \);  
(ii) the restriction (1) of the asymptotic growth;  
(iii) the square integrability on \( \mathbb{R} \).

None of these conditions can be abandoned, but variants of the sampling theorem have been established in which some of the conditions (i)–(iii) have been relaxed.

It is relatively easy to relax condition (iii). Although \( \mathcal{B}^p_\sigma \subset \mathcal{B}^q_\sigma \) for \( 1 \leq p < q \leq \infty \), see [26, Lemma 6.6], the conclusion of Theorem A remains valid for \( f \in \mathcal{B}^p_\sigma \) with \( p \in [1, \infty) \); see [26, Theorem 6.13] or [36, p. 50]. Moreover, if \( f \in \mathcal{E}_\varepsilon \), where \( 0 \leq \varepsilon < \sigma \), then a modification of (2) with a bandlimited convergence factor on the right-hand side allows us to reconstruct \( f \) from the samples \( f(n\pi/\sigma) \) even if \( |f| \) is unbounded on \( \mathbb{R} \) provided that it does not grow too fast on the sequence \( (n\pi/\sigma)_{n \in \mathbb{Z}} \); see, e.g., [38, Theorem 3 and Remark on p. 222] or Theorem B in §6 below.

The condition (ii) is crucial for the minimal density of the sampling points. It determines the so-called Nyquist rate. For \( \tau > \sigma \), there exists an \( f \in \mathcal{B}^2_\sigma \) which cannot be reconstructed from samples taken on the sequence \( (n\pi/\sigma)_{n \in \mathbb{Z}} \). However, if \( \tau \leq k\sigma \), where \( k \in \mathbb{N} \), then \( f \) can be reconstructed from samples of \( f, f', \ldots, f^{(k-1)} \) taken on the sequence \( (n\pi/\sigma)_{n \in \mathbb{Z}} \); see [26, §9.3] for a generalization which includes the case mentioned here.

The Wirtinger derivatives, described below, allow us to express condition (i) by the equation \( \partial f / \partial z = 0 \). In order to relax (i), we should admit functions \( f \) for which \( \partial f / \partial z \) need not be zero. In some sense, the simplest class of such functions are the complex-valued harmonic functions since they satisfy the equation \( \partial^2 f / \partial z \partial \bar{z} = 0 \).

The problem of reconstructing real-valued harmonic functions by sampling series was raised by Boas [9]. His paper has inspired numerous contributions; see [3,4,11,14–16,39–41,45,47,52]. In [42], we established an extension of Theorem A in which, instead of the analyticity, \( f \) is only required to be a planar complex-valued harmonic function. In the special case of an entire function, this extension reduces to Theorem A. For a generalization with nonuniform sampling points, see [43].

In this paper, we shall relax condition (i) further by going beyond harmonic functions. Instead of (i), we shall only require that a mixed Wirtinger derivative of order \( k \) vanishes. This includes sampling of polyharmonic functions.
Polyharmonic functions have been studied since the end of the nineteenth century; see [2], where the familiar Laplace operator is denoted by $\Delta^2$. Originally, they have been of interest mainly in mathematical physics. In particular, biharmonic functions are important in the theory of elasticity. In recent times, polyharmonic functions have gained interest in diverse branches of mathematics and engineering. For example, they have been used in image processing [31], in approximation of functions [5,32], in cubature [10], and for the construction of multivariate splines [25,35]. Polyharmonic splines have been much used as radial basis functions [28], and they have also been employed as pre-wavelets in multiresolution analysis [6]. In some of these applications, harmonic functions would be too special for obtaining reasonable results. Thus one profits from the fact that polyharmonic functions form a wider class. On the other hand, by turning to a wider class, one may lose certain favorable properties. From this point of view, it seems to be a natural and interesting question whether, under the usual growth conditions, polyharmonic functions, in general, can still be recovered from countable data. This paper contains an answer to this question.

2. Polyharmonic functions and their representations

For $z = x + iy$, where $x, y \in \mathbb{R}$, the Wirtinger differential operators are defined by

$$\frac{\partial}{\partial z} := \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

They are not proper partial derivatives since $z$ and $\bar{z}$ cannot vary independently. Nevertheless, it turns out that in calculations we may use them like partial derivatives [49]. This phenomenon is often referred to as Wirtinger calculus. To be precise, we may proceed as follows.

Identifying $\mathbb{C}$ with $\mathbb{R}^2$, we consider functions $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ from now on. Substituting

$$x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i},$$

we write

$$f(x, y) = f \left( \frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i} \right) =: f^w(z, \bar{z})$$

and call $f^w$ the Wirtinger form of $f$. Clearly, $f$ can be uniquely recovered from $f^w$.

If $f^w$ has as continuously differentiable extension from the points $(z, \bar{z})$ to arbitrary points $(\xi, \eta) \in \mathbb{C}^2$, then

$$\frac{\partial}{\partial \bar{z}} f(x, y) = \frac{\partial}{\partial \xi} f^w(\xi, \bar{z})|_{\xi = z}, \quad \frac{\partial}{\partial \bar{z}} f(x, y) = \frac{\partial}{\partial \eta} f^w(z, \eta)|_{\eta = \bar{z}}.$$

In other words, the Wirtinger derivatives of $f$ can be obtained as partial derivatives of $f^w$ treating $z$ and $\bar{z}$ as if they were independent variables.

Next, we define the following powers of Wirtinger’s differential operators

$$D_0 := \text{id}, \quad D_1 := \frac{\partial}{\partial \bar{z}}, \quad D_{2\ell} := \frac{\partial^{2\ell}}{\partial z^\ell \partial \bar{z}^\ell}, \quad D_{2\ell+1} := \frac{\partial^{2\ell+1}}{\partial z^\ell \partial \bar{z}^{\ell+1}} \quad (\ell \in \mathbb{N})$$
and introduce the function spaces
\[ \mathcal{H}_k := \left\{ f : \mathbb{R}^2 \to \mathbb{C} : f \in C^k(\mathbb{R}^2), D_k f = 0 \right\} \quad (k \in \mathbb{N}). \]

The space \( \mathcal{H}_1 \) consists of all entire functions. Since \( D_{2\ell} = 4^{-\ell} \Delta^\ell \), where \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is the Laplace operator, we see that \( \mathcal{H}_2 \) comprises all planar (complex-valued) harmonic functions and \( \mathcal{H}_4 \) comprises all planar biharmonic functions. Generally, the members of \( \mathcal{H}_{2\ell} \) for \( \ell > 1 \) are called polyharmonic functions of order \( \ell \). There seems to be no special name for the members of \( \mathcal{H}_{2\ell+1} \). For our purposes, we may consider them as polyharmonic functions of fractional order \( \ell + \frac{1}{2} \). We observe that
\[ \mathcal{H}_1 \subset \mathcal{H}_2 \subset \cdots \subset \mathcal{H}_k \subset \cdots. \]

The subclass of all functions \( f \in \mathcal{H}_k \) which are of exponential growth at most \( \sigma \), that is,
\[ \lim_{r \to \infty} \sup \frac{\log \max_{x^2 + y^2 = r^2} |f(x, y)|}{r} \leq \sigma, \]
shall be denoted by \( \mathcal{H}_{k,\sigma} \).

The following description of the spaces \( \mathcal{H}_k \) will be very useful later on. As usual, a summation over an empty range of indices shall give zero.

