Discretizing continuous-time controllers for infinite-dimensional linear systems

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Abstract
We investigate a fundamental question about the discretization of continuous-time controllers. We consider a linear feedback system that works in continuous time, and has satisfactory performance. We want to replace the controller with a combination of a discrete-time system (a digital processor), an analog-to-digital converter (a filter with a sampler) and a digital-to-analog converter (a hold device). The problem is to determine whether it is possible to achieve closed-loop performance arbitrarily close to the performance of the original feedback system. The performance is represented by the input-output maps of the closed-loop system, and the distance is measured in the operator norm. We show that arbitrarily close performance can be achieved, by choosing a sufficiently small sampling period and an appropriate controller structure, if the original controller is strictly proper.

We first introduce our terminology about transfer functions, which is fairly standard. A \( C^{p\times m} \)-valued analytic function \( G \) is said to be a well-posed transfer function if it is bounded and analytic on some right half-plane in \( \mathbb{C} \). Such a function defines an input-output operator from \( L^2_{\text{loc}}([0, \infty), C^m) \) to \( L^2_{\text{loc}}([0, \infty), C^p) \) via the formula \( \hat{y}(s) = G(s)\hat{u}(s) \), where \( \hat{u} \) and \( \hat{y} \) are the Laplace transforms of the input \( u \) and the output \( y \). This operator is time-invariant (i.e., right shift-invariant) and hence causal. In this paper we represent linear and time-invariant (LTI) systems by their transfer functions, i.e., we ignore their state space. By some abuse of notation, we use the same symbol to denote both the transfer function and the corresponding input-output operator. Finite-dimensional systems have rational transfer functions, but in this paper we shall never assume that the transfer functions we deal with are rational, because it would not simplify our arguments. A transfer function \( G \) is called strictly proper if \( \lim_{|s| \rightarrow \infty} \| G(s) \| = 0 \).

The space \( H^\infty(\mathbb{C}_+, C^{p\times m}) \) consists of bounded analytic \( C^{p\times m} \)-valued functions defined on the right half-plane \( \mathbb{C}_+ \). This is a Banach space with the norm \( \| G \| = \sup_{s \in \mathbb{C}_+} \| G(s) \| \), where \( \| G(s) \| \) denotes the greatest singular value of \( G(s) \). For transfer functions \( G \in H^\infty(\mathbb{C}_+, C^{p\times m}) \), the corresponding input-output operator is in \( \mathcal{L}(L^2([0, \infty), C^m), L^2([0, \infty), C^p)) \). In this case the norm of \( G \) as an input-output operator is equal to the \( H^\infty \)-norm of \( G \). Such transfer functions are called stable. The transfer function \( G \) is called exponentially stable if it is bounded and analytic on a right half-plane which strictly includes \( \mathbb{C}_+ \). In other words, there exists a \( \delta > 0 \)
such that the function $G_\delta$ defined by $G_\delta(s) = G(s - \delta)$ in $H^\infty(C_+, C^{p\times m})$.

Our terminology and conventions for discrete-time transfer functions are very similar. A $C^{p\times m}$-valued discrete-time well-posed transfer function is bounded and analytic on the complement of some disk centered at 0. Such a transfer function defines a time-invariant operator from $C^m$-valued sequences to $C^p$-valued sequences, via the $z$-transform. Again, we use the same symbol to denote the transfer function and the corresponding operator. The stable transfer functions are those in $H^\infty(D^c, C^{p\times m})$, where $D^c$ denotes the complement of the unit disk.

Figure 1: A plant with a stabilizing controller, both working in continuous time. The plant and controller are represented by their transfer functions $P$ and $C$.

Suppose that an LTI plant is given, and an LTI controller has been designed such that the feedback connection of the two would perform in a satisfactory way. This control system is shown in the familiar Figure 1, with $P$ and $C$ denoting the transfer functions of the plant and of the controller. The signals $v, u_p, u_c, y_p, y_c$ have values in $C^m$, $r, y_r, y_c$ have values in $C^p$ and all the signals have their components in $L^2_{loc}[0, \infty)$. The performance is evaluated by requirements imposed on (or cost functions depending on) the transfer function from $[v \, r]$ to $[u_p \, y_c]$. This transfer function is

$$
T = \begin{bmatrix} (I - CP)^{-1} & PC(I - PC)^{-1} \\ P(I - CP)^{-1} & (I - PC)^{-1} \end{bmatrix},
$$

and the performance may be evaluated, for example, according to some (e.g., mixed sensitivity) $H^\infty$ performance criterion. A basic requirement on the controller is that it should be stabilizing, i.e., the feedback system is stable, which means that

$$
T \in H^\infty(C_+, C^{(p+m)\times(p+m)}).
$$

For technological reasons, we would like to implement a controller based on digital technology, so that it should consist of a discrete-time system (e.g., a digital signal processor) preceded by a filter and a sampler and followed by a hold device. Such a controller cannot be described by the transfer function $C$, but nevertheless, we would like to obtain a control system with very similar behavior to the one in Figure 1. More precisely, we would like to replace the controller $C$ by the cascade connection shown in Figure 2, such that the operator from $v$ and $r$ to $u_p$ and $y_c$ should remain close to what it was, i.e., to $T$ (in the $L^2$-induced operator norm).

