Almost global asymptotic stability of a constant field current synchronous machine connected to an infinite bus

Vivek Natarajan and George Weiss

Abstract—We derive conditions for the almost global asymptotic stability of a system, modeling a constant rotor current synchronous generator connected to an infinite bus, in terms of the system parameters.

I. INTRODUCTION

It is well known that synchronous generators (SG), once they are synchronized, tend to remain synchronized even without any control unless very strong disturbances destroy the synchronism - this is a feature that enabled the development of the AC electricity grid at the end of the XIX century. Today, the inherent stability of networks of synchronous generators coupled with various types of loads and power sources (such as inverters) is an area of high interest and intense research. This is partly due to the proliferation of power sources that are not synchronous generators, which threatens the stability of the power grid. Analytical approaches are extremely difficult due to the high complexity of the grid, which can only be described by thousands of coupled nonlinear differential equations with lots of uncertainty. Usually, researchers use crude approximations and simplified models to get analytical conclusions. In this paper we look at one basic ingredient of this system, one synchronous generator, and investigate its stability when it is connected to a much more powerful grid, so that this one generator has practically no influence on the grid. Thus, we model the grid as an “infinite bus”, i.e., a three-phase AC voltage source. The question is: will the synchronous generator, driven by a prime mover with constant torque, having a constant field current (rotor current) and starting from an arbitrary initial state, converge to a state of synchronous rotation with a constant difference between the grid and the rotor angles? We do not employ any control apart from the standard frequency droop.

The importance of this problem has been recognized for a long time and it has been studied, for instance, in [1], [2], [3], [4], [5]. Relevant related results are in [6]. However, as far as we know, all the available studies are based on some crude approximations and simplified models which can only be described by thousands of coupled nonlinear differential equations with lots of uncertainty. Usually, researchers use crude approximations and simplified models to get analytical conclusions. In this paper we look at one basic ingredient of this system, one synchronous generator, and investigate its stability when it is connected to a much more powerful grid, so that this one generator has practically no influence on the grid. Thus, we model the grid as an “infinite bus”, i.e., a three-phase AC voltage source. The question is: will the synchronous generator, driven by a prime mover with constant torque, having a constant field current (rotor current) and starting from an arbitrary initial state, converge to a state of synchronous rotation with a constant difference between the grid and the rotor angles? We do not employ any control apart from the standard frequency droop.

The mathematical model of a synchronous machine can be found for instance in [9], [10], [4], [11], [7]. In this paper we consider that the rotor is round (non-salient) and, for the sake of simplicity, that the machine has one pair of poles per phase, the rotor current \( i_r \) is constant (or equivalently, the rotor is a permanent magnet) and the machine is “perfectly built”, meaning that in each stator winding, the flux caused by the rotor is a sinusoidal function of the rotor angle \( \theta \) (with shifts of \( \pm 2\pi/3 \) between the phases of course). The stator windings are connected in star, with no neutral connection, and there are no damper windings. We will quickly derive the equations that we need for our study, following the notation and sign conventions in [7].

The voltages are measured with the minus side at the center of the star and a current is considered positive if it flows outward. The stator windings have self-inductance \( L \), mutual inductance \(-M\) and resistance \( R_s \). (The typical value for \( M \) is \( L/2 \).) We denote \( L_s = L + M \) and \( e, v \) and \( i \) are the 3-dimensional vectors of electromotive force (EMF), voltage and current, respectively. Then (see [7, equation (3)])

\[
L_s i + R_s i = e - v.
\]

(2.1)

We mention that if the synchronous generator is connected to the infinite bus via an impedance that consists of a resistor in series, then these can be regarded as being parts of \( R_s \) and \( L_s \), respectively.

We apply to (2.1) the Park transformation, which is the unitary matrix

\[
U(\theta) = \sqrt{2} \begin{bmatrix}
\cos \theta & \cos(\theta - \frac{2\pi}{3}) & \cos(\theta + \frac{2\pi}{3}) \\
-\sin \theta & -\sin(\theta - \frac{2\pi}{3}) & -\sin(\theta + \frac{2\pi}{3}) \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1/\sqrt{2}
\end{bmatrix}.
\]

We denote \( e_{dq} = U(\theta)e \), whose components are \( e_d, e_q \) and \( e_0 \) in this order, and \( v_{dq}, i_{dq} \) are defined similarly. Then we obtain

\[
L_s U(\theta) i + R_s i_{dq} = e_{dq} - v_{dq}.
\]

It is easy to check that (regardless of the physical meaning of \( i \))

\[
\frac{d}{dt} \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = U(\theta) i + \omega \begin{bmatrix} i_q \\ -i_d \\ 0 \end{bmatrix},
\]
where $\omega = \dot{\theta}$. This combined with the previous equation yields

$$L_i d = -R_i d + \omega L_i q + e_d - v_d, \quad (2.2)$$
$$L_i q = -\omega L_i d - R_i q + m f - v_q, \quad (2.3)$$

while $i_0 = 0$ and hence $e_0 = v_0$. Because the rotor current $i_f$ is constant (usually negative), it can be shown that

$$e = M f i f \omega \left[ \begin{array}{c}
\sin \theta \\
\sin(\theta - \frac{2 \pi}{3}) \\
\sin(\theta + \frac{2 \pi}{3})
\end{array} \right],$$

where $M f > 0$ is the peak mutual inductance between the rotor and any one stator winding (see [7, equation (4)]). This implies by a short computation

$$e_d = 0, \quad e_q = -m o f i f, \quad (2.4)$$

where $m = \sqrt{2} M f$. What is still missing is the mechanical equation: according to Newton’s law,

$$J \ddot{\omega} = T_m - T_e - D_p \omega, \quad (2.5)$$

where $J$ is the moment of inertia of all the parts rotating with the rotor, $T_m$ is the active mechanical torque (from the prime mover), $T_e$ is the electromagnetic torque developed by the generator (which normally opposes the movement) and $D_p$ is a damping factor. $T_e$ can be found from energy considerations, see for instance [7, equation (7)]:

$$T_e = -m i_f q.$$

The damping factor is partly due to viscous friction, but mainly it is created by a feedback from $\omega$ to $T_m$, called the frequency droop (as explained in the cited references), which increases the active power in response to a drop of the grid frequency. Substituting the last expression into (2.5), we obtain

$$J \ddot{\omega} = m i_f q - D_p \omega + T_m. \quad (2.6)$$

We denote by $V$ and $\theta_r$ the grid voltage magnitude and angle, by which we mean that the components of $v$ are

$$v_a = \sqrt{\frac{2}{3}} V \cos \theta_r, \quad v_b = \sqrt{\frac{2}{3}} V \cos(\theta_r - \frac{2 \pi}{3}),$$
$$v_c = \sqrt{\frac{2}{3}} V \cos(\theta_r + \frac{2 \pi}{3}).$$

Finally, we introduce the angle difference

$$\delta = \theta - \theta_r - \frac{\pi}{2}.$$

Applying the Park transformation, we obtain

$$v_d = -V \sin \delta, \quad v_q = -V \cos \delta.$$

Substituting this and (2.4) into (2.2) and (2.3), we get

$$L_i d = -R_i d + \omega L_i q + V \sin \delta,$$
$$L_i q = -\omega L_i d - R_i q + m o f i_f + V \cos \delta.$$

Denoting $\delta_0 = \dot{\theta}_0$ (the grid frequency), it is clear from the definition of $\delta$ that

$$\delta = \omega - \delta_0.$$

The last three equations together with (2.6) can be written in matrix form:

$$\begin{bmatrix}
L_i d \\
L_i q
\end{bmatrix} =
\begin{bmatrix}
-R_i \omega L_i s + 0 & 0 \\
-R_i \omega L_i s + 0 & m o f i_f + V \cos \delta
\end{bmatrix}
\begin{bmatrix}
i_d \\
i_q
\end{bmatrix} -
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\omega \\
\delta
\end{bmatrix} +
\begin{bmatrix}
V \sin \delta \\
V \cos \delta
\end{bmatrix},$$

and these define our nonlinear dynamical system, with state variables $i_d, i_q, \omega$ and $\delta$. In a synchronous generator we may control $i_f$ indirectly via the rotor voltage (this adds $i_f$ as one more state variable to the system) and we may control also $D_p$ and $T_m$ (though not instantly). In a synchroconverter we may control $i_f, D_p, T_m$ and even $\omega$ instantly, but in this study they are considered to be constants ($i_f$ is usually negative, the others are always positive).

