Controllability and observability of coupled systems

Xiaowei Zhao and George Weiss

Abstract—We consider coupled systems consisting of a well-posed subsystem and a finite-dimensional subsystem connected in feedback. The external world interacts with the coupled system via the finite-dimensional subsystem, which receives the external input and sends out the output. We also consider the output of the infinite-dimensional part as an additional output. Under several assumptions, we derive well-posedness, exact (approximate) controllability and exact (approximate) observability results for such coupled systems.

I. INTRODUCTION

Consider a coupled system $\Sigma_c$, in which a well-posed linear system $\Sigma_d$ is connected to a finite-dimensional linear system $\Sigma_f$ as shown in Figure 1. The external world interacts with the coupled system $\Sigma_c$ via the finite-dimensional system $\Sigma_f$, which receives the input $v = u_e - y$, where $u_e$ is the input of $\Sigma_c$ and the signal $y$ comes from $\Sigma_d$. The system $\Sigma_f$ sends out the output $u$, which is also the output of the coupled system $\Sigma_c$. We consider $y$ as an additional output of $\Sigma_c$. The equations of $\Sigma_f$ are

\[
\begin{align*}
\dot{q}(t) &= aq(t) + bu_e(t) - by(t), \\
u(t) &= cq(t) + du_e(t) - dy(t),
\end{align*}
\]

where $a \in \mathbb{C}^{n \times n}$, $b \in \mathbb{C}^{n \times m}$, $c \in \mathbb{C}^{m \times n}$, $d \in \mathbb{C}^{m \times m}$ and $q(t) \in \mathbb{C}^n$ is the state of $\Sigma_f$. The equations of $\Sigma_f$ are

\[
\begin{align*}
\dot{y}(t) &= Ay(t), \\
y(t) &= Cy(t).
\end{align*}
\]

Let $p$ be a function defined on some domain in $\mathbb{C}$ that contains a right half-plane, with values in a normed space.

We say that $p$ is strictly proper if

\[
\lim_{\text{Re } s \to \infty} \|p(s)\| = 0,
\]

uniformly with respect to $\text{Im } s$.

A linear system is called strictly proper if its transfer function is strictly proper.

Let $\Sigma_d$ be a well-posed and strictly proper system, $\Sigma_f$ a finite-dimensional system, and $\Sigma_c$ a coupled system consisting of $\Sigma_d$ and $\Sigma_f$ in feedback.

The well-posed linear system $\Sigma_d$, with input function $u$, input space $\mathbb{C}^m$, state trajectory $z$, output function $y$, and output space $\mathbb{C}^n$, is the same as in (1.1). It is determined by its generating triple $(A, B, C)$ via

\[
\dot{z}(t) = Az(t) + Bu, \\
y(t) = Cz(t).
\]

Here $A$ is the semigroup generator of $\Sigma_d$, which generates a strongly continuous semigroup $T$ on the state space $X$ (a Hilbert space), $B \in \mathcal{L}(\mathbb{C}^m, X_1)$ is the control operator of $\Sigma_d$ and $C \in \mathcal{L}(X_1, \mathbb{C}^n)$ is its observation operator and $C_A$ is the $z$-extension of $C$. As $\Sigma_d$ is strictly proper, its feedthrough operator is zero. The transfer function $G$ of $\Sigma_d$ is given by

\[
G(s) = C_A(sI - A)^{-1}B, \quad \forall s \in \rho(A).
\]

For the terminology on regular systems that has been used here we refer to the background in Section II.

Our approach to proving controllability properties of $\Sigma_c$ is to consider it as a cascaded system $\Sigma_{casc}$ (the open loop system in Figure 2) with a feedback. The input of $\Sigma_{casc}$ is $v$ (see Figure 1), and its outputs are $u$ and $y$. We obtain $\Sigma_c$ via the feedback $v = u_e - y$. The cascaded system is easier to analyze than the coupled system and its controllability properties are invariant under feedback. Our approach to proving observability properties of $\Sigma_c$ is similar, but we use a different cascaded system (with the order reversed), as shown in Figure 4.

