The Leitmann–Schmitendorf advertising differential game

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ABSTRACT

The paper revisits the advertising differential game suggested by Leitmann and Schmitendorf [1]. We put the model into perspective and discuss the related issues of (i) state separable (or linear-state) games and (ii) open-loop equilibria that are feedback equilibria which are constant with respect to state.

1. Introduction

This paper revisits the advertising differential game suggested by Leitmann and Schmitendorf [1]. A brief outline of the model and the results of Leitmann and Schmitendorf [1] was published in Leitmann [2].

We put the Leitmann–Schmitendorf model into perspective and discuss the notions of state-separable and state-redundant games, open-loop equilibria, and feedback equilibria that are constant with respect to state. The paper proceeds as follows. Section 2 introduces the Leitmann–Schmitendorf game and relates the model to earlier works in the area of sales response models of competitive advertising. Extensions/modifications of the model are discussed. Section 3 considers the structural feature of differential game models (which is present in the Leitmann–Schmitendorf model), where an open-loop equilibrium is a state-independent feedback equilibrium. The concepts of state-separability (state linearity) and state-redundancy are discussed. Section 4 concludes.

2. Differential games in advertising

The Leitmann–Schmitendorf paper is concerned with optimal advertising in a duopolistic market. The model suggested in the paper belongs to a class of models called Sales-advertising Response Models. The idea of such a model is to let the rate of change of the sales rate of a firm be a function of the sales rate itself and the marketing instruments of the players.

A general sales response model for advertising competition in a duopolistic market is as follows. Let real time be denoted by \( t \geq 0 \) and let \( x_i(t) \) and \( u_i(t) \) represent the sales and advertising rates, respectively, of firm \( i \in \{1, 2\} \). The evolution of the rate of sales over time is given by the differential equation

\[
\dot{x}_i(t) = f_i(t, x_i(t), u_i(t), u_j(t)), \quad i, j \in \{1, 2\}, \quad i \neq j, \quad x_i(0) \geq 0.
\]

Sales response models are discussed in, e.g., Erickson [3], Jørgensen and Zaccour [4]. The Leitmann–Schmitendorf model has, as many sales response models, its origins in the Vidale–Wolfe [5] one-decision maker model and the Lanchester differential game of military combat.
The Vidale–Wolfe model has the sales dynamics
\[ x(t) = g(u(t))[N - x(t)] - \delta x(t) \]
where \( x(t) \) is the sales rate at time \( t \), \( N \) is the maximum sales potential, and \( g \) is a function such that \( g(0) = 0 \). Function \( g \) expresses the efficiency of advertising, the term \( N - x \) is the untapped portion of the market, and the term \( \delta x (\delta > 0 \text{ and constant}) \) reflects decay of sales. Decay may occur due to competitors’ marketing activities, but note that competitors’ actions are not explicitly modeled.

In the Vidale–Wolfe model, sales \( x(t) \) are fixed and the sales dynamics are
\[ x_i(t) = g_i(u_i(t))x_i(t) - h_i(u_j(t))x_j(t), \quad i, j \in \{1, 2\}, \quad i \neq j \]
where \( g_i \) and \( h_i \) are functions that express efficiency of own and the rival firm’s advertising, respectively. A simple choice of these functions is \( g_i(u_i) = u_i, h_i(u_j) = u_j \); see, e.g., Case [6]. Note in (1) that a firm’s advertising affects the rival firm’s customers, not the firm’s own customers. The hypothesis is that a firm’s advertising is designed to make the rival’s customers switch to the firm.

The sales rate in the Leitmann–Schmitendorf model evolves according to
\[ \dot{x}_i(t) = u_i(t) - \frac{c}{2} u_i^2(t) - k x_i(t) - \delta x_i(t) \]
where \( c, \delta, \) and \( k \) are positive parameters. Note that the effect of firm \( i \)’s own advertising \( u_i - c u_i/2 \) does not depend on the remaining market potential as in the Vidale–Wolfe model. However, the Leitmann–Schmitendorf dynamics include a Vidale–Wolfe decay term \( (\delta x_i) \) as well as a Lanchester-term \( k x_i \) reflecting the impact of the competitor’s advertising on the sales of firm \( i \).