**Theorem 1.** Let \( f \in \mathcal{H}_k \), where \( k \in \mathbb{N} \).

(i) If \( k = 2\ell - 1 \) is odd, then the Wirtinger form of \( f \) has a unique representation
\[ f^w(z, \bar{z}) = \sum_{j=0}^{\ell-2} (\bar{z}^j g_j(z) + z^j h_j(z)) + \bar{z}^{\ell-1} g_{\ell-1}(z), \]
where \( g_0, \ldots, g_{\ell-1}, h_0, \ldots, h_{\ell-2} \) are entire functions and
\[ h_j(0) = \cdots = h_j^{(\ell-1)}(0) = 0 \quad (j = 0, \ldots, \ell - 2). \]

(ii) If \( k = 2\ell \) is even, then the Wirtinger form of \( f \) has a unique representation
\[ f^w(z, \bar{z}) = \sum_{j=0}^{\ell-1} (\bar{z}^j g_j(z) + z^j h_j(z)), \]
where \( g_0, \ldots, g_{\ell-1}, h_0, \ldots, h_{\ell-1} \) are entire functions and
\[ h_j(0) = \cdots = h_j^{(\ell-1)}(0) = 0 \quad (j = 0, \ldots, \ell - 1). \]

These representations show that \( f^w \) has a continuation from the pairs \((z, \bar{z})\) to arbitrary pairs \((\xi, \eta) \in \mathbb{C}^2\).
Proof. We shall proceed by induction on \( k \). When \( k = 1 \), we have to verify statement (i) for \( \ell = 1 \). In this case, \( f^w \) is an entire function and therefore the conclusion is trivial.

Now assuming that the theorem holds for \( k \), we have to prove it for \( k + 1 \). Take any \( f \in \mathcal{H}_{k+1} \).

We distinguish the cases of odd and even \( k \).

When \( k = 2\ell - 1 \), then \( \partial f/\partial z \) belongs to \( \mathcal{H}_k \). Therefore, by the induction hypothesis, statement (i) applies to \( \partial f/\partial z \) and shows that

\[
\left( \frac{\partial f}{\partial z} \right)^w (z, \bar{z}) = \sum_{j=0}^{\ell-1} \bar{z}^j \varphi_j(z) + z^{\ell-1} \phi_{\ell-1}(z),
\]

(7)

where \( \varphi_0, \ldots, \varphi_{\ell-1}, \psi_0, \ldots, \psi_{\ell-2} \) are entire functions and

\[
\psi_j(0) = \cdots = \psi_j^{(\ell-1)}(0) = 0 \quad (j = 0, \ldots, \ell - 2).
\]

By virtue of the Wirtinger calculus, \( f^w \) can be obtained by integrating (7) with respect to \( z \), treating \( z \) and \( \bar{z} \) as if they were independent variables. The result of this integration is unique apart from an additive entire function in \( \bar{z} \). Thus

\[
f^w(z, \bar{z}) = \sum_{j=0}^{\ell-1} \bar{z}^j \phi_j(z) + \sum_{j=0}^{\ell-2} \frac{z^{j+1}}{j+1} \psi_j(\bar{z}) + \chi(\bar{z}),
\]

where \( \phi_j \) is a primitive of \( \varphi_j \) for \( j = 0, \ldots, \ell - 1 \) and \( \chi \) is an appropriate entire function. Now, defining

\[
g_j(z) := \phi_j(z) + \frac{\chi^{(j)}(0)}{j!} \quad (j = 0, \ldots, \ell - 1)
\]

and

\[
h_0(z) := \chi(z) - \sum_{j=0}^{\ell-1} \frac{\chi^{(j)}(0)}{j!} z^j, \quad h_j(z) := \frac{\psi_{j-1}(z)}{j} \quad (j = 1, \ldots, \ell - 1),
\]

we obtain the representation of statement (ii).

It remains to show the uniqueness. Assume that, aside from (5), \( f^w \) has another representation

\[
f^w(z, \bar{z}) = \sum_{j=0}^{\ell-1} \left( \bar{z}^j \tilde{g}_j(z) + z^j \tilde{h}_j(\bar{z}) \right)
\]

with entire functions \( \tilde{g}_0, \ldots, \tilde{g}_{\ell-1}, \tilde{h}_0, \ldots, \tilde{h}_{\ell-1} \) such that

\[
\tilde{h}_j(0) = \cdots = \tilde{h}_{j}^{(\ell-1)}(0) = 0 \quad (j = 0, \ldots, \ell - 1).
\]

Then \( (\partial f/\partial z)^w \) has two representations of a form as in statement (i). Since \( \partial f/\partial z \in \mathcal{H}_k \), the induction hypothesis applies to this function and guarantees a unique representation. Consequently,

\[
\tilde{g}_j = g_j' \quad (j = 0, \ldots, \ell - 1) \quad \text{and} \quad \tilde{h}_j = h_j \quad (j = 1, \ldots, \ell - 1).
\]
With these identities, the two representations of \( f^w \) imply that
\[
h_0(\bar{z}) + \sum_{j=0}^{\ell-1} \bar{z}^j g_j(0) = \tilde{h}_0(\bar{z}) + \sum_{j=0}^{\ell-1} \bar{z}^j \tilde{g}_j(0).
\]
Both sides of this equation are power series in \( \bar{z} \). Taking note of (6) and (8) for \( j = 0 \), we conclude that \( g_j(0) = \tilde{g}_j(0) \) for \( j = 0, \ldots, \ell - 1 \) and \( h_0 = \tilde{h}_0 \). Combining this with (9), we now find that the two representations of \( f^w \) are identical.

When \( k = 2\ell \), we have to show that the representation of statement (i) holds with \( \ell \) replaced by \( \ell + 1 \). This time, the induction hypothesis applies to \( D_1 f \). With these indications, the proof becomes analogous to the previous one. We therefore do not present the details. □

If \( f \in \mathcal{H}_{k,\sigma} \), we would like to have that each of the entire functions \( g_j \) and \( h_j \) appearing in the representations of Theorem 1 is of exponential type \( \sigma \). However, it is not so obvious how to deduce the growth of each term on the right-hand sides of (4) and (5) from the growth of the whole sum. In the case \( k = 2 \), we could follow Boas [9] who employed an inequality of Carathéodory. Unfortunately, this approach does not extend to \( k > 2 \). Nevertheless, the desired result holds.

**Theorem 2.** Let \( k \in \mathbb{N} \). For \( f \in \mathcal{H}_{k,\sigma} \), the representations of Theorem 1 hold with the functions \( g_0, \ldots, g_{\ell - 1} \) and \( h_0, \ldots, h_{\ell - 1} \) belonging to \( \mathcal{E}_\sigma \).

For the proof of Theorem 2, we shall need an extension of the integral formula
\[
\frac{1}{2\pi i} \oint_{|\zeta - z| = 1} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta = f'(z)
\]
from analytic functions \( f \) to polyharmonic functions; see (11) below. As an intermediate step, we first prove the following auxiliary result.