![Fig. 1. A coupled system $\Sigma_c$ consisting of a well-posed and strictly proper system $\Sigma_d$ and a finite-dimensional system $\Sigma_f$.](image1)

![Fig. 2. A cascaded system $\Sigma_{casc}$ consisting of a well-posed and strictly proper system $\Sigma_d$ and a finite-dimensional system $\Sigma_f$.](image2)

For the well-posedness, controllability and observability properties of the coupled system $\Sigma_c$ we have the following results:

**Theorem 1.1:** Let $\Sigma_d$ be a well-posed and strictly proper (hence regular) system with input space $\mathbb{C}^m$, state space $X$ (a Hilbert space), output space $\mathbb{C}^n$, semigroup $T$ and generating triple $(A, B, C)$. Let $a, b, c, d$ be matrices as in (1.1)–(1.2). Then the coupled system $\Sigma_c$ from Figure 1 described by (1.1), (1.2) and (1.3), with input $u_e$, state $z$ and output $y$, is well-posed and regular with the state space $X \times \mathbb{C}^n$.

Now assume additionally the following:

(i) $(A, B)$ is exactly controllable in time $T_0$;
(ii) \((a, b)\) is controllable;
(iii) \(d \in \mathbb{C}^{m \times m}\) is invertible;
(iv) Denote \(a^* = a - bd^{-1}c\). Then \(A^*\) and \(a^{*\times}\) have no common eigenvalue.

Then \(\Sigma_c\) is exactly controllable in any time \(T > T_0\).

**Theorem 1.2:** We use the assumptions and the notation from the first part of Theorem 1.1. We also assume the following:

(i) \((A, C)\) is exactly observable in time \(T_0\);
(ii) \((a, c)\) is observable;
(iii) \(d \in \mathbb{C}^{m \times m}\) is invertible;
(iv) \(A\) and \(a^{\times}\) have no common eigenvalue.

Then \(\Sigma_c\), with output \(u\) only, is exactly observable in any time \(T > T_0\).

For approximate controllability and approximate observability we have weaker results, in which we cannot tell the approximate controllability (or observability) time of the coupled system. We denote by \(\rho_\infty(A)\) the connected component of \(\rho(A)\) containing some right half-plane.

**Proposition 1.3:** We use the assumptions and the notation from the first part of Theorem 1.1. We also assume the following:

(i) \((A, B)\) is approximately controllable in some time;
(ii) \((a, b)\) is controllable;
(iii) \(d \in \mathbb{C}^{m \times m}\) is invertible;
(iv) Denote \(a^* = a - bd^{-1}c\). We have \(\sigma(a^*) \subset \rho_\infty(A)\).

Then \(\Sigma_c\) is approximately controllable in some time.

**Proposition 1.4:** We use the assumptions and the notation from the first part of Theorem 1.1. We also assume the following:

(i) \((A, C)\) is approximately observable in some time;
(ii) \((a, c)\) is observable;
(iii) \(d \in \mathbb{C}^{m \times m}\) is invertible;
(iv) Denote \(a^* = a - bd^{-1}c\). We have \(\sigma(a^*) \subset \rho_\infty(A)\).

Then \(\Sigma_c\), with output \(u\) only, is approximately observable in some time.

II. SOME BACKGROUND ON INFINITE-DIMENSIONAL SYSTEMS

In this section we introduce some concepts and results on infinite-dimensional linear time invariant systems, without proof. For the details we refer to the literature.

A. Admissible control and observation operators

The material of this section can be found (in much greater detail and with many references) in Tucsnak and Weiss [4, Chapter 4].

Let \(A\) be the generator of a strongly continuous semigroup \(T\) on a Hilbert space \(X\). Then \(T\) and \(X\) determine two additional Hilbert spaces: \(X_1 = D(A)\) with the norm \(\|z\|_1 = \|(\beta I - A)z\|\), and \(X_{-1}\) is the completion of \(X\) with respect to the norm \(\|z\|_{-1} = \|(\beta I - A)^{-1}z\|\), where \(\beta \in \rho(A)\) is fixed. These spaces are independent of the choice of \(\beta\), since different values of \(\beta\) lead to equivalent norms on \(X_1\) and \(X\). The norm \(\|z\|_1\) is equivalent to the graph norm of \(A\). We have \(X_1 \subset X \subset X_{-1}\) densely and with continuous embeddings. We can extend \(A\) to a bounded operator from \(X\) to \(X_{-1}\), still denoted by \(A\). The semigroup generated by this extended \(A\) is the extension of \(T\) to \(X_{-1}\), which is still denoted by \(T\).

In this section, \(U, X\) and \(Y\) are Hilbert spaces, \(\mathbb{T}\) is a strongly continuous semigroup on \(X\), with generator \(A, B \in \mathcal{L}(U, X_{-1})\) and \(C \in \mathcal{L}(X, Y)\).