We have to specify what the blocks appearing in Figure 2 are. The number $\tau > 0$ is the sampling period. The hold filter $H_\tau$ is an LTI system with transfer function

$$
H_\tau(s) = \frac{1 - e^{-\tau s}}{\tau s},
$$

so that at any given time, its output is the average of the input over the last time interval of length $\tau$. The input signal to the controller, $y_c$, is first processed by such a filter, which transforms it into a continuous function of time. The sampling operator $S_\tau$ is an operator from continuous (vector-valued) functions on $[0, \infty)$ to (vector-valued) sequences, defined for all $k \in \mathbb{N}$ by

$$
(S_\tau h)_k = \sqrt{\tau} \cdot h(k\tau).
$$

The central block in Figure 2 is an LTI system operating in discrete time, represented by its discrete-time transfer function $C^d$. The pulse generator $S_\tau^r$ is an operator from (vector-valued) sequences to distributions supported on $[0, \infty)$, defined by

$$
S_\tau^r a = \sqrt{\tau} \sum_{k \in \mathbb{N}} a_k \delta_{k\tau},
$$

where $a = (a_k)$ is a sequence and $\delta_t$ denotes the unit pulse (Dirac measure) at $t$. The last operator is denoted by $S_\tau^r$ because it is the adjoint operator of $S_\tau$ in some well-defined sense (see [7] for details). We see in Figure 2 that the train of pulses generated by the pulse generator is filtered by a second hold filter, so that it is transformed into a step function (constant on intervals of length $\tau$).

The controller shown in Figure 2 can be written as

$$
C^SD = H_\tau S_\tau^r C^d S_\tau^r H_\tau.
$$

We call a controller with this structure a sampled-data controller. The system obtained by putting $C^SD$ in place of $C$ in Figure 1 is called a sampled-data control system. The operator $C^SD$ is linear and causal, but not time-invariant. However, it has some time-invariance: it is invariant to right shifts by multiples of $\tau$ (i.e., it is $\tau$-periodic). When the signals are in a frequency range significantly below $\pi/\tau$ (rad/sec), $C^SD$ may appear to be time-invariant. We denote by $T^{SD}$ the operator corresponding to $T$ from (1), but with the new controller:

$$
T^{SD} = \begin{bmatrix} (I - C^{SD}P)^{-1} & C^{SD}(I - PC^{SD})^{-1} \\ P(I - C^{SD}P)^{-1} & (I - PC^{SD})^{-1} \end{bmatrix}.
$$

It might be argued that the presence of the pulse generator $S_\tau^r$ in (2) is unrealistic, since nothing of this sort appears in practical controller implementations. However, what does appear in practice is the product $H_\tau S_\tau^r$.
For the sake of some mathematical arguments (and some conceptual elegance) we prefer to keep the factors \( H \tau \) and \( S \tau \) separate. The relationship between our operators and the operators \( H \) (hold operator) and \( S \) (ideal sampler) used in [3] is as follows:

\[
H = \sqrt{\tau} H\tau S\tau, \quad S = \frac{1}{\sqrt{\tau}} S\tau.
\]

In [3], the proposed structure of a sampled-data controller is \( C^{SD} = H C^d S \). This is slightly different from (2): the hold filter at the input is missing. Such a system cannot receive \( L^2 \) inputs, because the sampled values of an \( L^2 \) function are not defined.

An important problem in modern control theory is the following: How to replace a continuous-time controller by a sampled-data controller, without significantly altering the performance of the control system. A solution to this problem is needed in the sampled-data controller design method called “analog design, SD implementation”. This is discussed in Chapter 1 of [3], along with other design methods (discrete-time design and direct sampled-data design). The interest in this design method (and hence, in the above problem) stems from the following facts:

1. It is easier and more natural to formulate control objectives and to design controllers in continuous time (as opposed to discrete time).

2. It is easier, cheaper and more reliable to implement controllers digitally (except for very simple controllers, such as stand-alone P or PI).