**Lemma 3.** Let \( g \) and \( h \) be entire functions, and let \( j \) be a nonnegative integer. Denote by \( \gamma \) the circle given by \( \zeta = z + e^{it} \) for \( 0 \leq t \leq 2\pi \). Then
\[
\frac{1}{2\pi i} \int_{\gamma} \frac{\bar{\zeta}^j g(\zeta)}{(\zeta - z)^2} \, d\zeta = \sum_{v=0}^{j} \left( \begin{array}{c} j \\ v \end{array} \right) \bar{z}^v g^{(j-v+1)}(z) (j - v + 1)!
\]
and
\[
\frac{1}{2\pi i} \int_{\gamma} \frac{\bar{\zeta}^j h(\zeta)}{(\zeta - z)^2} \, d\zeta = \sum_{v=0}^{j-1} \left( \begin{array}{c} j \\ v \end{array} \right) \bar{z}^v h^{(j-v-1)}(z) (j - v - 1)!
\]

**Proof.** Let \( \phi \) be an entire function, and let \( \mu \) be an integer. Then, by Cauchy’s integral formula,
\[
\frac{1}{2\pi i} \int_{0}^{2\pi} e^{-i\mu t} \phi(z + e^{it}) \, dt = \frac{1}{2\pi i} \oint_{\gamma} \frac{\phi(\zeta)}{(\zeta - z)^{\mu+1}} \, d\zeta
\]
\[
= \begin{cases} 
\frac{\phi^{(\mu)}(z)}{\mu!} & \text{if } \mu \geq 0, \\
0 & \text{if } \mu < 0.
\end{cases}
\] (10)
Next we note that, by using the binomial formula, the integrals of the lemma can be written as linear combinations of integrals of the form (10). Indeed,
\begin{align*}
\frac{1}{2\pi i} \int_\gamma \frac{\bar{\zeta}^j g(\zeta)}{(\bar{\zeta} - z)^2} \, d\zeta &= \frac{1}{2\pi} \int_0^{2\pi} \left( \bar{\zeta} + e^{-it} \right)^j g(z + e^{it}) e^{-it} \, dt \\
&= \sum_{v=0}^j \binom{j}{v} \bar{\zeta}^v \cdot \frac{1}{2\pi} \int_0^{2\pi} e^{-i(j-v-1)t} g(z + e^{it}) \, dt.
\end{align*}

Similarly, but with the additional transformation $t \mapsto 2\pi - t$, we find that
\begin{align*}
\frac{1}{2\pi i} \int_\gamma \frac{\bar{\zeta}^j h(\zeta)}{(\zeta - z)^2} \, d\zeta &= \sum_{v=0}^j \binom{j}{v} z^v \cdot \frac{1}{2\pi} \int_0^{2\pi} e^{-i(j-v-1)t} h(\bar{\zeta} + e^{it}) \, dt.
\end{align*}

Now the proof is easily completed by using formula (10) for calculating the integrals on the right-hand sides. □

**Proof of Theorem 2.** Again we proceed by induction on $k$. For $k = 1$, the statement is trivial. Now let $f \in H_{k+1, \sigma}$.

If $k = 2\ell - 1$, then the representation (5) holds for $f^w$. Applying Lemma 3, we find that
\begin{equation}
\frac{1}{2\pi i} \int_\gamma \frac{f^w(\zeta, \bar{\zeta})}{(\zeta - z)^2} \, d\zeta = \sum_{j=0}^{\ell-2} \left( \bar{\zeta}^j G_j(z) + z^j H_j(\bar{\zeta}) \right) + \bar{\zeta}^{\ell-1} G_{\ell-1}(z),
\end{equation}
where $G_j$ is a linear combination of
$$
g'_j, g''_j, \ldots, g^{(\ell-j)}_{\ell-1} \quad (j = 0, \ldots, \ell - 1)
$$
and $H_j$ is a linear combination of
$$
h_{j+1}, h'_{j+2}, \ldots, h^{(\ell-j-2)}_{\ell-1} \quad (j = 0, \ldots, \ell - 2).
$$

Clearly, the right-hand side of (11) may be written as $F^w(z, \bar{\zeta})$ for some $F \in H_k$. Using this and turning to the left-hand side of (11), we have
\begin{equation}
F^w(z, \bar{\zeta}) = \frac{1}{2\pi i} \int_\gamma \frac{f^w(\zeta, \bar{\zeta})}{(\zeta - z)^2} \, d\zeta = \frac{1}{2\pi} \int_0^{2\pi} e^{-it} f^w(z + e^{it}, \bar{\zeta} + e^{-it}) \, dt.
\end{equation}

Now take any $\varepsilon > 0$. Since $f$ is of exponential growth at most $\sigma$, we know that
\[|f^w(\zeta, \bar{\zeta})| \leq e^{(\sigma+\varepsilon)|\zeta|}\]
for all sufficiently large $|\zeta|$. Thus, it follows from (12) that
\[|F^w(z, \bar{\zeta})| \leq e^{(\sigma+\varepsilon)(|z|+1)}\]
for all sufficiently large $|z|$. This shows that $F \in H_{k, \sigma}$.

We now want to apply the induction hypothesis to the right-hand side of (11) but, unfortunately, the quantities
$$
H_j(0), \ldots, H^{(\ell-1)}_j(0) \quad (j = 0, \ldots, \ell - 2)
$$
need not be all zero. However, as is easily seen, the unique representation of $F^w$, guaranteed by statement (i) of Theorem 1, can be obtained by cancelling the first $\ell$ Taylor coefficients of $H_0, \ldots, H_{\ell-2}$ and correcting this change by modifying the first $\ell-1$ Taylor coefficients of $G_0, \ldots, G_{\ell-1}$ appropriately. This manipulation does not affect the exponential growth of any of these functions. Thus, we may apply the induction hypothesis and conclude that $G_0, \ldots, G_{\ell-1} \in \mathcal{E}_{\sigma}$ and $H_0, \ldots, H_{\ell-2} \in \mathcal{E}_{\sigma}$. Since an entire function and its derivatives and primitives are of the same exponential type [8, Theorem 2.4.1], we find recurrently that

$$g_{\ell-1}, g_{\ell-2}, \ldots, g_0 \in \mathcal{E}_{\sigma} \quad \text{and} \quad h_{\ell-1}, h_{\ell-2}, \ldots, h_1 \in \mathcal{E}_{\sigma}.$$ 

Now (5) implies that also $h_0 \in \mathcal{E}_{\sigma}$. This completes the proof for odd $k$.

If $k$ is even, then we consider the integral

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f^w(\zeta, \bar{z})}{(\zeta - \bar{z})^2} \, d\bar{z}$$

and argue analogously to the case of odd $k$. □

As a consequence of Theorem 2, we obtain the following representation which is very convenient for proofs by induction on $k$.