We define the operators \(\Phi_\tau\) (for \(\tau > 0\)) by

\[
\Phi_\tau u = \int_0^\tau T_{\tau-t}Bu(t)dt.
\]

where \(u(t) \in L^2_{loc}([0, \infty); U)\). Clearly \(\Phi_\tau \in \mathcal{L}(L^2([0, \infty); U), X_{-1})\). These operators are called the input maps of \((A, B)\).

**Definition 2.1:** \(B\) is said to be an admissible control operator for the semigroup \(T\) if \(\text{Ran} \Phi_\tau \subset X\) for some \(\tau > 0\).

The admissibility of \(B\) implies that the solutions \(z(\cdot)\) of

\[
\dot{z}(t) = Az(t) + Bu(t),
\]

with initial state \(z(0) = z_0 \in X\) and with \(u \in L^2_{loc}([0, \infty); U)\) remain in \(X\). Moreover it follows that \(z(\cdot)\) is a continuous \(X\)-valued function of \(\tau\) and

\[
z(t) = T_\tau z_0 + \Phi_\tau u.
\]

The operator \(B\) is said to be bounded if \(B \in \mathcal{L}(U, X)\) and unbounded otherwise.

We define the operators \(\Psi_\tau\) (for \(\tau > 0\)) by

\[
(\Psi_\tau z_0)(t) = \begin{cases} CT_\tau z_0 & \text{for } t \in [0, \tau], \\ 0 & \text{for } t > \tau. \end{cases}
\]

It is clear that \(\Psi_\tau \in \mathcal{L}(X_1, L^2([0, \infty); Y))\). These operators are called the output maps of \((A, C)\). \(C\) is said to be bounded if it can be extended such that \(C \in \mathcal{L}(X, Y)\) and unbounded otherwise.

**Definition 2.2:** \(C\) is said to be an admissible observation operator for the semigroup \(T\) if \(\Psi_\tau\) has a continuous extension to \(X\) for some \(\tau > 0\).

The admissibility of \(C\) is equivalent to the fact that for some (hence, for every) \(\tau > 0\) there is a \(K_\tau \geq 0\) such that

\[
\int_0^\tau \|CT_\tau z_0\|^2 dt \leq K_\tau^2 \|z_0\|^2 \quad \forall z_0 \in D(A).
\]

We regard \(L^2_{loc}([0, \infty); Y)\) as a Fréchet space with the seminorms being the \(L^2\) norms on the intervals \([0, n]\), \(n \in \mathbb{N}\). Then the admissibility of \(C\) means that there exists a continuous operator \(\Psi : X \to L^2_{loc}([0, \infty); Y)\) such that

\[
(\Psi z_0)(t) = CT_\tau z_0 \quad \forall z_0 \in D(A). \tag{2.3}
\]

We introduce the \(\Lambda\)-extension of \(C\), denoted \(C_\Lambda\), by

\[
C_\Lambda z_0 = \lim_{\lambda \to +\infty} C\lambda(\lambda I - A)^{-1}z_0, \tag{2.4}
\]

whose domain \(D(C_\Lambda)\) consists of all \(z_0 \in X\) for which the limit exists. If we replace \(C\) by \(C_\Lambda\), formula (2.3) becomes true for all \(z_0 \in X\) and for almost every \(t \geq 0\).
B. Well-posed linear systems

A well-posed linear system with input space \( U \), state space \( X \) and output space \( Y \) is a family of bounded linear operators (parametrized by \( \tau \geq 0 \)) that associates to every initial state \( z_0 \in X \) and every input signal \( u \in L^2([0, \tau]; U) \) a final state \( z(\tau) \) and an output signal \( y \in L^2([0, \tau]; Y) \). These operators have to satisfy certain natural functional equations, for the formal definition we refer to Weiss \[6\].

By continuous extension, for any well-posed linear system, we can define state trajectories and output signals for any initial state in the state space \( X \) and for any input signal in \( L^2_{\text{loc}}([0, \infty); U) \); the output signal is then in \( L^2_{\text{loc}}([0, \infty); Y) \).

For more detailed background about well-posed systems we refer to Salamon \[1\], Staffans \[2\], Staffans and Weiss \[3\], Weiss, Staffans and Tucsnak \[9\].