Consider the positive semidefinite function $W(t) = L_s i_d^2/2 + L_q i_q^2/2 + J \omega^2/2$ on the state space of (2.7). Its derivative along the trajectories of (2.7) is given by

$$W(t) = -R_i (i_d^2 + i_q^2) - D_p \omega^2 + V_i d \sin \delta + V_i q \cos \delta + T_m \omega,$$

which is negative for large values of $i_d, i_q$ and $\omega$. From this we get that along any trajectory of (2.7), $i_d$, $i_q$ and $\omega$ stay bounded. This implies the global (in time) existence of solutions to (2.7) for all initial conditions.

If $V$ is sufficiently large, then there exist two sequences of equilibrium points for (2.7). Any two equilibrium points in each of the sequences differ only in their value of $\delta$. This difference, like for the nonlinear pendulum equation, is an integer multiple of $2 \pi$. If we identify angles $\delta$ that differ by an integer multiple of $2 \pi$ (so that now $\delta$ is a point on the unit circle), then the system has two equilibrium points, one stable and one unstable. Extensive simulations have indicated that for a range of SG parameters, (2.7) is almost globally asymptotically stable, i.e., all the trajectories of (2.7), except those starting from a set of measure zero and converging to the unstable equilibrium point, converge to the stable equilibrium point. Our objective is to find an analytic test (in terms of the parameters of the SG) that implies this almost global asymptotic stability, see Theorem 5.1 and Remark 5.2.

## III. AN EXACT SWING EQUATION FOR THE SG

In this section, based on (2.7) we will derive an integrodifferential equation governing the angle difference $\delta$. We shall see that this equation closely resembles the nonlinear pendulum equation. It is not the same as the classical swing equation, which is an approximation obtained by making simplifying assumptions regarding the electromagnetic torque acting on the rotor (see [4], [5]). In contrast we use the exact expression for this torque.

The first two equations in (2.7), with the notation $p = R_i/L_i s$, can be rewritten as

$$\begin{bmatrix}
i_d \\
i_q
\end{bmatrix} =
\begin{bmatrix}
-p & \omega \\
-\omega & -p
\end{bmatrix}
\begin{bmatrix}
i_d \\
i_q
\end{bmatrix} -
\begin{bmatrix}
0 & m o f i_f + V \cos \delta \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\omega \\
\delta
\end{bmatrix} +
\begin{bmatrix}
V \sin \delta \\
V \cos \delta
\end{bmatrix}, \quad (3.1)$$

We regard $\omega$ and $\delta$ as continuous exogenous signals in (3.1). Then (3.1) is a linear time-varying system with state matrix

$$A(t) = \begin{bmatrix}
-p & \omega(t) \\
-\omega(t) & -p
\end{bmatrix}.$$
Clearly $A(t_1)A(t_2) = A(t_2)A(t_1)$ for all $t_1, t_2 \geq 0$. Hence the state transition matrix $\Phi(t, \tau)$ corresponding to $A$ is given by the following expression: for all $t, \tau \geq 0$

$$\Phi(t, \tau) = e^{\int_0^t A(\sigma)d\sigma} = e^{\left[-p(t-\tau) \int_0^t \omega(\sigma)d\sigma - \int_0^t \Omega^2(\sigma)d\sigma - p(t-\tau)\right]}.$$  

The unique solution of (3.1) is

$$\begin{cases}
\dot{i}_q(t) = \Phi(t, 0) i_q(0) \\
\dot{i}_q(t)
\end{cases} + \int_0^t \Phi(t, \tau) \left[-\frac{m_i f_{i'}^2}{L_i} + \frac{V}{L_i} \sin(\delta(\tau))\right] d\tau. \quad (3.2)

Next we will simplify the terms under the integral in (3.2). For the first term we have

$$\int_0^t \Phi(t, \tau) \left[\frac{0}{\omega(\tau)}\right] d\tau = \int_0^t e^{-p(t-\tau)} \left[\sin(\int_0^\tau \omega(\sigma)d\sigma) \omega(\tau) \cos(\int_0^\tau \omega(\sigma)d\sigma) \omega(\tau)\right] d\tau$$

$$= \int_0^t e^{-p(t-\tau)} \left[-\frac{d}{d\tau} \cos(\int_0^\tau \omega(\sigma)d\sigma) \right] d\tau$$

$$= e^{-pt} \left[-\cos(\int_0^t \omega(\sigma)d\sigma)\right]$$

$$+ p \int_0^t e^{-p(t-\tau)} \left[-\cos(\int_0^\tau \omega(\sigma)d\sigma)\right] d\tau. \quad (3.3)

We turn to the second term under the integral in (3.2). Noting that $\delta(\tau) = \delta(0) + \int_0^\tau \omega(\sigma)d\sigma - \omega_0 \tau$ for all $\tau \geq 0$, we have

$$\int_0^t \Phi(t, \tau) \left[\sin(\delta(\tau)) \cos(\delta(\tau))\right] d\tau$$

$$= \int_0^t e^{-p(t-\tau)} \left[\sin(\int_0^\tau \omega(\sigma)d\sigma + \delta(\tau)) \cos(\int_0^\tau \omega(\sigma)d\sigma + \delta(\tau))\right] d\tau$$

$$= \int_0^t e^{-p(t-\tau)} \left[\sin(\int_0^\tau \omega(\sigma)d\sigma + \delta(0) - \omega_0 \tau) \cos(\int_0^\tau \omega(\sigma)d\sigma + \delta(0) - \omega_0 \tau)\right] d\tau$$

$$= e^{-pt} \left[\sin(\int_0^\tau \omega(\sigma)d\sigma + \delta(0) - \omega_0 \tau) \cos(\int_0^\tau \omega(\sigma)d\sigma + \delta(0) - \omega_0 \tau)\right]$$

$$+ \omega_0 e^{-pt} \frac{p \sin(\int_0^\tau \omega(\sigma)d\sigma + \delta(0) - \omega_0 \tau)}{p^2 + \omega_0^2}\right]. \quad (3.4)$$

Introduce the angle $\phi$ determined by

$$\sin \phi = \frac{\omega_0}{\sqrt{p^2 + \omega_0^2}}, \quad \cos \phi = \frac{p}{\sqrt{p^2 + \omega_0^2}}.$$  

Then we obtain that

$$\int_0^t \Phi(t, \tau) \left[\sin(\delta(\tau)) \cos(\delta(\tau))\right] d\tau$$

$$= \frac{e^{-pt}}{\sqrt{p^2 + \omega_0^2}} \left[\sin(\int_0^\tau \omega(\sigma)d\sigma + \delta(0) - \omega_0 \tau + \phi) \cos(\int_0^\tau \omega(\sigma)d\sigma + \delta(0) - \omega_0 \tau + \phi)\right]$$