**Remark 1.** In the earlier work [7], Leitmann used the following simplification of the dynamics in (2):
\[ \dot{x}_i(t) = 12 u_i(t) - 2 u_i^2(t) - u_i(t) - x_i(t). \]

The objective functional in the Leitmann–Schmitendorf duopoly game is
\[ J(u) = \int_0^T (p x_i(t) - u_i(t))dt \]
where \( p > 0 \) is a constant price per unit sold and \( u_i \) is the rate of advertising expenditure. The authors omitted discounting of future profits as well as a salvage value at the horizon date \( T \).

Using (2) and (3), we define Hamiltonian functions by
\[ H_1 = p x_i - u_i + \xi_1 x_1 + \xi_2 x_2, \]
\[ H_2 = p x_2 - u_2 + \gamma_1 x_1 + \gamma_2 x_2. \]

Regarding the open-loop equilibria in the advertising game, Leitmann and Schmitendorf stated the following:

**Comment.** Note that the open-loop Nash controls . . . are independent of initial conditions. It is readily shown that these controls are indeed closed-loop Nash equilibrium controls by verifying sufficient conditions for such controls (strategies) . . . Thus we have the unusual situation in which closed-loop strategies are independent of the state and depend only on time. This occurs for the following reason. Because \( \xi_2 \) and \( \gamma_2 \) satisfy homogeneous linear differential equations as well as \( \lambda_2(T) = 0 = \gamma_1(T), \lambda_2 \) and \( \gamma_2 \) vanish identically. Then, since state \( x_i \) enters \( H_i \) linearly, the maximization of \( H_i \) allows one to express \( u_i \) as a function of \( \lambda_1 \) only and \( u_2 \) as a function of \( \gamma_2 \) only. Furthermore, since the state is unspecified at the terminal time, the boundary conditions on \( \lambda_1 \) and \( \gamma_2 \) are independent of the state. Thus, there is always a feedback control which is constant with respect to \( x_i \) since then the terms depending on the state derivatives of the feedback control in the adjoint equations vanish. This allows integration of the adjoint equations separately from the state equations, and hence a state-independent feedback control (Leitmann and Schmitendorf [1, p. 648]).

**Remark 2.** The Leitmann–Schmitendorf advertising game belongs to the class of State-Redundant differential games (see Mehlmann [8]). In this class of games, an open-loop Nash equilibrium has the property of being subgame perfect. The next section discusses the notion of state-redundancy.

Feichtinger [9] suggested a non-linear extension of the Leitmann–Schmitendorf model, supposedly with the purpose of applying phase diagram analysis. The sales (or market share) dynamics are
\[ \dot{x}_i(t) = g_i(u_i(t)) - h_i(u_i(t))x_i(t) - \delta x_i(t); \quad i = 1, 2 \]
where advertising effectiveness functions \( g_i, h_i \) satisfy \( g_i(0) = h_i(0) = 0, g_i'(u_i), h_i'(u_i) > 0, g_i''(u_i) < 0 \). Admissible advertising rates satisfy \( 0 \leq u_i(t) \leq U_i \), where \( U_i = \text{const.} > 0 \). The objective functional of firm \( i \) is
\[ \int_0^T e^{-\tau T}[q x_i(t) - u_i(t)]dt + e^{-\tau T} S x_i(T) \]
where \( q_i > 0 \) is a constant profit margin and \( S_i > 0 \) a constant unit salvage value. A steady state \((u_1, u_2)\) having \( u_i \in (0, U_i)\) is an unstable node or a saddle point, depending on functions \( g_i \) and \( h_i \), and their derivatives, profit margins \( q_i \), discount rates \( r_n \), and depreciation rates \( \delta_i \). Steady states at the boundary, i.e., \( u_i = U_i \), may also exist. The author also applies phase plane analysis to the Leitmann–Schmitendorf model in which isolines are straight lines. The unstable node and saddle point steady states are obtained under simpler conditions on functions and parameters.

