Improving the exponential decay rate by back and forth iterations of the feedback in time

Vivek Natarajan and George Weiss

Abstract—We consider the control system \( \dot{x} = Ax + Bu \), where \( A \) generates a strongly continuous semigroup \( \mathbb{T} \) on the Hilbert space \( X \) and the control operator \( B \) maps into the dual of \( \mathcal{D}(A^{*}) \), but it is not necessarily admissible for \( \mathbb{T} \). We prove that if the pair \((A,B)\) is both forward and backward optimizable (our definition of this concept is slightly more general than the one in the literature), then the system is exactly controllable. This is a generalization of a well-known result called Russell’s principle. Moreover the usual stabilization by state feedback \( u = Fx \), where \( F \) is an admissible observation operator for the closed-loop semigroup, can be replaced with a more complicated periodic (but still linear) controller. The period \( \tau \) of the controller has to be chosen large enough to satisfy an estimate. This controller can improve the exponential decay rate of the system to any desired value, including \( -\infty \) (deadbeat control). The corresponding control signal \( u \), generated by alternately solving two exponentially stable homogeneous evolution equations on each interval of length \( \tau \), back and forth in time, will still be in \( L^2 \). The better the decay rate that we want to achieve, the more iterations the controller needs to perform, but (unless we want to achieve \( -\infty \)) the number of iterations needed on each period is finite.

I. INTRODUCTION

We consider a linear infinite-dimensional system described by

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{1.1}
\]

where \( A \) generates a strongly continuous semigroup on the Hilbert space \( X \), the \( L^2_{loc} \) signal \( u \) takes values in another Hilbert space \( U \) and \( B \) is a possibly unbounded control operator, which means that it maps \( U \) into a space that contains \( X \) densely.

Suppose that the system is optimizable, a concept that is a generalization of stabilizability from finite-dimensional control theory. Roughly speaking, it means that there exists a state feedback operator \( F \) (densely defined on \( X \), with values in \( U \)) such that \( A + BF \) generates an exponentially stable semigroup on \( X \). The precise definition will be given in Section III. Also suppose that the above system is backward optimizable which, roughly speaking, means that there exists a backward state feedback operator \( F_b \) (densely defined on \( X \), with values in \( U \)) such that \(-A + BF_b\) generates an exponentially stable semigroup on \( X \). The purpose of this paper is to show that under these assumptions, for any desired exponential decay rate, we can construct a time-varying linear controller for the above system that ensures the decay of the system state at the desired rate. In particular, we are also able to achieve the decay rate \(-\infty\), which means that the state of the system becomes zero in a finite time (if no disturbance acts on the system). In any scenario, the control signal \( u \) generated by the controller is in \( L^2 \).

In Rebarber and Weiss [1], it was established that if a system is both forward and backward optimizable, then it is exactly controllable (see also Weiss and Rebarber [2]). This principle of optimizability implying controllability is a generalization of Russell’s principle, established in Russell [3] (in finite dimensions) and Russell [4] (for the wave equation), which states that if a system is forward and backward stabilizable, then it is exactly controllable. We use a slightly more general definition of optimizability as compared to [1], [2] (we drop the admissibility requirement on \( B \) and we adjust the definition accordingly) and hence we generalize Russell’s principle even further. We provide an algorithm in which a forward and a backward dynamical system are simulated alternately to construct a sequence of functions whose superposition is the desired control signal. This idea has already appeared in Cindea, Micu and Tucsnak [5] in the specific context of a system with skew-adjoint \( A \) and bounded \( B \), with a specified structure (that requires \( B \) to be “more than” bounded).

However, the main focus of this paper is not exact controllability, but a controller that can provide a desired decay rate for the system. We propose a linear periodic controller that performs several iterations of the back and forth algorithm mentioned earlier on each interval of the form \( [k - 1)\tau, k\tau) \) \((k \in \mathbb{N})\). Here \( \tau \), the period of the controller, has to be large enough for the algorithm to work.

In Auroux and Blum [6] a back and forth nudging algorithm is proposed for estimating the initial state of a linear system, see also Kuchment and Kunyansky [7]. Back and forth observers have been proposed in Shim, Tanwani, and Ping [8] for finite dimensional, possibly nonlinear systems and in Ramdani, Tucsnak, and Weiss [9] for linear infinite-dimensional systems (see also Ito, Ramdani and Tucsnak [10]). These observers are based on a dual of Russell’s principle, i.e., forward and backward detectability implies observability. In [9], detectability has been replaced with the more general notion of estimatability, which is defined in a more general way than in [2]. The concepts and terminology in this paper are dual to those in [1], [9].