Putting together (3.2), (3.3) and the last equation, and using the notation

$$i_v = \frac{V}{L_i \sqrt{p^2 + \omega_0^2}},$$

we obtain that

$$i_q(t) = i_v \cos \left(\int_0^t \omega(\sigma)d\sigma + \delta(0) - \omega_0 t + \phi\right)$$

$$- \frac{m_i f_{i'}^2}{L_i} \int_0^t e^{-p(t-\tau)} \sin \left(\int_0^t \omega(\sigma)d\sigma\right) d\tau + e^{-\gamma t} f(t),$$

where $f(t)$ is a bounded function of time. Substituting for $i_q(t)$ in the equations for $\omega$ and $\delta$ in (2.7) we obtain the following integro-differential equation for $\delta(t)$:

$$J \dot{\delta}(t) + D_p \delta(t) - m_i f_{i'} \cos(\delta(t) + \phi) = T_m - D_p \omega_k$$

$$- \frac{m_i f_{i'}^2}{L_i} \int_0^t e^{-p(t-\tau)} \sin \left(\int_0^t \omega(\sigma)d\sigma\right) d\tau + mi_f e^{-\gamma t} f(t).$$

If we introduce the new variable $\psi$ by

$$\psi(t) = \frac{\pi}{2} + \delta(t) + \phi, \quad (3.5)$$

then the above equation becomes

$$J \dot{\psi}(t) + D_p \psi(t) - m_i f_{i'} \sin \psi(t) = T_m - D_p \omega_k + mi_f e^{-\gamma t} f(t)$$

$$- \frac{m_i f_{i'}^2}{L_i} \int_0^t e^{-p(t-\tau)} \sin \left(\psi(t) - \psi(t) + \omega_0 (t - \tau)\right) d\tau. \quad (3.6)$$

Notice that the integral above may be regarded as the output of a first order low-pass filter (with corner frequency $\gamma$) driven by a bounded input, so that it is bounded. If we regard the right-hand side of (3.6) as a bounded forcing function, then this is a pendulum equation with time-varying forcing. In the next section, we will derive some results for the asymptotic response of a forced pendulum. These will help us to establish our main result.

IV. ASYMPTOTIC RESPONSE OF A FORCED PENDULUM

Consider the forced pendulum equation

$$\psi + \alpha \psi + \sin \psi = \beta + \gamma(t), \quad (4.1)$$

where $\alpha > 0$ and $0 < \beta < 1$ are constants and $\gamma \in L^\infty([0, \infty))$ is a continuous time-varying disturbance satisfying $||\gamma||_{L^\infty} < d$ for some constant $0 < d < \beta$. We assume that $\beta + d < 1$. Define $0 < \psi_1, \psi_2, \psi_3 < \pi/2$ using

$$\sin \psi_1 = \beta + d, \quad \sin \psi_2 = \beta - d, \quad \sin \psi_3 = \beta + ||\gamma||_{L^\infty}. \quad (4.2)$$

The existence of a unique solution to (4.1) on a maximal time interval $[0, t_{\max})$ follows from standard results (see Chapter 3 in [12]). Since $|\beta + \gamma(t) - \sin \psi| < \beta + d + 1$ and $\alpha > 0$, it follows that along any solution $(\psi, \dot{\psi})$ of (4.1), $|\psi(t)|$ must remain bounded by $\max\{|\psi(0)|, |\beta + d + 1|/\alpha\}$. Therefore $|\psi(t)|$ cannot blowup to infinity in a finite time which implies that $t_{\max} = \infty$.

We will often regard the solution $(\psi, \dot{\psi})$ of (4.1) as a trajectory in the phase plane. Recall that in the phase plane
the angle $\psi$ is the $x$–coordinate while the velocity $\psi$ is the $y$–coordinate. In this plane, the trajectories of (4.1) satisfy

$$\frac{d\psi}{d\varphi} = -\alpha + \frac{\beta + \gamma - \sin \psi}{\psi}$$

whenever $\psi \neq 0$. (4.3)

We will need the following lemma.

**Lemma 4.1:** Consider the system

$$\psi_t + \alpha \psi_\varphi + \sin \psi = \beta + d,$$  

where $\alpha$, $\beta$ and $d$ are as defined earlier in this section. Let $(\psi, \psi_\varphi)$ and $(\psi_\varphi, \psi_\vartheta)$ be the trajectories of (4.1) and (4.4), respectively, starting from the same initial conditions, i.e.,

$$\psi(0) = \psi_\varphi(0) = \psi_\vartheta = \psi_\varphi(0) = \psi_0.$$

For some $\tau > 0$, if $\psi(t) \geq 0$ for all $t \in [0, \tau]$ and $\psi(0) \neq \psi(\tau)$, then the trajectory of (4.4) lies above the trajectory of (4.1) in the phase plane on the interval $(\psi(0), \psi(\tau))$, i.e., for each $\varphi \in (\psi(0), \psi(\tau))$ we have $\psi_\varphi(t) > \psi_\varphi(\varphi)$. 

*Proof:* First we assume that $\psi(0) = 0$. It then follows from the assumption $\psi(t) > 0$ for all $t \in [0, \tau]$ and $\psi(0) > 0$. Since $d > \gamma(\varphi)$ for all $t \geq 0$, we obtain that $\psi_\varphi(0) > 0$. Hence there exists a $0 < \sigma < \tau$ such that for each $t \in (0, \sigma)$

$$\psi_\varphi(t) > 0, \quad \beta + d - \sin \psi_\varphi(t) > 0, \quad \psi_\varphi(\sigma) < \psi(\tau).$$

We claim that for each $\varphi_0 \in (\psi_0, \psi_\varphi(\sigma))$, there exists a $\varphi_1 \in (\varphi_0, \varphi_0)$ such that $\psi_\varphi(\varphi_1) > \psi(\varphi_1)$. Indeed, if our claim is false we have

$$\psi_\varphi(\varphi_1) \leq \psi(\varphi_1) \quad \forall \varphi \in (\psi_0, \varphi_0)$$

and it follows from (4.3) that

$$\frac{d\psi}{d\varphi} \leq \frac{d\psi_\varphi}{d\varphi}_|_{\varphi=\varphi_1} \quad \forall \varphi \in (\psi_0, \varphi_0).$$

Since $\psi_\varphi(\varphi_1) = \psi(\varphi_1)$, we see that (4.6) contradicts (4.5), which implies that our claim is true.

Next we assume that $\psi(0) \neq 0$. Then there exists a $0 < \sigma < \tau$ such that $\psi_\varphi(\sigma) < \psi(\tau)$ and $\psi_\varphi(t) > 0$ for each $t \in (0, \sigma]$. We claim that for each $\varphi_0 \in (\psi_0, \psi_\varphi(\sigma))$, there exists a $\varphi_1 \in (\varphi_0, \varphi_0)$ such that $\psi_\varphi(\varphi_1) > \psi(\varphi_1)$. Indeed, if our claim is false, we get that (4.5) holds and

$$\frac{d\psi}{d\varphi} < \frac{d\psi_\varphi}{d\varphi}_|_{\varphi=\varphi_1} \quad \forall \varphi \in (\psi_0, \varphi_2),$$

for some $\varphi_2 < \varphi_0$. Since $\psi_\varphi(\varphi_2) = \psi(\varphi_2)$, we see that (4.7) contradicts (4.5), which establishes our claim.