**Corollary 4.** Let $f \in \mathcal{H}_{k,\sigma}$, where $k \geq 2$. Then there exists a function $g \in \mathcal{E}_{\sigma}$ and a function $\varphi \in \mathcal{H}_{k-1,\sigma}$ such that

$$f^w(z, \bar{z}) = g(z) + \int_{\bar{z}}^z \varphi^w(\zeta, z) \, d\zeta. \quad (13)$$

**Proof.** If $k = 2\ell$, then the representation (5) holds with $g_0, \ldots, g_{\ell-1}$ and $h_0, \ldots, h_{\ell-1}$ belonging to $\mathcal{E}_{\sigma}$. Setting

$$g(z) := \sum_{j=0}^{\ell-1} z^j (g_j(z) + h_j(z)),$$

we can rewrite (5) as

$$f^w(z, \bar{z}) = g(z) + \sum_{j=0}^{\ell-1} \left[ (\bar{z}^j - z^j) g_j(z) + z^j (h_j(\bar{z}) - h_j(z)) \right]$$

$$= g(z) + \int_{\bar{z}}^z \sum_{j=0}^{\ell-1} (jz^{j-1} g_j(z) + z^j h'_j(z)) \, d\zeta.$$

Since

$$\varphi^w(z, \bar{z}) := \sum_{j=0}^{\ell-1} \left( jz^{j-1} g_j(\bar{z}) + \bar{z}^j h'_j(z) \right)$$

is the Wirtinger form of a function $\varphi \in \mathcal{H}_{2\ell-1,\sigma}$, we see that the desired representation holds.
The proof for $k = 2\ell + 1$ is analogous. □

3. Uniqueness theorems

An important step towards a sampling theorem are uniqueness theorems. In fact, for a function $f$ to be reconstructed it is necessary that $f$ is uniquely determined by the samples used in the reconstruction formula.

Here, we present two uniqueness theorems. The first one requires the square integrability of the sampled functions and is suited for sampling at a lowest rate. The second one does not need any square integrability but it can only be used for oversampling.

**Theorem 5.** For $k \in \mathbb{N}$, let $f \in \mathcal{H}_{k,\sigma}$, and suppose that $D_j f(\cdot, 0) \in L^2(\mathbb{R})$ for $j = 0, \ldots, k - 1$. If
\[
D_j f \left( \frac{n\pi}{\sigma}, 0 \right) = 0 \quad (n \in \mathbb{Z}, \ j = 0, \ldots, k - 1),
\]
then $f$ is identically zero on $\mathbb{R}^2$.

**Theorem 6.** For $k \in \mathbb{N}$, let $f \in \mathcal{H}_{k,\tau}$, where $0 \leq \tau < \sigma$. If (14) holds, then $f$ is identically zero on $\mathbb{R}^2$.

**Proof of Theorem 5.** Again we proceed by induction on $k$. When $k = 1$, the hypotheses imply that $f^w \in B^2_\sigma$, and therefore the statement is a consequence of Theorem A.

Next, let $f \in \mathcal{H}_{k+1,\sigma}$. Suppose that $D_j f(\cdot, 0) \in L^2(\mathbb{R})$ for $j = 0, \ldots, k$, and assume that (14) also holds for $j = k$. We have to show that $f$ is identically zero.

By Corollary 4, the Wirtinger form of $f$ has the representation (13) with $g \in \mathcal{E}_\sigma$ and $\varphi \in \mathcal{H}_{k,\sigma}$. Applying the operator $D_j$ on both sides of (13), we find that
\[
\left( D_j f \right)^w (z, \bar{z}) = \left( D_{j-1} \varphi \right)^w (\bar{z}, z) \quad (j = 1, \ldots, k),
\]
or equivalently,
\[
D_j f(x, y) = D_{j-1} \varphi(x, -y) \quad (j = 1, \ldots, k). \tag{15}
\]
This shows that the induction hypothesis applies to $\varphi$. It yields that $\varphi$ is identically zero. Hence, in the present situation, the representation (13) reduces to $f^w(z, \bar{z}) = g(z)$, where $g \in B^2_\sigma$ and $g(n\pi/\sigma) = 0$ for $n \in \mathbb{Z}$. Now, again, Theorem A may be used to conclude that $f = 0$. □

**Proof of Theorem 6.** We argue as in the proof of Theorem 5 except that instead of Theorem A we always use a theorem of Carlson [8, p. 153]. The latter implies that if $f \in \mathcal{E}_\tau$, where $\tau < \sigma$, and $f(n\pi/\sigma) = 0$ for $n \in \mathbb{Z}$, then $f$ is identically zero. □

4. Construction of fundamental functions

In view of the uniqueness theorems, we may think of reconstructing $f \in \mathcal{H}_{k,\sigma}$ from the samples
\[
D_j f \left( \frac{n\pi}{\sigma}, 0 \right) \quad (n \in \mathbb{Z}, \ j = 0, \ldots, k - 1).
\]
In order to simplify the situation a little, we want to restrict ourselves to the case where \( |\alpha| = |\beta| \), which makes the integers to be the sampling points. This is no loss of generality since, given \( f \in \mathcal{H}_{k, \sigma} \), where \( \sigma > 0 \), we may first reconstruct the function

\[
F : (x, y) \mapsto f \left( \frac{x}{\sigma}, \frac{y}{\sigma} \right),
\]

which belongs to \( \mathcal{H}_{k, \pi} \), and recover \( f \) by scaling the argument of \( F \) by \( \sigma/\pi \).

The fundamental functions for the envisaged sampling formula can be obtained by appropriate integrations of the sinc function

\[
sinc z := \begin{cases} 
\frac{\sin \pi z}{\pi z} & \text{if } z \in \mathbb{C} \setminus \{0\}, \\
1 & \text{if } z = 0.
\end{cases}
\]

The following theorem holds. It can be verified by straightforward calculations.

**Theorem 7.** Define

\[
A^w_0(z, \bar{z}) := \text{sinc } z, \quad A^w_j(z, \bar{z}) := \int_{\bar{z}}^z A^w_{j-1}(\zeta, z) \, d\zeta \quad (j \in \mathbb{N}).
\]

Then \( A^w_j \) is the Wirtinger form of a function \( A_j \in \mathcal{H}_{j+1, \pi} \). Furthermore, for odd \( \ell \in \mathbb{N} \),

\[
D_\ell A_j(x, y) = \begin{cases} 
0 & \text{if } j < \ell, \\
\text{sinc } \bar{z} & \text{if } j = \ell, \\
A^w_{j-\ell}(\bar{z}, z) & \text{if } j > \ell,
\end{cases}
\]

and, for even \( \ell \in \mathbb{N}_0 \),

\[
D_\ell A_j(x, y) = \begin{cases} 
0 & \text{if } j < \ell, \\
\text{sinc } z & \text{if } j = \ell, \\
A^w_{j-\ell}(z, \bar{z}) & \text{if } j > \ell.
\end{cases}
\]

This shows that the functions \( A_j \) may serve as fundamental functions. Indeed, for \( y = 0 \), we have \( z = \bar{z} = x \), and so Theorem 7 reduces to the following statement.

**Corollary 8.** For the functions \( A_j \), defined in Theorem 7, we have

\[
D_\ell A_j(x, 0) = \delta_{\ell j} \text{sinc } x \quad (x \in \mathbb{R}, \ell, j \in \mathbb{N}_0),
\]

where Kronecker’s delta is used on the right-hand side.