II. DISCUSSION IN FINITE DIMENSIONS

To make our ideas more easily understood, we first present them in the simple context of finite-dimensional control

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V. Natarajan (n.vivek.n@gmail.com) and G. Weiss (gweiss@eng.tau.ac.il) are with the School of Electrical Engineering, Tel Aviv University, Ramat Aviv 69978, Israel.
theory. Thus in this section $X = \mathbb{C}^n$, $U = \mathbb{C}^m$ and $A, B, F$ and $F_0$ are matrices of suitable dimensions such that $A + BF$ and $-A + BF_0$ are stable (Hurwitz). The basic idea is very simple and is described next.

On some interval $[0, \tau]$ (the choice of $\tau$ will be clarified later) we apply the control law $u^0(t) = F_x^0(t)$ to the system

$$
\dot{x}^0(t) = Ax^0(t) + Bu^0(t), \quad x^0(0) = x_0,
$$

which is (1.1) but with slightly changed notation. Then the state trajectory $x^0$ solves

$$
\dot{x}^0(t) = (A + BF)x^0(t),
$$

with $x^0(\tau) = e^{(A+BF)\tau}x_0$. However, if we are not satisfied with the rate of decay of the state with this control law, an alternative control law as discussed next can be used to improve the decay rate. To this end we solve the differential equation

$$
x^0(t) = (A - BF_0)x^0(t)
$$

backwards in time, with the initial state $x^0(\tau) = x^0(\tau)$. The final state $x^0(0)$ is then given by

$$
x^0(0) = e^{-(A+BF_0)\tau}x^0(\tau) = e^{-(A+BF_0)\tau}e^{(A+BF)\tau}x_0.
$$

Next, we go forward in time again, solving

$$
x^1(t) = (A + BF)x^1(t),
$$

with the initial state $x^1(0) = x^0(0)$. We obtain

$$
x^1(\tau) = e^{(A+BF)\tau}x^0(0) = e^{(A+BF)\tau}e^{-(A+BF_0)\tau}x^0(\tau).
$$

This is clearly better than going forward in time just once, i.e., $x^1(\tau)$ is usually smaller than $x^0(\tau)$, because the above product of stable matrix exponentials is usually small (it converges to zero when $\tau \to \infty$).

However, in reality, we cannot go back and forth in time. The true control input that is given to the system is the superposition of the three inputs corresponding to the three state trajectories mentioned above. This will cause the true state trajectory to be the superposition of the three state trajectories (backwards trajectories are counted with a minus sign). We now explain this in more detail.

It is easy to check that if we use the control signal

$$
u_{alt}(t) = F_x^0(t) + F_0x^0(t) + Fx^1(t)
$$

in (1.1), the resulting state trajectory is

$$
x_{alt}(t) = x^0(t) - x^0(t) + x^1(t)
$$

and it satisfies $x_{alt}(0) = x_0$ and $x_{alt}(\tau) = x^1(\tau)$. If we choose $\tau$ large enough such that

$$
\left\|e^{(A+BF)\tau}e^{-(A+BF_0)\tau}\right\| < 1,
$$

(2.1)

then $x^1(\tau)$ will be smaller than $x^0(\tau)$, the final state that we can achieve by using the feedback operator $F$ in the standard way. Hence we have increased the rate of decay of the state of (1.1). After this, we continue to the next time interval of the same length, $[\tau, 2\tau]$, and we apply the same process, and so on. This process is illustrated in Figure 1.

The above superposition argument hinges on the linearity of the plant. Hence the proposed control law does not apply, in general, to nonlinear systems. However, the generalization to time-varying linear systems is straightforward.

If we repeat the above back and forth iterations $N$ times on each time interval of length $\tau$, i.e., we perform $N$ backward simulations each followed by a forward simulation then, denoting the $n$-th forward state trajectory by $x^n$, we have

$$
x^N(\tau) = \left[e^{(A+BF)\tau}e^{-(A+BF_0)\tau}\right]^N x^0(\tau)
$$

and it follows from this and (2.1) that the control signal constructed analogously to $u_{alt}$ will result in an even better decay rate for the state trajectory. When $N = \infty$, we obtain dead-beat control which guarantees that $x_{alt}(\tau) = 0$.