We will complete the proof by showing that if $\psi_\varphi(\varphi_1) > \psi(\varphi_1)$ for some $\varphi_1 \in (\varphi_0, \psi(\tau))$, then $\psi_\varphi(\varphi_1) > \psi(\varphi_1)$ for all $\varphi \in (\varphi_0, \psi(\tau))$. Indeed, if this is not true, then let

$$\varphi_2 = \inf \{ \varphi : \psi(\varphi) = \psi_\varphi(\varphi_1), \varphi \in (\varphi_0, \psi(\tau)) \}.$$  

(4.8)

If $\varphi_2 \in ((2n - 1)\pi - \psi_\varphi, 2n\pi + \psi_\varphi)$ for some integer $n$, then it is easy to see that $\psi_\varphi(\varphi_2) > 0$. If $\varphi_2 \in (2n\pi + \psi_\varphi, 0)$, then we have $\psi(\varphi_2) > 0$. In the latter case, it follows from (4.3) that for some $\mu > 0$

$$\frac{d\psi}{d\varphi} \leq \frac{d\psi_\varphi}{d\varphi}_|_{\varphi_2 = \mu, \varphi_2}$$

for all $\varphi \in (\varphi_2, \varphi_2)$. (4.9)

Since $\psi_\varphi(\varphi_2 - \mu) > \psi(\varphi_2 - \mu)$, we get from (4.9) that $\psi(\varphi_2 - \mu) > \psi(\varphi_2 - \mu)$, which contradicts (4.8) and thus validates our claim.

The next lemma (stated without proof) is an analog of Lemma 4.1. It can be established using arguments similar to those in the proof of Lemma 4.1.

**Lemma 4.2:** Consider the system

$$\psi_t + \alpha \psi_\varphi + \sin \psi = \beta - d$$

where $\alpha$, $\beta$ and $d$ are as defined earlier in this section. Let $(\psi, \psi_\varphi)$ and $(\psi_\varphi, \psi_\vartheta)$ be the trajectories of (4.1) and (4.10), respectively, starting from the same initial conditions, i.e.,

$$\psi(0) = \psi_\varphi(0) = \psi_\vartheta = \psi_\varphi(0) = \psi_0.$$

For some $\tau > 0$, if $\psi(t) \leq 0$ for all $t \in [0, \tau]$ and $\psi(0) \neq \psi(\tau)$, then the trajectory of (4.10) lies below the trajectory of (4.1) in the phase plane on the interval $(\psi(0), \psi(\tau))$, i.e., for each $\varphi \in (\psi(0), \psi(\tau))$ we have $\psi_\varphi(\varphi) < \psi(\varphi)$.

The following result on the nonexistence of certain type of periodic solutions to the pendulum equation with a constant forcing, was established in [13].

**Lemma 4.3:** Consider the system

$$\psi + \alpha \psi_\varphi + \sin \psi = \sin \lambda,$$  

where $\alpha > 0$ and $0 < \lambda < \pi/2$. If $\alpha > 2\sin(\lambda/2)$, then there exist no solutions for (4.11) such that the velocity $\psi(t)$ is non-negative and is periodic of period $2\tau$ in $\psi$.

For the pendulum, we define the energy function as

$$E(t) = \frac{1}{2} \psi(t)^2 + (1 - \cos \psi(t)).$$

(4.12)

The time derivative of $E(t)$ along the trajectories of (4.1) is

$$E(t) = -\alpha \psi(t)^2 + (\beta + \gamma(\tau)) \psi(t).$$

(4.13)

Therefore we get that for any $t_2 > \gamma \geq 0$

$$E(t_2) - E(t_1) = \int_{t_1}^{t_2} E(s) ds$$

$$= \int_{t_1}^{t_2} \psi(s)(-\alpha \psi(s) + \beta + \gamma(s)) ds.$$

If $\psi(t) \neq 0$ for each $t \in [t_1, t_2]$, then using the change of variables $s \mapsto \psi$ and the equality $\psi(s) ds = d\psi$, we get

$$E(t_2) - E(t_1) = \int_{\psi(t)}^{\psi(t_2)} (-\alpha \psi(s) + \beta + \gamma(s)) d\psi.$$

(4.14)

**Lemma 4.4:** There exist no solutions for (4.1) such that the velocity $\psi(t)$ is non-positive for all $t \geq 0$ and $\psi$ is unbounded.

*Proof:* Assume that there exists a solution $(\psi, \psi)$ of (4.1) such that $\psi(t) \leq 0$ for all $t \geq 0$ and $\psi(t)$ is unbounded. Then there exists an angle $\psi_0$ and an increasing sequence of times $(t_n)_{n=1}^{\infty}$ such that $\psi(t_n) = 2n\pi + \psi_0$. From (4.12) and (4.13) we get that

$$E(t_n) - E(t_1) = \int_{t_1}^{t_n} \psi(\tau)(-\alpha \psi(\tau) + \beta + \gamma(\tau)) d\tau$$

$$\leq (\beta - d)(\psi(t_n) - \psi(t_1)) = -2(\beta - d)(n - 1)\pi.$$

(4.15)
This implies that $E(t_n) < 0$ for large $n$, which is impossible. Hence we have a contradiction.

In Lemma 4.4, we established a result for the trajectories of (4.1) with non-positive velocity. In the following lemma, we will establish a similar result for the trajectories of (4.1) with non-negative velocity.

Lemma 4.5: There exist no solutions for (4.1) such that the velocity $\psi(t)$ is non-negative for all $t \geq 0$ and $\psi$ is unbounded if

$$\alpha > 2\sin\frac{\psi_1}{2}. \quad (4.15)$$

Here $\psi_1$ is introduced in (4.2).

Proof: Assume that $\psi(t) \geq 0$ for all $t \geq 0$ and $\psi$ is unbounded. Clearly, if $\psi(\tau) \in (2m\pi + \psi_1, (2m+1)\pi - \psi_1)$ for some integer $m$ and some $\tau \geq 0$, then $\psi(\tau) > 0$. Let $\tau_1, \tau_2 > 0$ be such that $\psi(\tau_1) \in [2m\pi + \psi_1, (2m+1)\pi - \psi_1]$ and $\psi(\tau_2) = (2m+1)\pi - \psi_1$. Then, we get using (4.14) that

$$\frac{\psi(\tau_1)^2}{2} \geq \int_{\psi(\tau_1)}^{\psi(\tau_2)} (\sin\psi - \beta - \gamma)d\psi > e^2 > 0,$$

where $e$ is a constant independent of $\psi(\tau_1)$. Therefore

$$\inf\{\psi(0) : \psi \in [2m\pi + \psi_1, (2m+1)\pi - \psi_1], m \in \mathbb{Z}\} > e > 0.$$ 

Let $(\psi_0, \psi_\alpha)$ be the solution of (4.4) with $\psi_0(0) = \psi(\tau_1)$ and $\psi_\alpha(0) = \psi(\tau_1)$. From Lemma 4.1 it follows that the trajectory $(\psi_0, \psi_\alpha)$ lies above the trajectory $(\psi, \psi)$. This, in particular, means that $\psi_0$ is also unbounded and $\psi_\alpha(t) > 0$ for all $t \geq 0$. Consider the sequence $\{\psi_m(\psi_0(0) + 2m\pi)\}_{m=0}^{\infty}$. This is a strictly positive bounded (since the velocity remains bounded) sequence which is either increasing or decreasing (since no two trajectories of (4.4) can intersect). Hence this sequence has a unique limit $\psi_0 \geq e$. Using the continuous dependence of solutions of ODEs on initial conditions (or via energy considerations), we get that the solution of (4.4) with $\psi_0(0) = \psi_\alpha(0)$ and $\psi_\beta(0) = \psi_\alpha$ is such that the resulting velocity $\psi_0$ is non-negative and $2\pi$ periodic in $\psi$. Since (4.15) holds, this contradicts Lemma 4.3. This completes the proof of the lemma.

Definition 4.6: A point $(\phi^0,0)$ in the phase plane is called a positive acceleration point if $\psi_0$ computed at this point using (4.4) is positive. A point $(\phi^0,0)$ in the phase plane is called a negative acceleration point if $\psi_0$ computed at this point using (4.10) is negative.