We recall some facts about well-posed linear systems from \[6\], \[7\]. Let \( \Sigma \) be a well-posed system with input space \( U \), state space \( X \) and output space \( Y \). Then \( \Sigma \) is completely determined by its generating triple \((A, B, C)\) and its transfer function \( G \). Here, \( A \) is the semigroup generator of \( \Sigma \), which generates a strongly continuous semigroup \( T \) on \( X \), \( B \in \mathcal{L}(U, X, -1) \) is the control operator of \( \Sigma \) and \( C \in \mathcal{L}(X, Y) \) is its observation operator. The transfer function \( G \) satisfies

\[
G(s) - G(\beta) = C((sI - A)^{-1} - (\beta I - A)^{-1})B
\]

for all \( s, \beta \in \rho(A) \). The state trajectories of \( \Sigma \) satisfy (2.1), hence also (2.2). If \( u \in L^2_{\text{loc}}([0, \infty); U) \) is the input function of \( \Sigma \), \( z_0 \in X \) is its initial state and \( y \in L^2_{\text{loc}}([0, \infty); Y) \) is the corresponding output function, then

\[
y = \Psi z_0 + \mathbb{F} u.
\]

Here, \( \Psi \) is the operator from (2.3). The operator \( \mathbb{F} \) appearing above is continuous from \( L^2_{\text{loc}}([0, \infty); U) \) to \( L^2_{\text{loc}}([0, \infty); Y) \) (which we regard as Fréchet spaces, see the comments around (2.3)). It is easiest to represent \( \mathbb{F} \) using Laplace transforms, as follows: if \( u \in L^2([0, \infty); U) \) and \( y = \mathbb{F} u \), then \( y \) has a Laplace transform \( \hat{y} \) and

\[
\hat{y}(s) = C((sI - A)^{-1}B)
\]

for all \( s \in \mathbb{C} \) with \( \text{Re } s \) sufficiently large. This determines \( \mathbb{F} \), since \( L^2([0, \infty); U) \) is dense in \( L^2_{\text{loc}}([0, \infty); U) \). \( \mathbb{F} \) is proper which means that its domain contains a right half-plane \( \mathbb{C}_\alpha \) such that \( \mathbb{F} \) is uniformly bounded on \( \mathbb{C}_\alpha \).

Conversely, if \( \mathbb{F} \) is an analytic and proper \( \mathcal{L}(U, Y) \)-valued function, then \( \mathbb{F} \) determines a continuous operator \( \mathbb{F} \) from \( L^2_{\text{loc}}([0, \infty); U) \) to \( L^2_{\text{loc}}([0, \infty); Y) \) via (2.6) (see for example \[6\], Theorem 3.6)). (In (2.6) we only take \( u \in L^2([0, \infty); U) \), but this determines \( \mathbb{F} \), as explained earlier.) We define the input-output maps of \( G \), denoted by \( \mathbb{F}_\tau \) (\( \tau \geq 0 \)), by truncating the output to \([0, \tau]\):

\[
\mathbb{F}_\tau u = (\mathbb{F} u)|_{[0, \tau]}.
\]

The operator \( \mathbb{F} \) (defined above via \( G \)) is causal, which means that \( \mathbb{F}_\tau u \) depends only on the truncation \( u|_{[0, \tau]} \). It follows that we may regard \( \mathbb{F}_\tau \) as a bounded operator from \( L^2([0, \tau]; U) \) to \( L^2([0, \tau]; Y) \).

Let \( \Sigma \) be a well-posed linear system on \((U, X, Y)\) with generating triple \((A, B, C)\) and transfer function \( G \). An operator \( K \in \mathcal{L}(Y, U) \) is called an admissible feedback operator for \( \Sigma \) (or for \( G \)) if \( I - GK \) has a proper inverse (equivalently, if \( I - KG \) has a proper inverse). If this is the case, then the system with output feedback shown in Figure 3 is well-posed on \((U, X, Y)\) (its input is \( v \), its state and output are the same as for \( \Sigma \)). This new system is called the closed-loop system corresponding to \( \Sigma \) and \( K \), and it is denoted by \( \Sigma^K \). Its transfer function is

\[
G^K = (I - GK)^{-1} = (I - G)^{-1}G.
\]

We have that \(-K\) is an admissible feedback operator for \( \Sigma^K \) and the corresponding closed-loop system is \( \Sigma \). Let us denote by \((A^K, B^K, C^K)\) the generating triple of \( \Sigma^K \). Then for every \( x_0 \in D(A^K) \) and for every \( z_0 \in D(A) \),

\[
A^K x_0 = (A + BK C^K) x_0, \quad A z_0 = (A^K - B^K C) z_0.
\]

For more details on closed-loop systems we refer to \[7\].

![Fig. 3](image-url)
Proposition 2.4: Suppose that $\Sigma$ is a regular linear system on $(U, X, Y)$ with generating operators $A, B, C$ and $D$. We assume that $U$ is finite-dimensional. Let $K$ be an admissible feedback operator for $\Sigma$. Then the corresponding closed-loop system $\Sigma^K$ is regular.