In precise terms, the problem outlined earlier can be formulated as follows:

**Main Problem.** Two well-posed transfer functions \( P \) and \( C \) of dimensions \( p \times m \) and \( m \times p \) are given, such that \( T \) from (1) is stable. An admissible error \( \varepsilon > 0 \) is also given. Find \( \tau > 0 \) and a discrete-time well-posed transfer function \( C^d \) with the following properties: \( T^{SD} \) from (3) is a bounded operator on \( L^2([0, \infty), \mathbb{C}^{p \times m}) \) and

\[
\| T^{SD} - T \| < \varepsilon.
\]

We investigate the question of when this problem is solvable and how. Related issues of sampled-data stabilizability for infinite-dimensional systems, in a state space setting and using more complex hold operators, are described in Rebarber and Townley [8] and [9].

The process (or method) by which a given continuous-time controller is replaced by a suitable sampled-data controller, such that the performance of the feedback system is not significantly altered, is called discretization.

This sentence was formulated in vague terms. A precise (and hence restrictive) description of the concept would be to say that discretization is the process (or method) by which we solve the Main Problem stated above.

Various discretization methods are described in the literature, although the problem is not usually stated precisely. For example, the classic book of Franklin, Powell and Workman [5] gives (in Chapter 6) six methods, called the forward rectangular rule, the backward rectangular rule, Tustin’s method (also called the bilinear transformation), zero-pole matching, zero-order hold equivalent and first-order hold equivalent. The book [3] proposes (in Chapter 1) two methods, the first being the formula \( C^d = SCH \) and the second being Tustin’s method. An alternative to the discretization approach to finding a good sampled-data controller, is to use the techniques of direct sampled-data design, see, for example, Bamieh and Pearson [1], Kabamba and Hara [6] and Chen and Francis [3]. This direct approach, which we will not consider here, uses lifting techniques, see Yamamoto [10], Bamieh et al. [2] and references cited in [3], and the frequency response of a sampled-data control system, see Yamamoto and Khargonekar [11].

In this paper we use the discretization formula

\[
C^d = S \tau H \tau C \tau, \quad S \tau H \tau \quad \text{(4)}
\]

which is similar to the first discretization formula of [3]. The difference with respect to the formula of [3] is that we have an additional hold filter.

**Theorem 1.** Let \( P, C \) and \( T \) be as in the Main Problem and assume that \( C \) is strictly proper. Let \( C^d \) be as in (4), let \( C_{SD}^d \) be the corresponding sampled-data controller, as in (2), and let \( T_{SD} \) be the corresponding closed-loop operator, as in (3). Then

\[
\lim_{\tau \rightarrow 0} \| T_{SD} - T \| = 0.
\]

One of the main ingredients in the proof of this theorem is the introduction of a factorization

\[
C = K_{22}(I - K_{22})^{-1}K_{21},
\]

where \( K_{ij} \) are stable transfer functions of appropriate matrix dimensions and \( K_{12}, K_{21} \) are strictly proper.
Such a factorization corresponds to replacing our controller $C$ by a stabilizing controller with internal loop, namely
\[ K = \begin{bmatrix} 0 & K_{12} \\ K_{21} & K_{22} \end{bmatrix}, \]
as shown in Figure 3. For the general background on controllers of this type we refer to Weiss and Curtain [12] (mostly state space theory) and to Curtain, Weiss and Weiss [4] (mostly transfer function theory).

![Figure 3: A plant $P$ with a stabilizing controller with internal loop $K$. The closed-loop system is stable (from all three inputs to any other signal). Every stabilizing controller $C$ as in Figure 1 can be replaced by an equivalent stabilizing controller with internal loop, but not the other way round.](image)

While $C$ is represented by $K$, we need a similar representation for the sampled-data controller $C_{SD}$. For this, we define the impulse sampler $E_{\tau}$ by
\[ E_{\tau} = S_{\tau}^* S_{\tau}. \]

Then the internal loop type representation $K_{\tau}^{SD}$ of the sampled-data controller $C_{SD}$ is given by
\[ K_{\tau}^{SD} = \begin{bmatrix} 0 & H_{\tau} E_{\tau} H_{\tau} K_{12} \\ K_{21} H_{\tau} E_{\tau} H_{\tau} & K_{22} \end{bmatrix}. \]

Now we need the fact that the operator $E_{\tau}$ approximates the identity. More precisely, it was shown in [7] that $E_{\tau}$ is in $L(H^r, H^{-r})$ for any $r > 1/2$, where $H^r$ is a Sobolev space, and with respect to the corresponding operator norm,
\[ \lim_{\tau \to 0} E_{\tau} = I. \]

Using this result, it is possible to prove that
\[
\lim_{\tau \to 0} \| H_{\tau} E_{\tau} H_{\tau} K_{12} - K_{12} \| = 0, \\
\lim_{\tau \to 0} \| K_{21} H_{\tau} E_{\tau} H_{\tau} - K_{21} \| = 0,
\]
which implies that
\[
\lim_{\tau \to 0} \| K_{\tau}^{SD} - K \| \to 0.
\]

References