Clearly the set of positive acceleration points is

$$\{(\phi,0) : \phi \in ((2m-1)\pi - \psi_1, 2m\pi + \psi_1), m \in \mathbb{Z}\}$$

and the set of negative acceleration points is

$$\{(\phi,0) : \phi \in ((2m\pi + \psi_1, (2m+1)\pi - \psi_1)), m \in \mathbb{Z}\}.$$ 

Lemma 4.7: Let $(\psi_0, \psi_\alpha)$ be a trajectory of (4.4) with a positive acceleration point $(\phi^0,0)$ as its initial condition. Assume that $(2k-1)\pi - \psi_1 < \phi^0 < 2k\pi + \psi_1$ for some integer $k$ and that $\alpha$ satisfies the inequality (4.15). Then, the closure of this trajectory in the phase plane contains a negative acceleration point which belongs to the set

$$\{(\phi,0) : \phi \in [2k\pi + \psi_1, (2k+1)\pi - \psi_1]\}, \quad (4.16)$$

where $0 < \psi_4 < \pi/2$ satisfies

$$\sin\psi_4 > \beta + d, \quad \alpha > 2\sin\frac{\psi_4}{2}.$$ 

The first (in time) negative acceleration point will be called the first negative acceleration point for this trajectory.

Proof: Let $\tau = \sup\{T > 0 : \psi_0(t) > 0 \text{ for all } t \in (0,T)\}$. Assume that

$$\lim_{t \to \tau^-} \psi_0(t) > (2k+1)\pi - \psi_4.$$ 

Consider the trajectory $(\psi, \psi)$ of (4.11) with $\lambda = \psi_4$ and initial condition $(\psi(0), \psi(0)) = (\phi^0,0)$. Let $\tau_1 > 0$ be such that $\psi_0(\tau_1) = (2k+1)\pi - \lambda$. It then follows (like in the proof of Lemma 4.1) that for some $\tau_2 > 0, \psi(\tau_2) = (2k+1)\pi - \lambda$ and $\psi(\tau_2) > 0$. This implies that there exists an increasing sequence $(\tau_n)_{\infty}^{\infty}$ with $\tau_1 > \tau_2$ such that $\psi(\tau_n) = (2n\pi + \phi^0$ and $\psi(\tau_n) > 0$. We can now obtain a contradiction to Lemma 4.3, like we did in Lemma 4.5, using the increasing sequence $(\psi(\tau_n))_{\infty}^{\infty}$. Therefore

$$\lim_{t \to \tau^-} \psi_0(t) \leq (2k+1)\pi - \psi_4. \quad (4.17)$$

First assume that $\tau = \infty$. Then $\psi_0(t)$ is a non-decreasing bounded function of time and therefore $\psi_0(t) \to \psi_0$. From Barbalat’s lemma it follows that $\psi_0(t) \to 0$ as $t \to \infty$. which, along with (4.17), implies that

$$\lim_{t \to \infty} \psi_0(t), \psi_\alpha(t) = (2k\pi + \psi_1, 0). \quad (4.18)$$

The limit in (4.18) is a negative acceleration point.

Next we assume that $\tau$ is finite. In this case, $(\psi_0(\tau),0)$ is the required negative acceleration point and $\psi_\alpha(\tau) \in [2k\pi + \psi_1, (2k+1)\pi - \psi_4].$

Clearly $\psi_\alpha(\tau)$ $(\psi_0$ if $\tau = \infty$) is the first negative acceleration point for the trajectory $(\psi_0, \psi_\alpha)$.

Next we present a result analogous to Lemma 4.7.

Lemma 4.8: Let $(\psi_0, \psi_\alpha)$ be a trajectory of (4.10) with a negative acceleration point $(\phi^0,0)$ as its initial condition. Assume that $\phi^0 \in (2k\pi + \psi_2, (2k+1)\pi - \psi_2)$ for some integer $k$ and that $\alpha$ satisfies the inequality (4.15). Then, the closure of this trajectory in the phase plane contains a positive acceleration point which belongs to the set

$$\{(\phi,0) : \phi \in ((2k-1)\pi + \psi_2, 2k\pi + \psi_2)\}. \quad (4.19)$$

The first (in time) positive acceleration point will be called the first positive acceleration point for this trajectory.

Proof: The proof of this lemma is similar to the proof of Lemma 4.7. Hence we omit the details. The left endpoint of the interval in (4.19) follows via energy considerations. Indeed, let $\tau = \sup\{T > 0 : \psi_0(t) < 0 \text{ for all } t \in (0,T)\}$. If $\psi_0(\tau_1) = (2k-1)\pi + \psi_2$ for some $0 < \tau_1 \leq \tau$, then we get the following contradiction:

$$0 < E(\tau_1) - E(0) = \int_0^{\tau_1} -\alpha \phi^2 dx + \psi_0(\beta - d)ds < 0.$$

We will now define some special curves in the phase plane referred to as trajectory envelope. These curves act as an envelope for certain trajectories of (4.1) (see Lemma 4.14) and hence the name.

Definition 4.9: Assume that $\alpha$ satisfies (4.15). Let $(\phi^0,0)$ be a positive acceleration point with $\phi^0 = (2k-1)\pi + \psi_2$
for some integer $k$. Construct a sequence of points $(\phi^n, 0)$, $n = 0, 1, \ldots$, as follows: for each non-negative integer $n$, let $(\phi^{2n+1}, 0)$ be the first negative argument point for the trajectory of (4.4) starting from $(\phi^{2n}, 0)$ and let $(\phi^{2n+2}, 0)$ be the first positive argument point for the trajectory of (4.10) starting from $(\phi^{2n+1}, 0)$. Denote the trajectory of (4.4) from $\phi^n$ to $\phi^{n+1}$ by $\Gamma_{n+1}$ and the trajectory of (4.10) from $\phi^{n+1}$ to $\phi^{n+2}$ by $\Gamma_{n+2}$. A trajectory envelope $\Gamma$ starting from $(\phi^0, 0)$ is a continuous curve in the phase plane obtained by concatenating the curves $\Gamma_n$, $n = 1, 2, \ldots, \infty$.

Lemma 4.10: Assume that $\alpha$ satisfies (4.15). Fix an integer $k$ and consider the trajectory envelope $\Gamma$ starting from $\phi^0 = (2k-1)\pi - \psi$. There exists a simple closed curve $\Gamma_c$ in the phase plane to which $\Gamma$ converges, i.e., for any $\epsilon > 0$ there exists an integer $n > 0$ such that for each point $x \in \Gamma_n$,

\[ d(x, \Gamma_c) < \epsilon. \]

Here $d$ is the Euclidean distance and the curve $\Gamma_n$ is a subset of $\Gamma$ excluding the segments $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_{n-1}\}$.

Proof: Consider the sequence $(\phi^n, 0)$, $n = 0, 1, \ldots$, as introduced in Definition 4.9. It follows from Lemmas 4.7 and 4.8 (and the fact that no two trajectories of (4.4) or (4.10) can intersect) that the sequence $(\phi^{2n})_{n=0}^\infty$ is contained in the interval $[(2k-1)\pi + \psi, 2k\pi + \psi]$ and is increasing and the sequence $(\phi^{2n+1})_{n=0}^\infty$ is contained in the interval $[(2k\pi + \psi, (2k-1)\pi + \psi])$ and is decreasing. Let

\[ \phi_1 = \lim_{n \to \infty} \phi^{2n+1}, \quad \phi_2 = \lim_{k \to \infty} \phi^{2n}. \]

Then

\[ \phi_1 \in [2k\pi + \psi, (2k-1)\pi + \psi], \quad \phi_2 \in [(2k-1)\pi + \psi, 2k\pi + \psi]. \]

The trajectory of (4.4) starting from $(\phi_2, 0)$ will (eventually) reach $(\phi_1, 0)$; we call this curve $\Gamma_c$. Similarly the trajectory of (4.10) starting from $(\phi_1, 0)$ will (eventually) reach $(\phi_2, 0)$; we call this curve $\Gamma_b$. The union of the curves $\Gamma_c$ and $\Gamma_b$ is a simple closed curve $\Gamma_c$. The fact that $\Gamma$ converges to $\Gamma_c$ can be easily deduced either using the continuous dependence of solutions of ODEs on initial conditions or via energy arguments using (4.14).