In order to express these fundamental functions by known special functions, we may use the sine integral

\[
\text{Si}(z) = \int_0^z \frac{\sin t}{t} \, dt = \sum_{v=0}^{\infty} \frac{(-1)^v}{2v+1} \cdot \frac{z^{2v+1}}{(2v+1)!} =: \text{Si}_0(z),
\]
see [1, §5.221, p. 936, No. 8.230], and its successive primitives

\[ \text{Si}_\ell(z) := \int_0^z \text{Si}_{\ell-1}(t) \, dt = \sum_{v=0}^{\infty} \frac{(-1)^v}{2v+1} \cdot \frac{z^{2v+\ell+1}}{(2v+\ell+1)!} \quad (\ell \in \mathbb{N}). \]

By a calculation, we find that

\[ A^w_1(z, \bar{z}) = \frac{1}{\pi} \left[ \text{Si}(\pi \bar{z}) - \text{Si}(\pi z) \right], \]

\[ A^w_2(z, \bar{z}) = \frac{1}{\pi^2} \left[ \text{Si}_1(\pi z) - \text{Si}_1(\pi \bar{z}) \right] - \frac{z - \bar{z}}{\pi} \text{Si}(\pi z), \]

\[ A^w_3(z, \bar{z}) = \frac{2}{\pi^3} \left[ \text{Si}_2(\pi z) - \text{Si}_2(\pi \bar{z}) \right] + \frac{z - \bar{z}}{\pi^2} \left[ \text{Si}_1(\pi z) + \text{Si}_1(\pi \bar{z}) \right], \]

\[ A^w_4(z, \bar{z}) = \frac{3}{\pi^4} \left[ \text{Si}_3(\pi z) - \text{Si}_3(\pi \bar{z}) \right] - \frac{z - \bar{z}}{\pi^3} \left[ 2 \text{Si}_2(\pi z) + \text{Si}_2(\pi \bar{z}) \right] + \frac{(z - \bar{z})^2}{2\pi^2} \text{Si}_1(\pi z). \]

5. A sampling theorem

We are now ready for presenting the announced sampling theorem.

**Theorem 9.** For \( k \in \mathbb{N} \), let \( f \in \mathcal{H}_{k,\pi} \), and suppose that \( D_j f(\cdot, 0) \in L^2(\mathbb{R}) \) for \( j = 0, \ldots, k-1 \). Then

\[ f(x, y) = \sum_{n=-\infty}^{\infty} f(n, 0) \, \text{sinc}(x + iy - n) + \sum_{j=1}^{k-1} \sum_{n=-\infty}^{\infty} D_j f(n, 0) \, A_j(x - n, y) \quad (16) \]

for \((x, y) \in \mathbb{R}^2\). The series converge absolutely and uniformly on every strip \(|y| \leq K\) for any \( K > 0 \).

**Proof.** We proceed by induction on \( k \). For \( k = 1 \) the hypotheses are equivalent to \( f^w \in \mathcal{B}^2_{\pi} \), and therefore the statement reduces to Theorem A.

Now suppose that \( f \in \mathcal{H}_{k+1,\pi} \) and \( D_j f(\cdot, 0) \in L^2(\mathbb{R}) \) for \( j = 0, \ldots, k \). By Corollary 4, the Wirtinger form of \( f \) has the representation (13) with \( g \in \mathcal{E}_\pi \) and \( \varphi \in \mathcal{H}_{k,\pi} \). This shows that \( f(x, 0) = g(x) \) for \( x \in \mathbb{R} \). Hence \( g \in \mathcal{B}^2_{\pi} \), and so, by Theorem A,

\[ g(x + iy) = \sum_{n=-\infty}^{\infty} f(n, 0) \, \text{sinc}(x + iy - n). \quad (17) \]

Next, recalling (15), we see that the induction hypothesis applies to \( \varphi \). The corresponding reconstruction formula (16) for \( \varphi \) may now be written as

\[ \varphi(x, y) = \sum_{n=-\infty}^{\infty} D_1 f(n, 0) \, \text{sinc}(x + iy - n) + \sum_{j=1}^{k-1} \sum_{n=-\infty}^{\infty} D_{j+1} f(n, 0) \, A_j(x - n, y), \]
where the series converge absolutely and uniformly on every strip $|\Im z| \leq K$ with $K > 0$. From this we conclude that

$$
\int_{\bar{z}} \varphi^w(\zeta, z) \, d\zeta = \sum_{n=-\infty}^{\infty} \frac{D_1 f(n, 0)}{\pi} \int_{\bar{z}} A_0^w(\zeta - n, z - n) \, d\zeta \\
+ \sum_{j=1}^{k-1} \sum_{n=-\infty}^{\infty} \frac{D_j+1 f(n, 0)}{\pi} \int_{\bar{z}} A_j^w(\zeta - n, z - n) \, d\zeta \\
= \sum_{j=1}^{k} \sum_{n=-\infty}^{\infty} D_j f(n, 0) A_j^w(z - n, \bar{z} - n). \tag{18}
$$

Note that the integrated series are again absolutely and uniformly convergent for $|\Im z| \leq K$ since the integration extends along a line segment of length at most $2K$. Finally, combining (13), (17), and (18), we obtain the conclusion of Theorem 9 for $k + 1$. □

Remark 10. Alternatively, Theorem 9 can be proved by following a standard way of verifying an interpolation formula. First one shows that the right-hand of (16) exists and represents some function $\varphi \in H_{k,\pi}$ such that $D_j \varphi(\cdot, 0) \in L^2(\mathbb{R})$ for $j = 0, \ldots, k - 1$. Next, the properties of the fundamental functions $A_j$ imply that

$$
D_j \varphi(n, 0) = D_j f(n, 0) \quad (n \in \mathbb{Z}, j = 0, \ldots, k - 1).
$$

Hence Theorem 5 applies to $f - \varphi$ and yields that (16) holds.

The idea underlying Theorem 9 simply is that derivatives $D_j f$ which do not vanish identically have to be sampled and included in the reconstruction formula. Note that Theorem 9 is not an analogue but rather an extension of Theorem A. If $f$ belongs to $H_{1,\pi}$, which is a subspace of $H_{k,\pi}$, then (16) reduces to (2) even if $k$ is taken bigger than 1 since $D_j f = 0$ for $j \in \mathbb{N}$. Furthermore, for any $f \in H_{k,\pi}$, formula (16) coincides with (2) on the real line since $A_j(x, 0) = 0$ for $x \in \mathbb{R}$ and $j \in \mathbb{N}$.

6. Supplements, variants, generalizations, and an algorithm

For the subsequent considerations, we recall the following standard notation.

Let $p \in [1, \infty]$. For a sequence $c = (c_n)_{n \in \mathbb{Z}}$ of complex numbers, we say that $c$ belongs to $l^p$ and write $c \in l^p$ if

$$
\|c\|_p := \begin{cases} 
\left( \sum_{n=-\infty}^{\infty} |c_n|^p \right)^{1/p} & \text{for } p \in [1, \infty), \\
\sup_{n \in \mathbb{Z}} |c_n| & \text{for } p = \infty
\end{cases}
$$

is finite. A sequence belonging to $l^2$ is said to be square summable.

The same symbol $\| \cdot \|_p$ will also be used for the norm of functions $f \in L^p(\mathbb{R})$, that is,

$$
\|f\|_p := \left( \int_{-\infty}^{\infty} |f(x)|^p \, dx \right)^{1/p} \quad \text{for } p \in [1, \infty).
$$

Whenever $\|f\|_\infty$ occurs, the function $f$ will be from a subspace of $L^\infty(\mathbb{R})$ for which $\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$. 
From the context, it will always be clear whether $\| \cdot \|_p$ applies to a sequence or to a function.