Wang and Wu [10] considers a straightforward combination of the Vidale–Wolfe and Lanchester, as these models were extended by Deal [11] and Case [6]. The sales dynamics in Wang and Wu [10] are

\[
\dot{x}(t) = \left[ M - x(t)\right] k_i u_i(t) - k_i u_i(t) x(t) - \delta_i x(t)
\]

which includes the feature that the effect of firm \( i \)'s own advertising \( u_i \) depends on the untapped market potential. The authors use a numerical algorithm to identify open-loop and feedback equilibrium strategies; for unknown reasons they did not include neither the Leitmann–Schmitendorf paper [1] nor the paper by Feichtinger [9] in their references.

3. Linear and state-redundant differential games

It has often been stated in papers applying differential games in economics and management science that a feedback-Nash equilibrium (FNE) is conceptually more appealing than an open-loop Nash equilibrium (OLNE). The former is more appealing in the sense that smart players would not decide from the outset what to do throughout the game; rather they would apply a strategy that reacts to the observed value of the state of the system. Notwithstanding this, open-loop Nash equilibrium has been popular in applications, for the simple reason that it is generally much easier to compute than a FNE. Facing the trade-off between the strategic appeal of feedback equilibrium and the tractability of open-loop equilibrium, researchers have looked for game structures where an open-loop equilibrium is subgame perfect, in an attempt to obtain the best of two worlds. The first papers which initiated this search are Clemhout and Wan [12] and Leitmann and Schmitendorf [1]. The Clemhout and Wan model belongs to the class of trilinear games, the Leitmann–Schmitendorf model to the class of state-redundant games.

To define the notion of state-redundancy and compare it to other notions that lead to the same result (i.e., that an OLNE is subgame perfect), consider an \( N \)-player differential game played on the time-interval \([t_0, T]\). Denote the vector of controls of player \( i \in \{1, \ldots, N\}\) by \( u_i(t) \in U_i \subseteq \mathbb{R}^{n_i} \), and the vector of state variables by \( x(t) \in X \subseteq \mathbb{R}^n \). Here, \( X \) is the state space and \( U_i \) the set of admissible controls of player \( i \). The state equations describing the motion of the system are given by:

\[
\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0
\]

where \( u(t) = (u_1(t), \ldots, u_N(t)) \). The payoff functional of player \( i \) reads

\[
J_i(u(\cdot); t_0, x_0) = \int_{t_0}^{T} g_i(x(t), u(t), t) dt + S_i(x(T), T)
\]

where function \( g_i \) is player \( i \)'s instantaneous payoff and \( S_i \) the terminal payoff function. The Hamiltonian of player \( i \) is

\[
H_i(x, u, \lambda_i, t) = g_i(x, u, t) + \lambda_i f(x, u, t)
\]

where \( \lambda_i = \dot{\lambda}_i(t) \) is a \( p \)-dimensional vector of costate variables. Assuming an interior solution and continuity of controls, candidates for Nash equilibrium satisfy the following first-order necessary conditions:

\[
\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0
\]

\[
\dot{\lambda}_i(t) = - \frac{\partial H_i(x(t), u(t), \lambda_i(t), t)}{\partial x}, \quad \lambda_i(T) = \frac{\partial S_i(x(T), T)}{\partial x(T)}
\]

\[
\frac{\partial H_i(x(t), u(t), \lambda_i(t), t)}{\partial u_i} = 0, \quad i = 1, \ldots, N.
\]

We proceed by identifying some classes of differential games having the property that open-loop Nash equilibria are subgame perfect.

Linear-state games were introduced in Dockner et al. [13] as state-separable games (see also Dockner et al. [14]). In these games the state equations and the objective functional are linear in the state variables. State separability requires that the following conditions are satisfied:

\[
\frac{\partial^2 H_i}{\partial u_i \partial x} = 0 \quad \text{for} \quad \frac{\partial H_i}{\partial u_i} = 0, \quad \frac{\partial^2 H_i}{\partial x^2} = 0, \quad \frac{\partial^2 S_i}{\partial x^2} = 0.
\]

For a weaker version of the condition \( \frac{\partial^2 H_i}{\partial u_i \partial x} = 0 \) see Dockner et al. [14].