C. Controllability and observability

Let $U, X, Y, T, A, B, C, \Phi_r$ and $\Psi_r$ be as at Subsection II-A. We assume that $B$ and $C$ are admissible for $T$.

Definition 2.5: The pair $(A, B)$ is said to be exactly controllable in time $\tau > 0$ if $\text{Ran} \, \Phi_r = X$; $(A, B)$ is said to be approximately controllable in time $\tau > 0$ if $\text{Ran} \, \Phi_r$ is dense in $X$.

Definition 2.6: The pair $(A, C)$ is said to be exactly observable in time $T > 0$ if $\Psi_T$ is bounded from below, i.e., there exists $k_T > 0$ such that

$$\int_0^T \| CT_t z_0 \|_Y^2 \geq k_T^2 \| z_0 \|^2. \quad (2.9)$$

$(A, C)$ is said to be approximately observable in time $T > 0$ if $\ker \, \Psi_T = \{0\}$.

We often need the controllability concepts without specifying a time $\tau$. Therefore the following definition is introduced.

Definition 2.7: The pair $(A, B)$ is said to be exactly controllable if it is exactly controllable in some finite time $\tau > 0$. $(A, B)$ is said to be approximately controllable if it is approximately controllable in some finite time.

Observability concepts without a specified time are introduced in a similar way.

Proposition 2.8: The pair $(A, C)$ is exactly observable in time $\tau > 0$ if and only if $(A^*, C^*)$ is exactly controllable in time $\tau$. The pair $(A, C)$ is approximately observable in time $\tau > 0$ if and only if $(A^*, C^*)$ is approximately controllable in time $\tau$.

For much more details on the above concepts we refer to [4]. The following invariance result is taken from Section 6 of Weiss [7].

Proposition 2.9: Let $\Sigma$ be a well-posed linear system, let $K$ be an admissible feedback operator for $\Sigma$ and let $\Sigma^K$ be the corresponding closed-loop system. Let $(A, B, C)$ and $(A^K, B^K, C^K)$ be the generating triples of $\Sigma$ and $\Sigma^K$, respectively.

Then $(A, B)$ is exactly (approximately) controllable in time $T$, if and only if $(A^K, B^K)$ has the same property.

Moreover, $(A, C)$ is exactly (approximately) observable in time $T$, if and only if $(A^K, C^K)$ has the same property.

We quote the following definition and results on simultaneous controllability and simultaneous observability from [4, Chapters 6, 11].

Definition 2.10: For $i \in \{1, 2\}$, denote by $A^i$ the generators of the strongly continuous semigroups $T^i$ on the Hilbert spaces $X^i$. Let $U_i, Y_i$ be Hilbert spaces. Assume that $B^i \in \mathcal{L}(U_i, X^i_{-1})$ are admissible control operators for $T^i$ and that $C^i \in \mathcal{L}(\mathcal{D}(A^i), Y_i)$ are admissible observation operators for $T^i$.

The pairs $(A^i, B^i)$ are said to be simultaneously exactly controllable in time $T > 0$, if for every $x_1^i \in X^i$ there exists a function $u \in L^2([0, T]; U)$ such that

$$\int_0^T T^i_{-\sigma} B^i u(\sigma) d\sigma = x_1^i, \quad i \in \{1, 2\}.$$ 

The pairs $(A^i, B^i)$ are said to be simultaneously approximately controllable in time $T > 0$, if the equality above holds for $(x_1^1, x_2^2)$ in a dense subspace of $X \times X^2$.

The pairs $(A^i, C^i)$ are said to be simultaneously exactly observable in time $T > 0$, if there exists $k_T > 0$ such that for all $(z_0^1, z_0^2) \in \mathcal{D}(A^1) \times \mathcal{D}(A^2)$ the following inequality holds:

$$\int_0^T \| C_1 T^i_0 z_1^i + C_2 T^2_0 z_2^2 \|^2 \geq k_T^2 \left( \| z_0^1 \|_{X^1}^2 + \| z_0^2 \|_{X^2}^2 \right). \quad (2.10)$$

The pairs $(A^i, C^i)$ are said to be simultaneously approximately observable in time $T > 0$, if the fact that $(z_0^1, z_0^2) \in X^1 \times X^2$ satisfies

$$C_1 A T^1_0 z_0^1 + C_2 A T^2_0 z_0^2 = 0,$$

for almost every $t \in [0, T]$, implies that $(z_0^1, z_0^2) = (0, 0)$.