6.1. Polyharmonic continuation of a signal

Let $\phi : \mathbb{R} \to \mathbb{C}$ be a signal of finite energy, bandlimited to $[-\pi, \pi]$, and let $(c_{nj})_{n \in \mathbb{Z}}$, where $j = 1, \ldots, k - 1$, be any square summable sequences of complex numbers. Then

$$f(x, y) := \sum_{n=-\infty}^{\infty} \phi(n) \text{sinc}(x + iy - n) + \sum_{j=1}^{k-1} \sum_{n=-\infty}^{\infty} c_{nj} A_j(x - n, y)$$

is a continuation of $\phi$ to a function $f \in \mathcal{H}_{k, \pi}$ such that $f(x, 0) \equiv \phi(x)$ for $x \in \mathbb{R}$ and

$$D_j f(n, 0) = c_{nj} \quad (n \in \mathbb{Z}, j = 1, \ldots, k - 1).$$

This leads us to the following observation. While a signal of finite energy and bandlimited to $[-\pi, \pi]$ has a unique extension to a function in $\mathcal{E}_\pi$, it has at the same time a whole family of extensions to functions in $\mathcal{H}_{k, \pi}$, where the cardinality of this family increases with $k$.

6.2. The sampling operator seen as an isomorphism

Theorem A has an interesting supplement which we formulate for $\sigma = \pi$. If $c = (c_n)_{n \in \mathbb{Z}} \in l^2$, then, as is well known and can be easily verified, the series

$$\sum_{n=-\infty}^{\infty} c_n \text{sinc}(z - n)$$

converges in the $L^2$ norm and also absolutely and uniformly on strips $|\Im z| < K$, and represents a function $g \in \mathcal{B}^2_\pi$ such that $g(n) = c_n$ for all $n \in \mathbb{Z}$ and $\|g\|_2 = \|c\|_2$. This shows that the sampling operator

$$I : \{ \mathcal{B}^2_\pi \to l^2, \quad f \mapsto (f(n))_{n \in \mathbb{Z}} \}$$

is an isometric isomorphism and the sampling series yields an explicit representation of $I^{-1}$.

A similar supplement holds for Theorem 9. Let

$$\mathcal{H}^2_{k, \pi} := \left\{ f \in \mathcal{H}_{k, \pi} : D_j f(\cdot, 0) \in L^2(\mathbb{R}) \text{ for } j = 0, \ldots, k - 1 \right\}.$$

We may endow $\mathcal{H}^2_{k, \pi}$ with the norm

$$\|f\|_2 := \left( \sum_{j=0}^{k-1} \|D_j f(\cdot, 0)\|_2^2 \right)^{1/2}.$$

Analogously, we may endow $(l^2)^k$, the $k$-fold cartesian product of $l^2$, with the norm

$$\|(c_1, \ldots, c_k)\|_2 := \left( \sum_{j=1}^{k} \|c_j\|_2^2 \right)^{1/2}.$$
where \( \epsilon_j \in \ell^2 \) for \( j = 1, \ldots, k \). Then the sampling operator

\[
I_k : \begin{cases} \mathcal{H}^2_{k, \pi} \to (\ell^2)^k, \\ f \mapsto (D_0 f(n, 0))_{n \in \mathbb{Z}}, \ldots, (D_{k-1} f(n, 0))_{n \in \mathbb{Z}} \end{cases}
\]

is an isometric isomorphism. In particular,

\[
\| D_j f(\cdot, 0) \|_2 = \|(D_j f(n, 0))_{n \in \mathbb{Z}}\|_2 \quad (j = 0, \ldots, k - 1).
\]

Furthermore, the formula (16) yields an explicit representation of \( I_{k-1}^{-1} \).

### 6.3. Oversampling with a convergence factor

For \( \epsilon \in (0, \pi) \), let \( \psi \in \mathcal{E}_\epsilon \) such that \( \psi(0) = 1 \). A variant of Theorem A with \( \psi \) as a convergence factor may be stated as follows.

**Theorem B.** Let \( f \in \mathcal{E}_\tau \), where \( 0 \leq \tau < \pi - \epsilon \). Suppose that the sequence

\[
(f(n)\psi(w - n))_{n \in \mathbb{Z}}
\]

is square summable for each \( w \in \mathbb{C} \). Then

\[
f(z) = \sum_{n = -\infty}^{\infty} f(n)\psi(z - n) \operatorname{sinc}(z - n) \quad (z \in \mathbb{C}).
\]

The series converges absolutely and uniformly on all compact subsets of \( \mathbb{C} \).

Probably Maria Theis [46, §§4–6] in 1919 was the first to incorporate a convergence factor into the classical sampling series. She was inspired by the Fejér means in the theory of Fourier series, which motivated her to use the sinc function as a multiplier \( \psi \). Since then, numerous special cases, modifications or generalizations of Theorem B have appeared in the literature; see, e.g., [12,13,20,29,33,38]. For the reader’s convenience, we give a short proof of Theorem B although it is essentially known.

**Proof of Theorem B.** By a theorem in [8, p. 197, Theorem 10.6.4], the square summability of the sequence (19) implies that the restriction to \( \mathbb{R} \) of \( f \psi(w - \cdot) \in \mathcal{E}_{\tau + \epsilon} \), and so, by Theorem A,

\[
f(z)\psi(w - z) = \sum_{n = -\infty}^{\infty} f(n)\psi(w - n) \operatorname{sinc}(z - n) \quad (z, w \in \mathbb{C}).
\]

The proof is completed by substituting \( w = z \). \( \square \)

The isomorphism described in §6.2 in connection with Theorem A does not extend to the sampling operator of Theorem B, but the freedom in the choice of \( \psi \) allows us to gain other desirable properties. If

\[
\psi(x) = O \left( |x|^{-1} \right) \quad \text{as } x \to \pm \infty,
\]

then...
then, for each \( p \in [1, \infty] \), the operator
\[
C_\psi : \begin{cases}
lp \rightarrow L^p(\mathbb{R}), \\
(c_n)_{n \in \mathbb{Z}} \mapsto \sum_{n=-\infty}^{\infty} c_n \psi(\cdot - n) \text{sinc}(\cdot - n)
\end{cases}
\]
is well defined and bounded.

Now, let
\[
M := \sup_{x \in \mathbb{R}} \sum_{n=-\infty}^{\infty} |\psi(x - n) \text{sinc}(x - n)|
\]
and suppose that the samples \( f(n) \) have deviations bounded in modulus by \( \delta \). Such deviations may come from round-off errors or from tolerances of a device that produces the samples. Then Theorem A fails while (20) produces a function \( \tilde{f} \), say, such that
\[
\sup_{|z| \leq K} |f(z) - \tilde{f}(z)| \leq \delta M e^{(\pi+\varepsilon)K}.
\]
However, membership in a Bernstein class is not preserved under such perturbations. If \( f \in B^\infty_{\tau} \), where \( 0 \leq \tau < \pi - \varepsilon \), then \( \tilde{f} \) can only be guaranteed to belong to \( B^\infty_{\tau+\varepsilon} \). This is not really a problem in practice.

For an extension of Theorem B to polyharmonic functions, we define fundamental functions \( B_j \) by
\[
B_0^w(z, \bar{z}) := \psi(z) \text{sinc} z, \quad B_j^w(z, \bar{z}) := \int_{\bar{z}}^z B_{j-1}^w(\zeta, z) \, d\zeta \quad (j \in \mathbb{N}).
\]
Now the desired result can be stated as follows.