We do not really need that both $A + BF$ and $-A + BF_0$ should be stable. What we really need is the condition (2.1). For example, if $A$ is skew-adjoint and $A + BF$ is stable, then we do not need any backward stabilization, we may take $F = 0$ and (2.1) will hold for $\tau$ large enough.

In the particular case when we can take $F_0 = 0$, the feedback system becomes much simpler. Assuming that we do $N$ back and forth iterations in each time interval of length $\tau$, we get from a simple reasoning that for $t \in [0, \tau]$,

$$
u_{alt}(t) = \sum_{n=0}^{N} Fx^n(t) = \sum_{n=0}^{N} F[e^{(A+BF)\tau}e^{-(A+BF_0)\tau}]^n x_0.
$$

Thus, in the interval $t \in [0, \tau]$, the control signal $u_{alt}$ is the output of the controller

$$
\dot{z}(t) = (A + BF)z(t), \quad u_{alt}(t) = Fz(t),
$$

(2.2)

with initial state $z(0) = Tz_0$, where

$$
T = \sum_{n=0}^{N} \left(e^{-A\tau}e^{(A+BF)\tau}\right)^n.
$$
where we have denoted $A$ at the same moments). It is sort of an impulsive system. Indeed, (2.1) (with $F$ used in the later sections). In this figure and in the remaining set to be $\frac{d}{dt}x = Ax + Bu$, the story is similar, and so on. Thus, the controller is linear $[\begin{array}{c} \begin{array}{c} T \\ B \\ A \\ \end{array} \\ \begin{array}{c} u \\ x \\ \end{array} \end{array}] = \left( I - e^{-At} e^{(A+BF)t} \right)^{-1}$. Indeed, (2.1) (with $F_0 = 0$) implies that the spectral radius of $e^{(A+BF)t} e^{-At}$ is less than 1, but this is the same as the spectral radius of $e^{-At} e^{(A+BF)t}$, so that the Neumann series giving $T$ converges.

In the interval $[\tau, 2\tau)$ the controller will work in the same way as in $[0, \tau)$: its initial state is set as $z(\tau) = x_0$, and then it works according to (2.2). For the interval $[2\tau, 3\tau)$ the story is similar, and so on. Thus, the controller is linear and periodic. There is one disturbing aspect in this controller which is that its state $z$ may jump at the moments $k\tau$ (it is set to be $x_0$, regardless of the limit from the left of $z(t)$ at the same moments). It is sort of an impulsive system.

For the case $F_0 = 0$, the closed-loop system described above (with the controller as in (2.2)) is shown in Figure 2, where we have denoted $A^F = A + BF$ (because this notation is used in the later sections). In this figure and in the remaining part of this paper, the subscript alt is dropped without risk of ambiguity.

For finite-dimensional systems, the existence of $F$ and $F_0$ as discussed implies that the pair $(A, B)$ is controllable. Hence the closed loop poles can be placed at any desired values and the back and forth procedure as outlined is unnecessary to improve the rate of decay. However, for infinite-dimensional systems this is not the case. Typically, even if one linear stabilizing controller $F$ can be identified, finding others with better decay rates is a non-trivial problem.

III. FORWARD AND BACKWARD OPTIMIZABILITY

Let $X$ and $U$ be Hilbert spaces. We consider a control system $(A, B)$ described by (1.1), where $A : D(A) \to X$ is the infinitesimal generator of a strongly continuous semigroup $\mathbb{T}$ on $X$. As is customary, we denote by $X_{-1}$ the closure of $X$ with respect to the norm $\|z\|_{-1} = \| (\beta I - A)^{-1} z \|$, for some $\beta \in \rho(A)$ (the resolvent set of $A$). We assume that the control operator satisfies $B \in \mathcal{L}(U, X_{-1})$ (i.e., $B$ is a bounded linear operator from $U$ to $X_{-1}$), but we do not assume that $B$ is an admissible control operator for $\mathbb{T}$ (we refer to Tucsnak and Weiss [11] for details on admissibility).

Definition 3.1: The pair $(A, B)$ is (forward) optimizable if the following conditions hold:

(i) There exists an operator $A^F : \mathcal{D}(A^F) \to X$, that generates an exponentially stable semigroup $\mathbb{T}^F$ on $X$. We denote $X^F := \mathcal{D}(A^F)$, with the norm $\|z\|_1 = \| (\beta I - A^F)^{-1} z \|$, for some $\beta \in \rho(A^F)$.