A subclass of linear-state games is the trilinear games which were introduced Clemhout and Wan [12]. This paper has a great historical value, being the very first model in which an open-loop Nash equilibrium is subgame perfect. A game is
called trilinear when the Hamiltonians are linear in the state and costate variables as well as functions of the control variables.

Applications of linear-state games are numerous in economics and management science. Examples of recent contributions include Breton et al. [15], Jørgensen and Zaccour [16], Jørgensen et al. [17], Martín-Herrán and Zaccour [18] in environmental economics and management, Kemp et al. [19] in macroeconomics, Jørgensen et al. [20], Jørgensen and Zaccour [21], Viscolani and Zaccour [22] in advertising management, and Cellini and Lambertini [23,24] in R&D.

Another subclass of linear-state differential games are exponential games. In this class the objective function of player $i$ is given by

$$
\int_{0}^{T} g_i(u(t), t)e^{-\mu_1(t)}dt
$$

where $\mu_i \in \mathbb{R}^p$ are constants. The state equations are

$$
\dot{x}(t) = f(u(t), t), \quad x(t_0) = x_0.
$$

Note that the state dynamics depend on the control variables only and the state variables enter the objective functions in an exponential way. It can be shown (see, e.g., Dockner et al. [14]) that an exponential game can be transformed into a linear-state game by the state variable transformation $y_i(t) = e^{-\mu_i(t)}$. Reinganum [25] was the first to analyze an exponential game and illustrated it with a game of R&D.

Jørgensen [26] let the control variables enter into the dynamical equations and the objective functionals in an exponential way and uses the same procedure as Leitmann–Schmitendorf to show that the OLNE is a FNE in which feedback strategies depend only on time. The procedure is as follows:

1. Derive open-loop equilibrium controls that are independent of the initial values of the state variables.
2. Treat open-loop controls as if they were feedback strategies being constant with respect to the state variables.
3. Show that a sufficient optimality condition is satisfied for the optimization problems of all players.

Yeung [27] extends the differential game in Jørgensen [26] to a setting where the Hamiltonians are not required to be linear in the state, and obtains an open-loop Nash equilibrium which is subgame perfect.

Let us now move to the class of state-redundant differential games, initiated by the Leitmann–Schmitendorf paper. A differential game is state-redundant if the following condition holds:

**Condition 1** If after substitution for the solution of the costate Eq. (6) in the Hamiltonian-maximization conditions (7), the latter are independent of the state variables and of their initial values, then the game is state-redundant.

The following remark clarifies the relationships between linear-state and state-redundant games.

**Remark 3.** A game satisfying the conditions in (8) satisfies also the condition for state-redundancy and therefore is a state-redundant game. The reverse implication does not hold in general, that is, there are state-redundant games which do not satisfy the conditions in (8) and therefore are not state-separable games.

To see why an open-loop Nash equilibrium is subgame perfect for the class of state-redundant games, consider the Leitmann–Schmitendorf model. The state equation is

$$
x_i(t) = u_i(t) - \frac{C_i}{2}u_i^2(t) - k_iu_i(t)x_i(t) - \delta x_i(t).
$$

One feature of this model is that state variable $x_i$ does not occur on the right-hand side of the state equation for $x_j, i \neq j$. Neither does $\lambda_i$ occur in the integrand of the objective functional in (3) for player $i \neq j$. In [13] this feature is referred to as Non-interacting dynamics and objectives with respect to state. The game structure, and the transversality conditions $\lambda_i(t) = 0 = \gamma_i(T), \text{ imply } \lambda_i(t) \equiv 0, \gamma_i(t) \equiv 0.$ The Hamiltonians in (4) become

$$
H_1 = p_1x_1 - u_1 + \lambda_1 x_1,
H_2 = p_2x_2 - u_2 + \gamma_2 x_2,
$$

which implies

$$
\frac{\partial^2 H_1}{\partial u_1 \partial x_1} = \frac{\partial^2 H_1}{\partial u_1 \partial x_2} = 0, \quad \frac{\partial^2 H_2}{\partial u_2 \partial x_1} = \frac{\partial^2 H_2}{\partial u_2 \partial x_2} = 0,
\frac{\partial^2 H_2}{\partial x_1^2} = \frac{\partial^2 H_2}{\partial x_1 \partial x_2} = 0, \quad \frac{\partial^2 H_2}{\partial x_2^2} = \frac{\partial^2 H_2}{\partial x_1 \partial x_2} = 0.
$$