**Theorem 11.** For \( k \in \mathbb{N} \), let \( f \in \mathcal{H}_{k, \tau} \), where \( 0 \leq \tau < \pi - \varepsilon \). Suppose that the sequences
\[
(D_j f(n) \psi(w - n))_{n \in \mathbb{Z}} \quad (j = 0, \ldots, k - 1)
\]
are square summable for each \( w \in \mathbb{C} \). Then
\[
f(x, y) = \sum_{j=0}^{k-1} \sum_{n=-\infty}^{\infty} D_j f(n, 0) B_j(x - n, y) \quad ((x, y) \in \mathbb{R}^2).
\]
The series converge absolutely and uniformly on all compact subsets of \( \mathbb{R}^2 \).

The proof is essentially the same as that of Theorem 9 except that Theorem B takes the role of Theorem A.

The aforementioned stability properties of the sampling series (20) are inherited by that of Theorem 11. Also note that if \( |\psi(x)| \) decays rapidly as \( x \to \pm \infty \), then the sequences \( (D_j f(n, 0))_{n \in \mathbb{Z}} \) need not be bounded. Furthermore, the speed of convergence of the series (22) improves.
Theorem 11 may be refined by using an individual convergence factor $\psi_j$ for each of the functions $D_j f$. For example, we may define

$$B^w_{0,\ell}(z, \bar{z}) := \psi_\ell(z) \text{sinc} \bar{z}$$

$$B^w_{j,\ell}(z, \bar{z}) := \int_\bar{z}^z B^w_{j-1,\ell}(\zeta, z) d\zeta \quad (\ell, j \in \mathbb{N})$$

and set $B_j := B^j_{j,j}$. If, instead of (21), the sequences $(D_j f(n)\psi_j(w-n))_{n \in \mathbb{Z}}$ are square summable for each $w \in \mathbb{C}$, then the conclusion of Theorem 11 also holds.

6.4. Nonuniform sampling

First we introduce a class of sampling points which has been frequently used in nonuniform sampling and which may be attributed to Duffin and Schaeffer [17].

**Definition 12.** Let $L$ be a positive real number and let $\delta \in (0, 1]$. Then $\mathcal{N}(L, \delta)$ denotes the set of all sequences $(t_n)_{n \in \mathbb{Z}}$ with the following properties: for each $n \in \mathbb{Z}$,

(i) $t_n \in \mathbb{R}$ and $t_n \neq 0$ if $n \neq 0$;

(ii) $|t_n - n| \leq L$;

(iii) $t_{n+1} - t_n > \delta$.

We may consider the sequences in $\mathcal{N}(L, \delta)$ as displaced integers which are separated by $\delta$.

When $L \in [0, 1/2)$, condition (iii) is automatically satisfied for any $\delta \in (0, 1 - 2L)$. In this case, we drop (iii) and simply write $\mathcal{N}(L)$ for the set of all sequences satisfying (i) and (ii).

For every sequence $t := (t_n)_{n \in \mathbb{Z}} \in \mathcal{N}(L, \delta)$, the infinite product

$$G(t; z) := (z - t_0) \prod_{n=1}^{\infty} \left( 1 - \frac{z}{t_n} \right) \left( 1 - \frac{z}{t_{n-1}} \right)$$

exists and

$$G_n(t; z) := \frac{G(t; z)}{(z - t_n)G'(t; t_n)} \quad (n \in \mathbb{Z})$$

represent fundamental functions of Lagrange interpolation with respect to the sequence of nodes $t$, that is,

$$G_n(t; t_\ell) = \delta_{n\ell} \quad (n, \ell \in \mathbb{Z})$$

see [8, §§2.6 and 10.5].

The following Theorem C is a generalization of Theorem A to nonuniform sampling points. In a preliminary form, it is due to Paley and Wiener [37, p. 115, Theorem 39] who had a stronger restriction on the sampling points. In the presented form, it follows from a result of Levinson [34, p. 48]. Its full power shows in conjunction with a result by Kadec [30]; also see [26, p. 108, Theorem 10.11], [51, §3.1].

**Theorem C.** Let $f \in B^2_n$ and let $t := (t_n)_{n \in \mathbb{Z}} \in \mathcal{N}(L)$, where $L \in [0, 1/4)$. Then

$$f(z) = \sum_{n=-\infty}^{\infty} f(t_n)G_n(t; z) \quad (z \in \mathbb{C}).$$

The series converges in the $L^2$ norm and uniformly on every strip $|\Im z| \leq K$ for any $K > 0$. 
Theorem of Kadec guarantees that the set of sampling points specified in Theorem C is both a set of stable sampling and a set of interpolation for $B^2_n$; see [26, §10.2]. This allows us to conclude that the sampling operator

$$I: \left\{ B^2_n \to l^2, \quad f \mapsto (f(t_n))_{n \in \mathbb{Z}} \right\}$$

is again an isomorphism (not necessarily an isometric one) and the sampling series of Theorem C yields an explicit representation of $I^{-1}$. The restriction $L < 1/4$ is best possible. It is often referred to as the “Kadec one quarter condition”.

Analogously to the functions $A_j$ in Theorem 7, we now define

$$C_{0,n}(t; z, \bar{z}) := G_n(t; z), \quad C_{j,n}(t; z, \bar{z}) := \int_{\bar{z}}^{z} C_{j-1,n}(t; \zeta, z) \, d\zeta \quad (n \in \mathbb{Z}, j \in \mathbb{N}).$$

An extension of Theorem C to polyharmonic functions may be stated as follows.

**Theorem 13.** For $k \in \mathbb{N}$, let $f \in \mathcal{H}_{k,\pi}^2$ and let

$$t_j := (t_{j,n})_{n \in \mathbb{Z}} \in \mathcal{N}(L_j),$$

where $L_j \in [0, 1/4)$ for $j = 0, \ldots, k - 1$. Then

$$f(x, y) = \sum_{j=0}^{k-1} \sum_{n=-\infty}^{\infty} D_j f(t_{j,n}, 0) C_{j,n}(t_j; x, y) \quad ((x, y) \in \mathbb{R}^2).$$

The double series converges in the norm $1 \cdot 1_2$ and uniformly on every strip $|y| \leq K$ for any $K > 0$.

We omit the proof since it is analogous to that of Theorem 9 with Theorem C taking the role of Theorem A.

The sampling operator that maps $f$ to

$$\left( (D_0 f(t_{0,n}, 0))_{n \in \mathbb{Z}}, \ldots, (D_{k-1} f(t_{k-1,n}, 0))_{n \in \mathbb{Z}} \right)$$

is an isomorphism from $\mathcal{H}_{k,\pi}^2$ to $(l^2)^k$, and the sampling formula of Theorem 13 represents its inverse.