(ii) There exists $F \in \mathcal{L}(X^F, U)$ that is an admissible observation operator for $\mathbb{T}^F$, and the exponential stability of $\mathbb{T}^F$ that the function $u : [0, \infty) \to U$ defined (for almost every $t \geq 0$) by

\[ u(t) = F_A x_0 \quad (3.2) \]

is in $L^2([0, \infty); U)$. Here $F_A$ is the $L^2$-valued $L^2$ function from (3.2).

Proposition 3.2: The state trajectory $x(t) = \mathbb{T}^F x_0$ (defined earlier as the solution of a Cauchy problem) is the (strong) solution in $X_{-1}$ of the non-homogeneous equation

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \]

where $u$ is the $L^2$ function from (3.2).

Proof. Indeed, this follows directly from (3.1) when $x_0 \in \mathcal{D}(A^F)$, and it implies that for such $x_0$ we have

\[ x(t) - x_0 = \int_0^t [Ax(\sigma) + Bu(\sigma)] d\sigma \quad \forall t \in [0, \infty). \]

Both terms under the integral, when regarded as $X_{-1}$-valued $L^2$ functions, depend continuously on $x_0$ and $X_{-1}$ of $X$. Hence the integral, as an element of $X_{-1}$, depends continuously on $x_0$. It follows by continuous extension that the equality remains valid for every $x_0 \in X$. Hence, for every $x_0 \in X$, $x$ is the (strong) solution in $X_{-1}$ of the non-homogeneous equation under consideration, in the sense of Definition 4.1.1 in [11].

Remark 3.3: In finite-dimensional control theory, the property in Definition 3.1 is called stabilizability. When dealing
with unbounded operators, stabilizability has several (non-
equivalent) generalizations. A more restrictive one has been
given (and called stabilizability) in [12], where there are
additional assumptions, for example, \((A,B,F)\) has to be a
regular triple. The definition of optimizability in this paper
is close to, but not equal to the one in [2]. Indeed, in [2] \(B\)
is assumed to be an admissible control operator for \(\mathbb{T}\),
while here we have dropped this requirement. Even for admissible
\(B\), the definition of optimizability in [2] looks different from
the one given here, but they are equivalent, see Propositions
3.2–3.4 in [2].

Definition 3.1 is the precise dual of the definition of
estimatability given in [9], and Remark 3.3 is in fact a dual
formulation of Remark 2.2 from [9]. To save space, we shall
not formulate the dual versions of the other remarks in [9],
instead we only mention that if \(B\) is an admissible control
operator for \(\mathbb{T}\), then the exact controllability of \((A,B)\)
implies its optimizability. We also mention that it is far from simple
to find a stabilizing state feedback operator for \((A,B)\), see
the comments in [9].

Definition 3.4: As in Definition 3.1, we assume that \(A\) is
the generator of a semigroup \(\mathbb{T}\) on \(X\) and \(B \in \mathcal{L}(U,X)\).
The pair \((A,B)\) is backward optimizable if the following
conditions hold:

(i) There exists an operator \(A^F B : \mathcal{D}(A^F) \to X\), that generates
an exponentially stable semigroup \(\mathbb{T}^F_b\) on \(X\). We
denote \(X^F_b = \mathcal{D}(A^F)\), with the norm
\[ ||z||_1 = ||(\beta I - A^F)z||, \]
for some \(\beta \in \rho(A^F)\).

(ii) There exists \(F_b \in \mathcal{L}(X^F_b, U)\) that is an admissible
observation operator for \(\mathbb{T}^F_b\), such that
\[ A^F B z = -Az + BF_b z \quad \forall z \in \mathcal{D}(A^F). \]  

We call \(F_b\) as introduced above a backward stabilizing
state feedback operator for \((A,B)\). Let \(x_0 \in X\) be the initial
state \((t=0)\) for the Cauchy problem \(\dot{x}(t) = A^F x_t\). This
problem has the mild solution \(x^F(t) = \mathbb{T}^F_b x_0\). As in
the case of forward optimizability, there exists \(u \in L^2([0,\infty);U)\)
defined (for almost every \(t \geq 0\)) by
\[ u(t) = F^T_b \mathbb{T}^F_b x_0. \]  

Here \(F^T_b A\) is the \(A\)-extension of \(F_b\).

The concept of backward optimizability is dual to that of
backward estimatability as introduced in [9]. We could make
similar remarks about this concept as for optimizability. Next
we state the equivalent of Proposition 3.2.