Proposition 2.11: With the notation of Definition 2.10, the pairs $(A^1, C^1)$ and $(A^2, C^2)$ are simultaneously exactly observable in time $T$ if and only if $(A^1, C^1)$ and $(A^2, C^2)$ are simultaneously exactly controllable in time $T$. A similar statement holds for simultaneous approximate observability.

Theorem 2.12: Denote by $A$ the generator of the strongly continuous semigroup $T$ on the Hilbert space $X$. We assume that $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $T$ and that $(A, C)$ is exactly observable in time $T_0$. Let $a \in \mathbb{C}^{n \times n}$ and $c \in \mathbb{C}^{m \times n}$ be matrices such that $(a, c)$ is observable. Further, assume that $A$ and $a$ have no common eigenvalues. Then the pairs $(A, C)$ and $(a, c)$ are simultaneously exactly observable in any time $T > T_0$.

Using the duality from Propositions 2.8 and 2.11, we can easily find the dual version of the above theorem, which we leave to the reader to formulate.

Proposition 2.13: Let $A$ be the generator of a strongly continuous semigroup on $X$. Let $\rho_\infty(A)$ be the connected component of $\rho(A)$ containing some right half-plane. We assume that $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $T$ and that $(A, C)$ is approximately observable. Let $a \in \mathbb{C}^{n \times n}$ and $c \in \mathbb{C}^{m \times n}$ be such that $(a, c)$ is observable. Assume that $\sigma(a) \subset \rho_\infty(A)$. Then $(A, C)$ and $(a, c)$ are simultaneously approximately observable in some time.

Again, using the duality from Propositions 2.8 and 2.11, the reader can easily formulate the dual version of the above proposition. The following result is taken from Weiss and Zhao [10]:

Proposition 2.14: Let $A$ be the generator of the strongly continuous semigroup $T$ on $X$. Let $B \in \mathcal{L}(\mathbb{C}^m, X_{-1})$ be an admissible control operator for $T$. Let $a \in \mathbb{C}^{n \times n}$ and $b \in \mathbb{C}^{n \times m}$. Suppose that there exists $T > 0$ such that the pairs $(A, B)$ and $(a, b)$ are simultaneously approximately controllable in time $T$. 

5523
Then for every \( z \in X, q \in \mathbb{C}^n \) and \( \varepsilon > 0 \) there exists \( u \in L^2([0, T]; \mathbb{C}^m) \) such that
\[
\left\| \int_0^T T_{T-t} Bu(t) dt - z \right\| \leq \varepsilon, \quad \int_0^T e^{u(T-t)} Bu(t) dt = q. 
\]

III. WELL-POSEDNESS, CONTROLLABILITY AND OBSERVABILITY OF COUPLED SYSTEMS

In this section first we analyse the well-posedness and controllability of \( \Sigma_{\text{casc}} \) introduced in Section 1 (see Figure 2). Then we show our main results on coupled systems \( \Sigma_c \).

Recall that \( \Sigma_{\text{casc}} \) is described by:
\[
\begin{align*}
\dot{q}(t) &= aq(t) + bv(t), \\
u(t) &= cq(t) + dv(t), \\
\dot{\vartheta}(t) &= Az(t) + Bu(t), \\
y(t) &= C_\vartheta z(t).
\end{align*}
\]

Here (3.1)–(3.2) describe the finite-dimensional subsystem \( \Sigma_f \) with input space \( \mathbb{C}^n \), state space \( \mathbb{C}^n \), output space \( \mathbb{C}^m \) and matrices \( a \in \mathbb{C}^{n \times n}, b \in \mathbb{C}^{n \times m}, c \in \mathbb{C}^{m \times n}, d \in \mathbb{C}^{m \times m} \).

The equations (3.3)–(3.4) describe the well-posed and strictly proper system \( \Sigma_d \) with input space \( \mathbb{C}^m \), state space \( \mathbb{C}^m \), output space \( \mathbb{C}^m \), semigroup \( T \), semigroup generator \( A \), control operator \( B \in \mathcal{L}(\mathbb{C}^m, X) \) and observation operator \( C \in \mathcal{L}(X, \mathbb{C}^m) \).

\( \Lambda \) is the \( \Lambda \)-extension of \( C \), defined in (2.4). \( q(t) \in \mathbb{C}^n \) is the state of \( \Sigma_f \), while \( z(t) \in \mathbb{C}^n \) is the state of \( \Sigma_d \).