(9)

The equations in the first line of (9) say that the Hamiltonian maximization conditions are independent of the state $(x_1, x_2)$ and hence strategies can be expressed as functions $u_1(\lambda_1), u_2(\gamma_2)$ (cf. the comment by Leitmann and Schmitendorf quoted above). The equations in the second line say that the right-hand sides of the costate equations for $\lambda_1$ and $\gamma_2$ do not depend on the state $(x_1, x_2)$. The conclusion is that strategies $u_1, u_2$ are constant with respect to state (cf. the comment...
by Leitmann and Schmitendorf quoted above), and hence an open-loop Nash equilibrium is a degenerated feedback equilibrium.

Remark 4. The Leitmann–Schmitendorf game is a linear-state game and hence one knows that an OLNE is subgame perfect. The procedure provided in the Leitmann–Schmitendorf paper has the additional merit of detecting if this property holds for non-linear-state games.

To illustrate further the distinction between linear-state and state-redundant games we look at two recent papers, Calzolari and Lambertini [28] and Cellini and Lambertini [29], in which the model is a state-redundant game. The papers share the following characteristics. First, the state equations are not linear in the state variables. Second, the feedback effects at any instant of time during the game are endogenously nil and therefore the open-loop Nash equilibrium is subgame perfect. This is due to the fact that the first-order optimality conditions for the control variables become independent of the state variables and initial conditions after replacing the optimal values of the costate variables. Calzolari and Lambertini [28] analyze voluntary export restraints in a differential game of an international duopoly. The two state variables are the physical capital of each firm and the control variable of a firm is output (under Cournot competition) or price (under Bertrand competition).

For the case of Cournot competition the problem reads:

\[ J_i = \int_0^\infty e^{-\rho t}(a - u_i(t) - su_j(t))u_i(t)dt \]

s.t. : \[ \dot{x}_i(t) = f(x_i(t)) - u_i(t) - \delta x_i(t) \]

where \( a \) and \( \delta \) are positive constants and \( f(x_i) \) is a strictly increasing concave function. The model implies state-redundancy by virtue of the additive separability of the Hamiltonian functions with respect to the state and control variables. Moreover, the state equations are decoupled because the dynamics of variable \( x_i \) are independent of \( x_j \) and the state variables appear in the state equations, but not in the profit functions. These properties imply

\[ \frac{\partial u_i}{\partial x_j} = \frac{\partial u_i}{\partial x_i} = 0, \quad i \neq j. \]

Since firm \( i \)'s instantaneous payoff is independent of firm \( j \)'s state, and the dynamics of firm \( i \)'s state are independent of the state and control of the rival, the costate variable that each firm associates with the state variable of its counterpart is zero. The implication is that the Hamiltonian of player \( i \) depends only on the costate variable \( \lambda_i \) associated with its own state variable:

\[ H_i = (a - u_i - su_j)u_i + \lambda_i[f(x_i) - u_i - \delta x_i]. \]

Partial differentiation leads to

\[ \frac{\partial H_i}{\partial u_i} = a - 2u_i - su_i - \lambda_i, \quad \frac{\partial H_i}{\partial x_i} = \lambda_i[f'(x_i) - \delta], \]

\[ \frac{\partial^2 H_i}{\partial u_i \partial x_i} = \frac{\partial^2 H_i}{\partial u_i \partial x_j} = 0, \quad i, j = 1, 2, \quad i \neq j, \]

\[ \frac{\partial^2 H_i}{\partial x_i^2} = \lambda_i f''(x_i), \quad \frac{\partial^2 H_i}{\partial x_i \partial x_j} = 0, \quad i, j = 1, 2, \quad i \neq j \]

and the costate equation reads

\[ \dot{\lambda}_i(t) = \lambda_i[\rho + \delta - f'(x_i(t))]. \]