We want to estimate the magnitude of $|\phi_1 - \phi_2|$. In this regard, the following lemma will be useful.

Lemma 4.11: Let $\alpha$ satisfy (4.15). Fix an integer $k$ and consider the corresponding trajectory envelope $\Gamma$. Recall the definitions of $\Gamma_a$, $\Gamma_b$, $\Gamma_c$, $\phi_1$, and $\phi_2$ from the proof of Lemma 4.10 and note that $\phi_1$ and $\phi_2$ satisfy (4.20) and (4.21).

If $\phi_1 \neq 2k\pi + \psi_1$, then the curve $\Gamma_b$ and the $x$-axis $\psi = 0$ enclose a convex set $\Delta_b$ in the phase plane. Similarly, if $\phi_2 \neq 2k\pi + \psi_2$, then $\Gamma_b$ and the $x$-axis $\psi = 0$ enclose a convex set $\Delta_b$. In the phase plane.

Proof: The trajectory $(\psi_a, \psi_b)$ of (4.4), corresponding to the curve $\Gamma_a$, satisfies the equation

\[ \frac{d\psi_a}{d\psi} = -\alpha + \beta + \frac{d}{d\psi} \sin \psi_a, \]

at each $\psi_a \in (\phi_1, \phi_2)$ (this follows from (4.3)). We claim that $d\psi_a/d\psi_b \leq 0$ along the curve $\Gamma_a$. Let us define

\[ F = -\alpha + \beta + \frac{d}{d\psi} \sin \psi_a. \]

It then follows that

\[
\begin{align*}
\frac{dF}{d\psi_a} &= -F(F + \alpha) + \cos \psi_a \quad \text{and} \\
\frac{d^2F}{d\psi_a^2} &= -2F \frac{dF}{d\psi_a} - (F + \alpha) \frac{dF}{d\psi_a} + \sin \psi_a. \quad \text{(4.23)}
\end{align*}
\]

Assume that our claim is not true. Then $\frac{dF}{d\psi_a} \mid_{\psi_a = \psi_0} > 0$ for some $\psi_0 \in (\phi_1, \phi_2)$. It is easy to verify that for a sufficiently small $\epsilon > 0$, $\frac{dF}{d\psi_a} \mid_{\psi_a = \psi} < 0$ for each $\psi \in (\phi_2, \phi_2 + \epsilon)$. Therefore there exists a $\eta_2 \in (\phi_2, \phi_2)$ such that

\[ \frac{dF}{d\psi_a} \mid_{\psi_a = \eta_2} = 0, \quad \frac{d^2F}{d\psi_a^2} \mid_{\psi_a = \eta_2} \geq 0. \]

This, using (4.23), implies that $\sin \eta_2 \geq 0$ or equivalently $\eta_2 \geq [2k\pi, \psi_0]$ and therefore $\psi_0 > 2k\pi$. Since $\phi_1 > 2k\pi + \psi$, we can verify that for a sufficiently small $\epsilon > 0$, $\frac{dF}{d\psi_a} \mid_{\psi_a = \psi} < 0$ for each $\psi \in (\phi_1 - \epsilon, \phi_1)$. Therefore there exists $\eta_1 \in (\psi_0, \phi_1)$ such that

\[ \frac{dF}{d\psi_a} \mid_{\psi_a = \eta_1} = 0, \quad \frac{d^2F}{d\psi_a^2} \mid_{\psi_a = \eta_1} \leq 0. \]

This, using (4.23), gives us that $\sin \eta_1 \leq 0$ or equivalently $\eta_1 \leq 2k\pi$, which implies that $\psi_0 < 2k\pi$. This is a contradiction. Hence $d\psi_a/d\psi_b \leq 0$ along the curve $\Gamma_a$ which implies that $\Delta_a$ is a convex set.

Similarly $\Delta_b$ can be shown to be a convex set. ■

Lemma 4.12: Fix an integer $k$ and let $\alpha$, $\Gamma_a$, $\Gamma_b$, $\phi_1$ and $\phi_2$ be as in Lemma 4.11. Define

\[ v_a = \max_{(\psi_a, \psi_b) \in \Gamma_a} \psi_a, \quad v_b = \max_{(\psi_a, \psi_b) \in \Gamma_b} \psi_b \]

Then the following relations hold:

\[ \phi_1 \neq 2k\pi + \psi_1, \quad \phi_2 \neq 2k\pi + \psi_2 \implies v_a + v_b \leq 4d/\alpha \quad \text{(4.24)} \]

\[ \phi_1 \neq 2k\pi + \psi_1, \quad \phi_2 \neq 2k\pi + \psi_2 \implies v_a \leq 4d/\alpha \quad \text{(4.25)} \]

\[ \phi_1 = 2k\pi + \psi_1, \quad \phi_2 \neq 2k\pi + \psi_2 \implies v_b \leq 4d/\alpha \quad \text{(4.26)} \]

Proof: First assume that $\phi_1 \neq 2m\pi + \psi_1$. Then the curve $\Gamma_c$ can be regarded as a trajectory $(\psi, \psi)$ of (4.1) with $\gamma$ defined as follows:

\[ \gamma(t) = d \text{ if } \psi(t) > 0, \quad \gamma(t) = -d \text{ if } \psi(t) \leq 0. \]

Let this trajectory (which starts from $(\phi_2, 0)$, passes through $(\phi_1, 0)$ and then returns to $(\phi_2, 0)$) be defined on a maximal time interval $[0, \tau)$, where $\tau$ can be finite or infinite. Since $E(0) = \lim_{t \to \tau} E(t)$, it follows from (4.14) that

\[ -\int_{\phi_2}^{\phi_1} \alpha(\psi_{\Gamma_b} - \psi_{\Gamma_a})d\psi + 2d(\phi_1 - \phi_2) = 0, \quad \text{(4.27)} \]

where $\psi_{\Gamma_a}$ and $\psi_{\Gamma_b}$ denote the velocity along the curves $\Gamma_a$ and $\Gamma_b$, respectively. Define

\[ (\psi_a, v_a) = \arg \max_{(\psi_a, \psi_b) \in \Gamma_a} \psi_a. \]
Consider the triangle in the phase plane with vertices \((\phi_1, 0), (\phi_2, 0)\) and \((\psi, v_\alpha)\). It follows from Lemma 4.11 that this triangle lies completely inside the convex set \(\Delta_n\). Therefore
\[
\int_{\phi_2}^{\phi_1} \psi_{\alpha} \cdot d\psi \geq \frac{v_\alpha (\phi_1 - \phi_2)}{2}. \tag{4.28}
\]
This along with (4.27) gives us
\[
2d(\phi_1 - \phi_2) \geq \frac{v_\alpha (\phi_1 - \phi_2)}{2}. \tag{4.29}
\]
which implies (4.25).

If \(\phi_2 \neq 2m\pi + \psi_2\) and \(\phi_1 \neq 2m\pi + \psi_2\), then the inequalities in (4.28) and (4.30) hold and together with (4.27) give us
\[
2d(\phi_1 - \phi_2) \geq \frac{(v_\alpha + v_\beta)(\phi_1 - \phi_2)}{2}, \tag{4.30}
\]
which implies (4.26).

In Lemma 4.12 we have obtained upper bounds for the velocity along the curve \(\Gamma_n\). In the next lemma we will use these bounds to obtain a bound for \(\phi_1 - \phi_2\).