Proposition 3.5: Consider \(x^F(t) = \mathbb{T}^F_b x_0\) (defined earlier
as the solution of a Cauchy problem) and the \(L^2\) function
\(u\) from (3.4). On any interval \([0,T]\), \(x(t) = x^F(T-t)\) is the
(strong) solution in \(X_{-1}\) of the non-homogeneous equation
\[ \dot{x}(t) = Ax(t) - Bu(T-t), \quad x(T) = x_0. \]

Proof. It follows from (3.3) that when \(x_0 \in \mathcal{D}(A^F)\),
\[ x^F(T-t) - x^F(T) = \int_{T-t}^T [Ax^F(\sigma) - Bu(\sigma)]d\sigma, \]
for all \(t \in [0,T]\). Using the change of variables \(\sigma = T - \gamma\)
and some simplifications we get
\[ x(t) - x(0) = \int_0^t [Ax(\gamma) - Bu(T - \gamma)]d\gamma \quad \forall t \in [0,T]. \]

Hence when \(x_0 \in \mathcal{D}(A^F)\), \(x(t)\) is the strong solution in \(X_{-1}\)
of the non-homogeneous equation under consideration. Using
a continuity argument, contained in the proof of Proposition
3.2, the same can be established for any \(x_0 \in X\). \(\square\)

IV. THE TIME-VARYING CONTROLLER

We assume that the infinite dimensional system (1.1) is
both forward and backward optimizable. Theorem 4.1 formalizes
the ideas presented in the finite dimensional context earlier
by proposing a time-varying controller that stabilizes (1.1). Any
desired rate of decay can be obtained for this system by choosing
the controller parameters appropriately.

Theorem 4.1: With notation as in Definitions 3.1 and 3.4
choose \(\tau > 0\) such that
\[ \|F^T_b \mathbb{T}^F_b \| < 1. \]  

Fix \(N \geq 0\). Let \(x\) be the state of the system (1.1). Define
the control signal \(u(t)\) as
\[ u(t) = \sum_{n=0}^N b_t(t) + \sum_{n=1}^N \mu^F_n(t), \]
where, for each integer \(k \geq 0\), every \(t \in [k\tau, (k+1)\tau]\)
each \(n \in \{1,\ldots,N\},
\[ u^F_t(f) = F^T_b \mathbb{T}^F_{b,\tau} x(t\tau), \]
\[ u_t^F(f) = F^T_b \mathbb{T}^F_{b,\tau} (F^T_b \mathbb{T}^F_{b,\tau})^{-1} x(t\tau), \]
\[ u^F_{\mu}(t) = F^T_b \mathbb{T}^F_{b,\tau} (F^T_b \mathbb{T}^F_{b,\tau})^{-1} x(t\tau). \]

This control signal \(u\) stabilizes (1.1). Furthermore the rate
of decay of \(x\) improves with \(N\). In particular, for any \(k \geq 0\)
\[ \|x((k+1)\tau)\| \leq \|F^T_b \mathbb{T}^F_{b,\tau} \| \|F^T_b \mathbb{T}^F_{b,\tau}\| \|x(k\tau)\|. \]

Proof. Fix \(k \geq 0\) and consider the following set of evolution
equations on the interval \([k\tau, (k+1)\tau)\):
\[ x^F(t) = Ax^F + Bu, \quad x^F(t\tau) = (F^T_b \mathbb{T}^F_{b,\tau} \mathbb{T}^F_{b,\tau})^n x(t\tau), \]
for \(n = 0, \ldots, N\). From Proposition 3.2, it follows that each of
these equations has a strong solution on \([k\tau, (k+1)\tau)\) given
by
\[ x^F(t) = \mathbb{T}^F_{b,\tau} (F^T_b \mathbb{T}^F_{b,\tau})^n x(k\tau). \]

Let \(x_f = \sum_{n=0}^N x^F_n\). Then
\[ x_f(t\tau) = \sum_{n=0}^N x^F_n(t\tau) = \sum_{n=0}^N (F^T_b \mathbb{T}^F_{b,\tau} \mathbb{T}^F_{b,\tau})^n x(t\tau), \]
and
\[ x_f((k+1)\tau) = \sum_{n=0}^N x^F_n((k+1)\tau) = \sum_{n=0}^N (F^T_b \mathbb{T}^F_{b,\tau} \mathbb{T}^F_{b,\tau})^n x(t\tau). \]