As in the Leitmann–Schmitendorf paper, this differential equation admits the solution \( \lambda_i(t) = 0 \) for all \( t \). Hence it admits a solution which is independent of the initial states for all \( t \in [0, \infty) \). Taking this into account in the Hamiltonian-maximization conditions the game turns out to be state-redundant and hence the open-loop equilibrium is subgame perfect. A similar argument proves the result in the case of Bertrand competition.

Cellini and Lambertini [29] revisit an oligopoly Cournot game with capital accumulation à la Ramsey. Following the same steps as above the authors show that there exists an open-loop Nash equilibrium which is a degenerate feedback equilibrium and hence subgame perfect. The costate variables are identically equal to zero for all \( t \) if all firms' initial capacity endowments are large enough.

Other developments in differential games which were illustrated in the Leitmann–Schmitendorf model are the sufficiency conditions in Jørgensen [30] and the method proposed by Fershtman [31] to identify open-loop Nash equilibria which are degenerate feedback-Nash equilibria. Jørgensen shows that for state-separable games, the first and second order conditions for an extremum of the Hamiltonian of player \( i \) also constitute a set of sufficient conditions in a Nash open-loop differential game. Fershtman characterizes games for which the open-loop Nash equilibrium does not depend on the initial conditions and hence is a candidate for being a degenerate feedback equilibrium. The author shows that his technique works for linear-state games and for the advertising model of Leitmann and Schmitendorf.

Recently, Cellini et al. [32] study games having the features that an open-loop Nash equilibrium is subgame perfect and an open-loop Stackelberg equilibrium is time-consistent. They call these games "perfectly uncontrollable" games and show that
if the game is linear-state and the Hamiltonian of each player is additively separable in state and control variables, the game is perfectly uncontrollable. Furthermore, they show that the Leitmann–Schmitendorf model is also a perfectly uncontrollable game, even if the additive separability condition does not hold.

Finally, we make the following remark which points to another interesting feature of the Leitmann–Schmitendorf model.

**Remark 5.** We have assumed controls that are interior to the sets of admissible controls, but it is worth mentioning that Leitmann and Schmitendorf do not make such assumption to derive equilibrium strategies. The assumption of interior controls is often made in economic and management differential games, mainly to avoid technical difficulties. Formally speaking, the assumption is justified if

1. The sets of admissible controls are open.
2. There are no control constraints (unbounded control sets).
3. For nonnegativity control constraints, Inada conditions are imposed, e.g., for a cost function $c(u_i)$ it holds that $c'(u_i) \to 0$ for $u_i \to 0$ or for a production function $f_i(u_i)$ it holds that $f_i'(u_i) \to +\infty$ for $u_i \to 0$.

Also note that one can use the following condition for $H_i$ – maximization

$$
\frac{\partial H_i}{\partial u_i} =
\begin{cases}
0 & \text{if } u_i^* = u_{i0} \\
0 & \text{if } u_i^* \in (u_{i0}, u_{i1}) \\
\geq 0 & \text{if } u_i^* = u_{i1}
\end{cases}
$$

when there are control constraints of the type $u_i \in [u_{i0}, u_{i1}]$, $u_{i0} < u_{i1}$ being real numbers. The assumption is that the Hamiltonians $H_i$ are concave in $u_i$, given the state and costates.

## 4. Conclusions

This short paper has put the Leitmann–Schmitendorf 1978-paper into perspective. We have demonstrated that the paper has influenced theoretical as well as applied work on games with linear-state and state-redundant structure, open-loop equilibria and degenerate feedback equilibria, subgame perfectness, and tractability.

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### References


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2 This method does not work if the sets of admissible controls are given by, say, $U_i(r,x)$, $x$ being the state vector.