**Lemma 4.13:** Fix an integer \(k\) and let \(\alpha, \Gamma_n, \Gamma_{n+1}, \Gamma_{n+2}, \phi_1\) and \(\phi_2\) be as in Lemma 4.11. Then
\[
\phi_1 - \phi_2 \leq \psi_1 - \psi_2 + \frac{4d}{\alpha^2}. \tag{4.31}
\]

**Proof:** If \(\phi_1 = 2k\pi + \psi_1\) and \(\phi_2 = 2k\pi + \psi_2\), then (4.31) holds trivially. Hence with no loss of generality we assume that \(\phi_1 \neq 2k\pi + \psi_1\). For simplicity of notation, we assume that \(k = 0\).

Consider the curve \(\Gamma_n\), regarded as a trajectory \((\psi_n, \psi_n)\) of (4.4). Let \(\tau_1 > 0\) and \(\tau_2 > 0\) be the distinct time instants such that \(\psi_n(\tau_1) = \psi_1\) and \(\psi_n(\tau_2) = \phi_1\). Using (4.14), we get
\[
\frac{\psi_n(\tau_1)^2}{2} = E(\tau_1) - E(\tau_2) + \cos \psi_1 - \cos \phi_1
\]
\[
= \int_{\psi_n}^{\phi_1} \alpha \psi_n \cdot d\psi_n + \int_{\psi_n}^{\psi_1} (\sin \psi_n - \beta - d) d\psi_n
\]
\[
\geq \frac{\alpha}{2} (\phi_1 - \phi_2)(\psi_n(\tau_1) + \int_{\psi_n}^{\psi_1} (\sin \psi_n - \beta - d) d\psi_n). \tag{4.32}
\]
To derive the last inequality, we have used the convexity of the set \(\Delta_n\). Since the integral term in (4.32) is positive, it follows from Lemma 4.12 that
\[
\phi_1 - \psi_1 \leq \frac{\psi_n(\tau_1)}{\alpha} \leq \frac{v_\alpha}{\alpha} \leq \frac{4d}{\alpha^2}. \tag{4.33}
\]
If \(\phi_2 = \psi_2\), then (4.31) holds. Hence assume that \(\phi_2 \neq \psi_2\). Consider the curve \(\Gamma_n\), regarded as a trajectory \((\psi_2, \psi_2)\) of (4.10). Then as above, we can conclude using Lemma 4.12 and the convexity of the domain \(\Delta_n\) that
\[
\psi_2 - \phi_2 \leq \frac{v_\beta}{\alpha}. \tag{4.34}
\]
Clearly (4.31) follows from (4.24), (4.33) and (4.34).}

The next lemma justifies the name trajectory envelope for the curve introduced in Definition 4.9.

**Lemma 4.14:** Let \(\alpha\) satisfy (4.15). Fix an integer \(k\) and consider the corresponding trajectory envelope \(\Gamma\). Let \(\Gamma_n\) and \(\phi^n\) be as introduced in Definition 4.9. For each integer \(n \geq 1\), let \(\Delta_n\) be the closure of the set enclosed by the curves \(\Gamma_{2n-1}\), \(\Gamma_{2n}\) and the line \(L_n\) joining the points \((\phi^{2n-2}, 0)\) and \((\phi^{2n}, 0)\). Then for any trajectory \((\psi, \psi)\) of (4.1) and any integer \(n \geq 1\), if \((\psi(0), \psi(0)) \in \Delta_n\), then \((\psi(t), \psi(t)) \in \Delta_n\) for all \(t > 0\).

**Proof:** Fix \(n \geq 1\). Let \((\psi, \psi)\) be a trajectory of (4.1) with \((\psi(0), \psi(0)) \in \Delta_n\). It follows from Lemma 4.1 that this trajectory cannot leave \(\Delta_n\) by crossing the curve \(\Gamma_{2n-1}\) and from Lemma 4.2 that it cannot leave \(\Delta_n\) by crossing \(\Gamma_{2n}\). It is easy to check that if the trajectory is on the line \(L_n\), then its velocity increases and it is forced to re-enter \(\Delta_n\).

In Theorem 4.16 (the main result of this section) we will derive some estimates for the asymptotic response of the forced pendulum equation.

**Definition 4.15:** A trajectory \((\psi, \psi)\) of (4.1) is called oscillating if for each \(T > 0\) there exists \(\tau_1, \tau_2 > T\) such that \(\psi(\tau_1) > 0\) and \(\psi(\tau_2) < 0\).

**Theorem 4.16:** Assume that \(\alpha\) satisfies (4.15). Consider a trajectory \((\psi, \psi)\) of (4.1). For each \(\epsilon > 0\), there exists a \(T > 0\) such that for any \(\tau_1, \tau_2 > T\)
\[
|\psi(\tau_1) - \psi(\tau_2)| \leq \psi_1 - \psi_2 + \frac{4d}{\alpha^2} + \epsilon. \tag{4.35}
\]

**Proof:** If the trajectory \((\psi, \psi)\) is not oscillating, it then follows from Lemmas 4.4 and 4.5 that \(\psi\) must remain bounded. Therefore \(\psi(t)\) must converge to a finite limit, which trivially implies that the claim of this theorem holds. Hence we will assume that \((\psi, \psi)\) is oscillating. Then, either (i) \(\psi(t) \in \{\mathbb{2k+1}\pi - \psi_1, (2k+1)\pi - \psi_2\}\) for some integer \(k\) and all \(t\) sufficiently large, or (ii) \(\psi(t) \notin \{\mathbb{2k+1}\pi - \psi_1, (2k+1)\pi - \psi_2\}\) for all integers \(k\) and all \(t\) sufficiently large. This can be established via elementary (but tedious) arguments using Lemmas 4.1, 4.2, 4.7, 4.8 and 4.14. If (i) holds, then (4.35) is trivially satisfied. To complete this proof, we therefore assume that (ii) holds.

Next we will find a trajectory envelope \(\Gamma\) and show via mathematical induction that given any integer \(m \geq 1\), there exists a \(T_m > 0\) such that \((\psi(t), \psi(t))\) is \(\Delta_n\) for all \(T \geq T_m\) (\(\Delta_n\) has been defined in Lemma 4.14). This, along with Lemmas 4.10 and 4.13 will imply that the claim of this theorem holds.

Let \(\tau_1 > 0\) be such that \(\psi(\tau_1) > 0\) and \(\psi(t) \notin \{\mathbb{2k+1}\pi - \psi_1, (2k+1)\pi - \psi_2\}\) for all integers \(k\) and all \(t \geq \tau_1\). Define \(\tau_2 = \sup\{s : \psi(t) \geq 0 \forall t \in [\tau_1, s]\}, \tau_1 = \sup\{s : \psi(t) \leq 0 \forall t \in [\tau_1, s]\}\). Clearly \(\psi(\tau_1) = 0\) and \(\psi(\tau_2) \leq 0\). Hence for some integer \(k\), \(\psi(\tau_2) \in \{\mathbb{2k+1}\pi + \psi_2, (2k+1)\pi - \psi_2\}\). We also have \(\psi(\tau_1) = 0\) and \(\psi(\tau_2) \geq 0\) which via Lemmas 4.2 and 4.8 implies that \(\psi(t) \in \{(2k-1)\pi + \psi_1, 2\pi + \psi_1\}\). Consider the trajectory envelope \(\Gamma\) corresponding to this \(k\). Let \(T_1 = \tau_1\), then \((\psi(T_1), \psi(T_1)) \in \Delta_1\). It follows from Lemma 4.14 that \((\psi(T_1), \psi(T_1)) \in \Delta_1\) for all \(t \geq T_1\). Assume that for some integer \(m > 0\) and some \(T_m > 0\), \((\psi(t), \psi(t)) \in \Delta_m\) for all \(t \geq T_m\). Let \(\tau_1 > T_m\) be such that \(\psi(\tau_1) < 0\). Define \(T_{m+1} = \sup\{s : \psi(t) < 0 \forall t \in [\tau_1, s]\}\).
Then \(\psi(T_{m+1}) = 0\). As \((\psi(T_{m+1}), \psi(T_{m+1})) \in \Delta_m\), we have \(\psi(T_{m+1}) \in [0, 2\pi].\) So \((\psi(T_{m+1}), \psi(T_{m+1})) \in \Delta_{m+1}\) and (from Lemma 4.14) \((\psi(t), \psi(t)) \in \Delta_{m+1}\) for all \(t \geq T_{m+1}\).