\( \Sigma_{\text{casc}} \) = \( \Sigma_f \) \( \times \) \( \Sigma_d \).

\( \Theta \) is controllable in any time \( T > T_0 \).

\textbf{Lemma 3.2:} If \( \Sigma_f \) is a finite-dimensional system described by
\[
\begin{align*}
\dot{x}(t) &= ax(t) + bu(t), \\
y(t) &= cx(t) + du(t),
\end{align*}
\]

where \( a, b, c, d \) are matrices of appropriate dimensions and \( d \) is invertible, then \( \Sigma_f \) is flow-invertible. Its flow-inverse system \( \Sigma_f^x \) is described by
\[
\begin{align*}
\dot{x}(t) &= (a - bd^{-1}c)x(t) + bd^{-1}y(t), \\
u(t) &= -d^{-1}cx(t) + d^{-1}y(t).
\end{align*}
\]

If \( u, x, y \) are functions satisfying (3.6), then the same functions satisfy also (3.7) and vice versa. The system (3.6) is controllable (observable) iff the system (3.7) is controllable (observable).

We omit the simple proof.

\textbf{Proposition 3.3:} With the assumptions (i)–(iv) of Theorem 1.1, the cascaded system \( \Sigma_{\text{casc}} \) described by (3.1)–(3.4) (with the state space \( X \times \mathbb{C}^n \)) is exactly controllable in any time \( T > T_0 \).

\textbf{Proof.} Set the initial state \( z(0) = 0 \) and \( q(0) = 0 \). The exact controllability of \( \Sigma_{\text{casc}} \) means that for any time \( T > T_0 \) and for any \( \left[ \frac{z_1}{q_1} \right] \in X \times \mathbb{C}^n \), there exists an input signal \( u \in L^2([0, T]; \mathbb{C}^m) \) such that the solution of (3.1)–(3.3) satisfies \( z(T) = z_1 \) and \( q(T) = q_1 \).

By assumption (ii) and Lemma 3.2, we know that \( \Sigma_f \) described by (3.1)–(3.2) is flow-invertible and that its flow-inverse system, denoted by \( \Sigma_f^x \), is described by
\[
\begin{align*}
\dot{q}(t) &= (a - bd^{-1}c)q(t) + bd^{-1}u(t), \\
v(t) &= -d^{-1}cq(t) + d^{-1}u(t).
\end{align*}
\]

Recall that \( a^x = a - bd^{-1}c \). From assumption (ii) and Lemma 3.2 it follows that \( \Sigma_f^x \) is controllable. Combining this fact with the assumptions (i) and (iv), and the dual version of Theorem 1.2, it follows that \( \Sigma_d \) and \( \Sigma_f^x \) (more precisely, the pairs \( (A, B) \) and \( (a^x, bd^{-1}) \)) are simultaneously exactly controllable in any time \( T > T_0 \). Therefore for any \( \left[ \frac{z_1}{q_1} \right] \in X \times \mathbb{C}^n \), and for the systems (3.3) and (3.8), we can find \( u \in L^2([0, T]; \mathbb{C}^m) \) such that \( z(T) = z_1 \) and \( q(T) = q_1 \).

Let \( u \) and \( v \) be the state trajectory and the output signal (on the time interval \([0, T]\)) of the system (3.8)–(3.9) corresponding to the input signal \( u \) found above and \( q(0) = 0 \). Obviously \( v \in L^2([0, T]; \mathbb{C}^m) \). By Lemma 3.2 these functions also satisfy (3.1) and (3.2) (and \( q(T) = q_1 \)). Let \( z \) be the solution of (3.3) with the signal \( u \) found above and with \( z(0) = 0 \), so that \( z(T) = z_1 \).

Thus we have found \( u \in L^2([0, T]; \mathbb{C}^m) \) such that the solution of the equations (3.1)–(3.3) satisfies \( z(T) = z_1 \) and \( q(T) = q_1 \).
Proposition 3.4: With the assumptions (i)-(iv) of Proposition 1.3, $\Sigma_{casc}$ described by (3.1)-(3.4) (with state space $X \times C^m$) is approximately controllable.