Next consider the evolution equations
\[ x^F_{\mu}(t) = A x^F_{\mu}(t) - Bu, \quad x^F_{\mu}((k+1)\tau) = (F^T_b \mathbb{T}^F_{b,\tau} \mathbb{T}^F_{b,\tau})^n x(t\tau), \]
for $n = 1, \ldots, N$, again on the interval $[k\tau, (k+1)\tau)$. From Proposition 3.5, it follows that each of these equations has a strong solution

$$x_n^b(t) = \mathbb{P}_{b,1}^{\tau} \mathbb{P}_{b,\tau}^{\tau} T_{b,\tau}^{T_{b,\tau}^{n-1}} x(k\tau).$$

Let $x_b = \sum_{n=1}^{N} x_n^b$. Then

$$x_b(k\tau) = \sum_{n=1}^{N} x_n^b(k\tau) = \sum_{n=1}^{N} \mathbb{P}_{b,\tau}^{\tau} T_{b,\tau}^{n-1} x(k\tau),$$

and

$$x_b((k+1)\tau) = \sum_{n=1}^{N} x_n^b((k+1)\tau) = \sum_{n=1}^{N} \mathbb{P}_{b,\tau}^{\tau} T_{b,\tau}^{n} x(k\tau).$$

It is easy to check that $x_f - x_b$ is the strong solution to

$$\dot{x} = Ax + Bu,$$

on the interval $[k\tau, (k+1)\tau)$ and that $(x_f - x_b)(k\tau) = x(k\tau)$. Therefore it is equal to the state of (1.1) on $[k\tau, (k+1)\tau)$. Hence for all $t \in [k\tau, (k+1)\tau)$

$$x(t) = \sum_{n=0}^{N} \mathbb{P}_{b,\tau}^{\tau} T_{b,\tau}^{n} x(k\tau) - \sum_{n=1}^{N} \mathbb{P}_{b,\tau}^{\tau} T_{b,\tau}^{n} \mathbb{P}_{b,\tau}^{\tau} T_{b,\tau}^{n-1} x(k\tau).$$

Clearly

$$x((k+1)\tau) = (\mathbb{P}_{b,\tau}^{\tau} T_{b,\tau}^{n}) x(k\tau).$$

It follows that the proposed control law is stabilizing and

$$\|x((k+1)\tau)) \| \leq \|\mathbb{P}_{b,\tau}^{\tau} T_{b,\tau}^{n}\| \|\mathbb{P}_{b,\tau}^{\tau} x(k)\|.$$

As in finite dimensions, if we can take $F_b = 0$, the feedback system becomes much simpler (Figure 2) and if we do $N$ back and forth iterations in each time interval of length $\tau$, then the operator $T$ is given by

$$T = I + \mathbb{P}_{b,\tau}^{\tau} T_{b,\tau}^{n} + \mathbb{P}_{b,\tau}^{\tau} T_{b,\tau}^{n-1} \mathbb{P}_{b,\tau}^{\tau} T_{b,\tau}^{n-2} \mathbb{P}_{b,\tau}^{\tau} T_{b,\tau}^{n-3} \mathbb{P}_{b,\tau}^{\tau} T_{b,\tau}^{n-4} \ldots \mathbb{P}_{b,\tau}^{\tau} T_{b,\tau}^{n-1}.$$\n
If $N = \infty$ we get the dead-beat control and by the Neumann series we obtain

$$T = (I - \mathbb{P}_{b,\tau}^{\tau} T_{b,\tau}^{n})^{-1}.$$

As noted before, given a stabilizing controller, finding other controllers that improve the decay rates is a non-trivial task. The theoretical contribution of this work is providing an control law that can get any desired rate of convergence for the state. From a practical standpoint, we recognize that the controller is complicated and hard to implement (it is infinite-dimensional, it requires instantaneous computations and periodically changing the state of the controller in a discontinuous way). However, similar objections could be raised against most controllers in infinite-dimensional control theory. Using the proposed algorithm to construct open loop controls for stable systems is more practical. The rigorous numerical analysis of the back and forth algorithm contained in [5] (for exact controllability) and in Haine and Ramdani [13] (for exact observability) could be useful while implementing our algorithm.

V. CONCLUSIONS AND FUTURE WORKS

In this work we have proposed a time-varying control law to obtain any desired rate of decay for the state for forward and backward optimal systems. In practice, implementing this control law would involve a certain computational delay. The behavior of the control law with delay will be of interest.

REFERENCES