This completes the proof of the theorem.

**V. STABILITY OF THE SG CONNECTED TO THE BUS**

Assume with no loss of generality that \(m_{ij} < 0\). Define

\[
P = \frac{m^2 \rho^2}{L_s} \lim_{t \to \infty} \int_0^t e^{-\rho(t-\tau)} \sin(\omega_s(t-\tau)) d\tau.
\]

Clearly \(P\) is finite. Then (3.6) can be rewritten as

\[
J\gamma(t) + D_{\rho} \psi(t) - m_{ij} \sin(\psi(t)) = T_m - D_P \omega_s - P + g(t)
\]

\[
-\frac{m^2 \rho^2}{L_s} \int_0^t e^{-\rho(t-\tau)} \cos(\omega_s(t-\tau)) \sin(\psi(t) - \psi(\tau)) d\tau
\]

\[
+ \frac{m^2 \rho^2}{L_s} \int_0^t e^{-\rho(t-\tau)} \sin(\omega_s(t-\tau)) (1 - \cos(\psi(t) - \psi(\tau))) d\tau
\]

where \(g\) decays exponentially. Let \(P = \sqrt{\frac{J}{m_{ij}}}\). Using a new time variable \(s = t/P\) and the scaled function \(s \to \psi(p s)\) (denoted as \(\psi(s)\) to simplify notation), we rewrite the above equation as

\[
\psi'(s) + \alpha \psi(s) + \sin \psi(s) = \beta + \gamma(s).
\]

Here \(prime\) denotes derivative with respect to \(s\),

\[
\alpha = \frac{D_{\rho}}{\sqrt{-m_{ij} J}}, \quad \beta = \frac{T_m - D_P \omega_s - P}{-m_{ij}}.
\]

\[
\gamma(s) = \frac{m_{ij} \rho P}{L_s} \left[ \frac{\rho g(s)}{\rho^2} \right] \sin(\psi(s) - \psi(\tau)) d\tau
\]

\[
+ 2 \int_0^t e^{-\rho(s-\tau)} \sin(\omega_s(s-\tau)) \sin^2 \frac{\psi(s) - \psi(\tau)}{2} d\tau.
\]

From the global existence of solution to (2.7) (see end of Section II), the global existence of solution to (5.1) follows.

**Theorem 5.1:** Consider the SG model (2.7) and its corresponding forced pendulum equation (5.1), where clearly \(\alpha > 0\) and \(\gamma \in L^\infty([0, \infty))\) is a continuous function. Assume that \(0 < \beta < 1\) and \(|\gamma|_{L^\infty} < d\) for some constant \(0 < d < \beta\) with \(\beta + d < 1\). For each \(\mu \in (0, d)\), let \(0 < \psi_2(\mu) < \psi_1(\mu) < \pi/2\) be determined by \(\sin(\psi_1(\mu)) = \beta + \mu\) and \(\sin(\psi_2(\mu)) = \beta - \mu\). Assume that \(\alpha > 2 \sin(\psi_1(d)/2)\). Denote \(\kappa(\mu) = |\psi_1(\mu) - \psi_2(\mu)| + 5\mu/\alpha^2\) for all \(\mu \in [0, d]\).

If for some constant \(0 < \eta < 1\) and all \(\mu \in (0, d]\)

\[
\kappa(\mu) > \frac{\pi}{2}, \quad \eta > \frac{|m_{ij}|}{L_s} \left( \sin(\kappa(\mu)) + 2 \sin^2 \frac{\kappa(\mu)}{2} \right). (5.2)
\]

then (2.7) is almost globally asymptotically stable.

**Proof:** Consider a trajectory \((i_1, i_2, \omega, \delta)\) of (2.7) and the corresponding trajectory \((\psi, \psi)\) of (5.1) (they are related via (3.5)). Under the assumptions of the theorem, we will show that \(\gamma = \limsup \gamma(s) = 0\). This will imply that (2.7) is almost globally asymptotically stable. Indeed, if \(\gamma = 0\), we can apply Theorem (4.16) to (5.1) to conclude that for any \(\mu \in (0, d]\) and \(\epsilon > 0\), there exists a \(T_{\mu, \epsilon} > 0\) such that for each \(s_1, s_2 > T_{\mu, \epsilon}\) we have

\[
|\psi(s_1) - \psi(s_2)| \leq \psi_1(\mu) - \psi_2(\mu) + 4\mu/\alpha^2 + \epsilon.
\]

This implies that \(\lim_{t \to \infty} \psi(s) = \psi_f\) for some finite \(\psi_f\). It now follows using Barbalat’s lemma that \(\lim_{t \to \infty} \delta(t) = 0\). From (3.5) we get that \(\lim_{t \to \infty} \delta(t) = 0\). It is straight forward to infer from (2.7) that

\[
\lim_{t \to \infty} (i_{1,t}(i), i_{2,t}(i), \omega(t), \delta(t)) = (i_{1,0}, i_{2,0}, \omega_0, \delta_0)
\]

for some constants \(i_{1,0}\) and \(i_{2,0}\). \((i_{1,0}, i_{2,0}, \omega_0, \delta_0)\) is necessarily an equilibrium point for (2.7). Hence the almost globally asymptotic convergence of (2.7) follows.

We will now show that \(\gamma = 0\). Assume not. Fix \(\mu_0 = \gamma + \epsilon\) for some \(\epsilon > 0\) such that \(\epsilon \mu_0 + \epsilon < \sqrt{T\mu_0}\). We get from Lemma 4.16 that there exists \(T_1 > 0\) such that for all \(s_1, s_2 > T_1\), \(|\psi(s_1) - \psi(s_2)| < k(\mu_0)\). It follows that there exists \(T_2 > T_1\) such that for all \(s > T_2\), the first term in the expression for \(\gamma(s)\) is bounded by \(\epsilon/3\), the second term by \(m_{ij} s \kappa(\mu_0)/(L_i l_s) + \epsilon/3\) and the third term by \(2m_{ij} s \kappa(\mu_0)/(L_i l_s) + \epsilon/3\). Thus for any \(s > T_2\), we get

\[
\gamma(s) < \frac{|m_{ij}|}{L_i l_s} \left( \sin(\kappa(\mu_0)) + 2 \sin^2 \frac{\kappa(\mu_0)}{2} \right) + \epsilon
\]

\[
< \epsilon \mu_0 + \epsilon < \sqrt{T\mu_0}
\]

This contradicts \(\gamma = \limsup |\gamma(s)|\). Hence \(\gamma = 0\).

**Remark 5.2:** Theorem 5.1 enables us to identify a large set of (not necessarily practical) SG parameters for which the model (2.7) is almost globally asymptotically stable (for example the conditions in the theorem hold if \(V\) is large and \(J\) is small). Due to the complexity of characterizing the global dynamics of nonlinear systems, this result is nontrivial. In the journal version of this paper we will present less restrictive conditions on the SG parameters, under which (2.7) is almost globally asymptotically stable. These conditions are obtained by applying Theorem 4.16 to (5.1).

**REFERENCES**