If we use the sampling formula of Theorem 13 for $f \in \mathcal{H}_{k,\tau}^2$, where $\tau \in [0, \pi)$, then we have the situation of oversampling. In this case, the sampling operator cannot be an isomorphism, but now we can gain other desirable properties. For entire functions of exponential type (the case $k = 1$), it is known that we can incorporate a convergence factor $\psi$ into the sampling formula of Theorem C and obtain stability properties as described in §6.3. Moreover, as sampling points we may take any sequence from $\mathcal{N}(L, \delta)$ for arbitrary $L \geq 0$ and $\delta > 0$ provided that we use a convergence factor

$$\psi(z) = \left[ \text{sinc} \left( \frac{\varepsilon}{\pi} z \right) \right]^\gamma$$
such that $\gamma \geq L + 1$ and $\varepsilon \gamma < \pi - \tau$. Precise statements with complete proofs, even for a more general setting, can be found in [48, §2.3, §3.1]; also see Chapter3/phd_thesis_voss.zip on the CD-ROM in [36]. For earlier contributions of this kind, see [44,27]. These results may serve as a basis for polyharmonic extensions analogous to Theorem 13. For a contribution in the case $k = 2$, see [43, Theorem 4.2].

For theoretical backgrounds concerning nonuniform sampling, we refer to [7].

6.5. An algorithm

Despite its attractive mathematical properties, Theorem C seems to be not very suitable for numerical work; see Gröchenig [22, §2.2] who lists four disadvantages. These disadvantages are inherited by Theorem 13.

In a series of papers, H.G. Feichtinger and K. Gröchenig have proposed various efficient iterative algorithms for nonuniform sampling of bandlimited signals or entire functions of exponential type. These algorithms were specially designed for the needs in practice. Moreover, the sampling points are less restricted than in the aforementioned sampling theorems except that their density must be above the Nyquist rate; see, e.g., [18,19,22–24] where further references can be located. As a typical example, we consider a “weighted frame operator method” since it extends to polyharmonic functions without much efforts.

Let $L(\delta)$ be the set of all sequences $(t_n)_{n \in \mathbb{Z}}$ such that $t_{n-1} < t_n$ for all $n \in \mathbb{Z}$ and

$$\sup_{n \in \mathbb{Z}} (t_{n+1} - t_n) = \delta.$$ 

The weighted frame operator is defined by

$$S[f](z) := \frac{1}{1 + \delta^2} \sum_{n=-\infty}^{\infty} \frac{t_{n+1} - t_{n-1}}{2} f(t_n) \text{sinc}(z - t_n).$$

Then an algorithm described in [19, Theorem 8.14] and in a slightly modified form in the earlier paper [22, pp. 329–330] may be formulated as follows.

**Theorem D.** Let $f \in B^2_n$ and let $(t_n)_{n \in \mathbb{Z}} \in L(\delta)$, where $\delta \in (0, 1)$. Define

$$f_0 := S[f], \quad f_n := f_{n-1} + S[f - f_{n-1}] \quad (n \in \mathbb{N}).$$

Then $f = \lim_{n \to \infty} f_n$ in $L^2(\mathbb{R})$ and

$$\|f - f_n\|_2 \leq \left( \frac{2\delta}{1 + \delta^2} \right)^{n+1} \|f\|_2.$$ 

It is remarkable that the functions $f_n$ are obtained as expansions with respect to shifted sinc functions. No such complicated functions like $G_n(t, \cdot)$ in Theorem C are needed. Numerical experiments in [19, §8.9] show the efficiency of this approach in comparison with four other methods.
In a polyharmonic extension of Theorem D, the role of the sinc function is taken by the functions \( A_j \) defined in Theorem 7 and expressed in terms of iterated sine integrals.

For \( j = 0, \ldots, k - 1 \), define

\[
S_j[f](x, y) := \frac{1}{1 + \delta_j^2} \sum_{n=-\infty}^{\infty} \frac{t_{j,n+1} - t_{j,n-1}}{2} D_j f(t_{j,n}, 0) A_j(x - t_{j,n}, y).
\]

By induction on \( k \), the following theorem can be proved.

**Theorem 14.** For \( k \in \mathbb{N} \), let \( f \in \mathcal{H}^2_{k,\pi} \) and let

\[
(t_{j,n})_{n \in \mathbb{Z}} \in \mathcal{L}(\delta_j),
\]

where \( \delta_j \in (0, 1) \) for \( j = 0, \ldots, k - 1 \). Define

\[
f_{j,0} := S_j[f], \quad f_{j,n} := f_{j,n-1} + S_j[f - f_{j,n-1}] \quad (n \in \mathbb{N}, j = 0, \ldots, k - 1),
\]

and

\[
f_n := \sum_{j=0}^{k-1} f_{j,n}.
\]

Then \( f = \lim_{n \to \infty} f_n \) in the \( \mathbf{1} \cdot \mathbf{1}_2 \) norm and

\[
\| f - f_n \|_2 \leq \left( \frac{2\delta}{1 + \delta^2} \right)^{n+1} \| f \|_2,
\]

where \( \delta := \max\{\delta_0, \ldots, \delta_{k-1}\} \).

Inequality (24) is obtained as a consequence of the \( k \) inequalities

\[
\| D_j f(\cdot, 0) - D_j f_n(\cdot, 0) \|_2 \leq \left( \frac{2\delta_j}{1 + \delta_j^2} \right)^{n+1} \| D_j f(\cdot, 0) \|_2 \quad (j = 0, \ldots, k - 1).
\]

However, instead of norm estimates on \( \mathbb{R} \), one is rather interested in uniform estimates on strips in the case of planar polyharmonic functions. When \( f \in \mathcal{B}^2_2 \), uniform estimates on horizontal strips are a consequence of norm estimates; see [50, p. 107, inequality (2)]. When \( f \in \mathcal{H}^2_{k,\pi} \), we can again make use of this fact by employing Corollary 4. Defining

\[
(e^z)_0 := e^z \quad \text{and} \quad (e^z)_j := e^z - \sum_{\ell=0}^{j-1} \frac{z^\ell}{\ell!} \quad (j \in \mathbb{N}),
\]

we find that

\[
\sup_{|y| \leq K} |f(x, y)| \leq \sum_{j=0}^{k-1} \left( \frac{2}{\pi} \right)^j \left( e^{\pi K} \right)_j \| D_j f(\cdot, 0) \|_2.
\]
The algorithm of Theorem 14 shows that the functions $f_{j,n}$ have representations

$$f_{j,n}(x, y) = \sum_{v=-\infty}^{\infty} c_{j,n,v} A_j(x - t_{j,v}, y).$$

For computing the coefficients $c_{j,n,v}$, the functions $A_j$ are not needed. In fact, applying $D_j$ and setting $y = 0$, we have, by Theorem 7,

$$D_j f_{j,n}(x, 0) = \sum_{v=-\infty}^{\infty} c_{j,n,v} \text{sinc}(x - t_{j,v}).$$

Applying $D_j$ also to the recurrence formula of the algorithm and introducing

$$w_{j,v} := \frac{t_{j,v+1} - t_{j,v-1}}{2(1 + \delta_j^2)},$$

we find that

$$c_{j,n,v} = c_{j,n-1,v} + c_{j,0,v} - w_{j,v} \sum_{\mu=-\infty}^{\infty} c_{j,n-1,\mu} \text{sinc}(t_{j,v} - t_{j,\mu})$$

$$(v \in \mathbb{Z}, n \in \mathbb{N}, j = 0, \ldots, k - 1)$$

with the initial values

$$c_{j,0,v} = w_{j,v} D_j f(t_{j,v}, 0) \quad (v \in \mathbb{Z}, j = 0, \ldots, k - 1).$$

The structure of these formulae does not depend on $j$. This simplifies an implementation.

References