Proof. It have to prove the following fact: for any $[z_1] \in X \times C^m$ and $\delta > 0$, there exists an input function $v \in L^2([0,T];C^m)$ such that if $q$ and $z$ are as in (3.1)-(3.3) with $z(0) = 0$, $q(0) = 0$, then
$$\left\| \begin{bmatrix} z(T) \\ q(T) \end{bmatrix} - \begin{bmatrix} z_1 \\ q_1 \end{bmatrix} \right\|_{X \times C^m} \leq \delta.$$  
If we can achieve $q(T) = q_1$, then (using the definition of the norm on $X \times C^m$) the above estimate reduces to
$$\|z(T) - z_1\|_X \leq \delta.$$  
(3.10)

Thus, it will be enough to show that we can find $v \in L^2([0,T];C^m)$ such that $q(T) = q_1$ and (3.10) holds.

The remaining part of the proof is similar to that of Proposition 3.3. First, by Lemma 3.2 and assumptions (ii) and (iii), we get that the flow-inverse system of $\Sigma_f$, denoted by $\Sigma_f^\times$ (see (3.8)-(3.9)) is controllable. From this fact, assumptions (i) and (iv) and the dual version of Proposition 2.13, we get the simultaneous approximate controllability of $\Sigma_d$ and $\Sigma_f^\times$. Now by Proposition 2.14, we can find a suitable $u \in L^2([0,T];C^m)$ to achieve $q(T) = q_1$ and (3.10). Following the same procedure as at the end of the proof of Proposition 3.3, we can show that there exists $v \in L^2([0,T];C^m)$ such that if $q$ and $z$ are the solutions of (3.1)-(3.3) corresponding to $z(0) = 0$, $q(0) = 0$, then $q(T) = q_1$ and (3.10) holds.

Now we consider a new cascaded system $\Sigma_{casc}$ as shown in Figure 4, to study the observability of the coupled system $\Sigma_c$ described by (1.1), (1.2) and (1.3) with output $u$. Since we are only interested in observability, we assume that the external input $u_c$ of $\Sigma_c$ is zero, so that $v = -y$. The output of $\Sigma_{casc}$ is $u$. We can obtain $\Sigma_c$ from $\Sigma_{casc}$ via the feedback $u_0 = u$. The system $\Sigma_{casc}$ is described by:
$$\begin{align*}
\dot{z}(t) &= Az(t) + Bu_0(t), \\
v(t) &= -C_dz(t), \\
\dot{q}(t) &= aq(t) + bv(t), \\
u(t) &= cq(t) + dv(t).
\end{align*}$$
(3.11)-(3.14)

The well-posedness and regularity of $\Sigma_{casc}$ can be proved similarly as for $\Sigma_{casc}$, see the comments after Proposition 3.1. Its exact (approximate) observability can be proved by showing the simultaneous exact (approximate) observability of $\Sigma_d$ and the flow-inverse system of $\Sigma_f$, and using equation (2.10) (the equation below (2.10)). We omit the detail.

Proof of Theorem 1.1 and Proposition 1.3. The coupled system $\Sigma_c$ can be considered as being obtained from $\Sigma_{casc}$ via output feedback with the feedback operator $K = [0 \ -I]$ (as in Figure 3). From Proposition 3.1 we know that $\Sigma_{casc}$ is regular with the state space $X \times C^m$. From (3.5) we know that the transfer function of $\Sigma_{casc}$ is $G_{casc} = [g_G]$, where $g$ is the transfer function of $\Sigma_f$, while $G$ is the transfer function of $\Sigma_d$.

Since $g$ is proper and $G$ is strictly proper, it follows that $(I - KG_{casc})^{-1} = (I + Gg)^{-1}$ is proper, which means that $K = [0 \ -I]$ is an admissible feedback operator for $\Sigma_{casc}$. Thus the well-posedness and regularity part of Theorem 1.1 follows from Proposition 2.4. Now we prove the exact controllability part of Theorem 1.1. According to Proposition 3.3, the assumptions (i)-(iv) in Theorem 1.1 imply that the cascaded system $\Sigma_{casc}$ (with state space $X \times C^m$) is exactly controllable in any time $T > T_0$. According to Proposition 2.9, it follows that $\Sigma_c$ is also exactly controllable (with state space $X \times C^m$) in any time $T > T_0$.

The proof of Proposition 1.3 is similar. According to Proposition 3.4, the assumptions (i)-(iv) in Proposition 1.3 imply that the cascaded system $\Sigma_{casc}$ (with state space $X \times C^m$) is approximately controllable. According to Proposition 2.9, it follows that $\Sigma_c$ is also approximately controllable (in the same state space).

The proof of Theorem 1.2 and Proposition 1.4 are similar to the above proof but now we use $\Sigma_{casc}$ and Propositions 3.5, 3.6. Because of the page limitation, we omit it. The journal version of this work will contain all the proofs and illustrative example.

References: