Strong stabilization of a wind turbine tower model

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Abstract—We derive a model for a wind turbine tower in the plane of the turbine blades, that comprises a non-uniform SCOLE system and a two-mass drive-train model (with gearbox). The input of this model is the electric torque created by the electrical generator. We show the impedance passivity and well-posedness of this model in the natural state space. We also prove that generically, this system can be strongly stabilized by feedback using information about the angular velocity of the nacelle and the angular velocity of the generator rotor.

I. INTRODUCTION

The aim of this paper is to develop a wind turbine model in the plane of turbine blades and investigate its well-posedness and strong stabilization. As a source of renewable and clean energy, wind power is rapidly increasing its share of the energy state space using either force (contained in the natural energy space) and its approximate angle, named pitch angle control. The exact controllability of the tower as a beam. The SCOLE (NASA Spacecraft Control Laboratory Experiment) system is a well known model for a wind turbine tower; we cannot ignore it and simply model the tower as a beam. The SCOLE system to suppress the vibrations of the turbine tower without affecting the reliability of the power supply.

To design a vibration suppression controller, we need to find a suitable model for the wind turbine tower and then to study its controllability and stabilization. The current wind turbine tower models are mostly based on finite-element approximations, which divide the tower into many slices and consider these slices as interconnected finite-dimensional systems. These models are very complex and are used mainly for simulation.

Think of a wind turbine tower clamped in the ocean floor that carries at its top the nacelle (with a weight of several hundred tons). This heavy nacelle has a big effect on the wind turbine tower; we cannot ignore it and simply model the tower as a beam. The SCOLE (NASA Spacecraft Control Laboratory Experiment) system is a well known model for a flexible beam with one end clamped and the other end linked to a rigid body. It has been originally developed to understand the dynamics of a mast carrying an antenna on a satellite, see Littman and Markus [7], [8]. In the vertical plane of the turbine axis, it is sensible to model the wind turbine tower as a non-uniform SCOLE system with only force control. The force control can be obtained by modulating the turbine pitch angle, named pitch angle control. The exact controllability of the non-uniform SCOLE model on certain smoother spaces (contained in the natural energy space) and its approximate controllability on the energy state space using either force or torque control has been shown in Guo [4] and in Guo and Ivanov [5]. An exponential stabilizing high order feedback for the non-uniform SCOLE model on the energy state space has been found in [5]. The strong stabilization of this system on the energy state space via feedback from either the velocity or the angular velocity of the rigid body has been shown in Zhao and Weiss[20].

In the plane of the turbine blades we have to use a more complex model. The gearbox transfers most of the torque caused by the wind to the nacelle. Our model must take the gearbox and the flexible (low-speed) turbine shaft into account. We use a two-mass drive-train model (see Figure 1) studied in Hansen et al. [6], Lubosny [10], and Wang and Weiss [16], to analyze the mechanical system within the nacelle.

For a wind turbine tower moving in the plane of the turbine blades, we denote by \( l \) the height of the tower and denote by \( w \) the transverse displacement of the tower, the model is

\[
\begin{align*}
\rho(x)w_{tt}(x,t) + (EI(x)w_{xx}(x,t))_{xx} &= 0, \\
(x,t) &\in (0,l) \times [0,\infty), \\
w(0,t) &= 0, \quad w_x(0,t) = 0, \\
Jw_{ttt}(l,t) - (EIw_{xx})(l,t) &= 0, \\
Jw_{ttt}(l,t) + EI(w_{xx})(l,t) &= T_{lss}(t) + T_hss(t) - b_m(\theta_m)(t) - T_e(t), \\
(\theta_T)_{tt}(t) &= \frac{1}{J_T} (T_a(t) - T_{lss}(t)) - w_{tt}(l,t), \\
(\theta_m)_{tt}(t) &= \frac{1}{J_g} (T_hss(t) - T_{lss}(t)) - b_m(\theta_m)(t) - T_e(t) + w_{tt}(l,t), \\
\theta_b(t) &= \theta_T(t) - \theta_m(t), \\
T_{lss}(t) &= K_s\theta_k(t) + C_s(\theta_k)(t) = n_g T_{hss}(t),
\end{align*}
\]

Fig. 1. The two-mass drive-train model with gearbox. \( T_a \) is the active torque from the turbine and \( T_e \) is the electric torque of the generator, which acts between the rotor (connected to the high speed shaft) and the stator (connected to the nacelle).
where the subscripts $t$ and $x$ denote derivatives with respect to the time $t$ and the position $x$, respectively. The equations (1.1)-(1.4) are a non-uniform SCOLE model. The equations (1.5)-(1.8) are a two-mass drive-train model.

In this model, $EI$ and $\rho$ are the flexural rigidity function and mass density function. $m > 0$ and $J > 0$ are the mass and the moment of inertia of the nacelle. $-\left( EI(x)w_{xx}(x,t) \right)_{xx}dx$ is the total lateral force acting on a slice of the tower of length $dx$, located at the position $x$ and the time $t$. $(EIw_{xx})_x(l,t)$ and $-\left( EI(l)w_{xx}(l,t) \right)$ are the force and the torque acting on the nacelle from the tower at the time $t$. $T_{hss}$ and $T_{Iss}$ are the torque acting on the gearbox from the high speed shaft and the low speed shaft, respectively. $T_e$ is the torque control created by the electrical generator. We assume that $\rho, EI \in C^4[0,l]$ are strictly positive functions.

$\theta_T$ and $\theta_m$ are the angles of the turbine rotor and generator rotor with respect to the nacelle. $\theta_k$ is the angular difference between the endpoints of the low-speed shaft. $J_T > 0$ is the rotational inertia of the turbine blades and other low-speed components (for example the hub) while $J_G > 0$ is the rotational inertia of the generator rotor. $K_s > 0$ and $C_s \geq 0$ are the torsional stiffness and torsional damping coefficient of the low-speed shaft. $b_m > 0$ is the damping coefficient of the high speed shaft. $T_n$ is the active torque from the turbine, which is a disturbance from our point of view. $n_g$ is the gearbox ratio. To derive this model, we have taken into account $w_{xxl}$, the angular acceleration of the nacelle: $(\theta_T)_x(t) + w_{xxl}$ and $(\theta_m)_x(t) - w_{xxl}$ are the angular accelerations of the turbine rotor and generator rotor with respect to the earth. We assume that $J_T > J_G n_g$, which is always true in reality (for example from Wang [15, Table 3.1] we can see that $J_T$ is 1713 times bigger than $J_G n_g$). Note that we allow $b_m > 0$ and $C_s \geq 0$, so that our results do not rely on damping (they hold also without damping).

The natural state at the time $t$ of this wind turbine tower model $\Sigma$ is

$$z(t) = \left[ z_1(t), z_2(t), z_3(t), z_4(t), z_5(t), z_6(t), z_7(t) \right]^T,$$

where the superscript $T$ means transpose, $z_1(x,t) = w(x,t)$, $z_2(x,t) = \theta_T(x,t)$, $z_3(t) = w_t(t)$, $z_4(t) = w_{xl}(t)$, $z_5(t) = (\theta_m)_x(t) - w_{xl}(t)$ and $z_7(t) = \theta_k(t)$. The energy state space of $\Sigma$ is

$$X = \mathcal{H}^2_1(0,l) \times L^2[0,l] \times \mathbb{C}^5.$$  \hfill (1.9)

Here

$$\mathcal{H}^2_1(0,l) = \{ h \in \mathcal{H}^2(0,l) \mid h(0) = 0, h_x(0) = 0 \},$$  \hfill (1.10)

where $\mathcal{H}^n$ ($n \in \mathbb{N}$) denote the usual Sobolev spaces. The natural norm on $X$ is

$$\|z(t)\|^2 = \int_0^l \left[ \left( EI(x)z_{xx}(x,t) \right)^2 + \rho(x)|z_2(x,t)|^2 \right] dx + m_7|z_3(t)|^2 + J|z_4(t)|^2 + J_T|z_5(t)|^2 + J_G|z_6(t)|^2 + K_s|z_7(t)|^2,$$  \hfill (1.11)

which represents twice the physical energy.

In this paper we show the following generic strong stabilization result: for every time $T > 0$ and for every choice of the strictly positive functions $\rho_n, EI_n \in C^4[0,l]$ and of the parameters $l > 0, m_n > 0, J_n > 0, K_s > 0, J_T > 0, J_G > 0, n_g > 0, C_s \geq 0$ and $b_m \geq 0$, there are at most 3 values $\mu > 0$ such that the system $\Sigma_c$ with

$$\rho = \mu \rho_n, \quad EI = \mu EI_n, \quad m = \mu m_n, \quad J = \mu J_n$$

is not strongly stabilizable on $X$, by the static output feedback $T_e = -ky + v$, where $v$ is the new input (typically zero) and

$$y(t) = -\frac{1}{J} w_{xt}(l,t) - \frac{1}{J_G} ((\theta_m)_x(t) - w_{xl}(l,t)).$$

The feedback gain $k$ can take any positive number and the resulting closed-loop system is well-posed, see Theorem 3.5.

II. SOME BACKGROUND ON INFINITE-DIMENSIONAL SYSTEMS

In this section we introduce some concepts and results on infinite-dimensional linear time invariant systems, without proof. For the details we refer to the literature.

A. Well-posed linear systems

In this section we introduce some concepts and results concerning well-posed linear systems. First we introduce the concepts of admissible control and observation operators, which can be found (in much greater detail and with many references) in Tucsnak and Weiss [14, Chapter 4].

Let $A$ be the generator of a strongly continuous semigroup $\mathbb{T}$ on a Hilbert space $X$. Then $\mathbb{T}$ and $X$ determine two additional Hilbert spaces: $X_1 = D(A)$ with the norm $\|z\|_1 = \| (\beta I - A)z \|$ and $X_{-1}$ is the completion of $X$ with respect to the norm $\|z\|_{-1} = \| (\beta I - A)^{-1}z \|$, where $\beta \in \rho(A)$ is fixed. These spaces are independent of the choice of $\beta$, since different values of $\beta$ lead to equivalent norms on $X_1$ and $X$. The norm $\|z\|_1$ is equivalent to the graph norm of $A$. We have $X_1 \subset X \subset X_{-1}$ densely and with continuous embeddings. We can extend $A$ to a bounded operator from $X$ to $X_{-1}$, still denoted by $A$. The semigroup generated by this extended $A$ is the extension of $\mathbb{T}$ to $X_{-1}$, which is still denoted by $\mathbb{T}$.

In this section, $U$ and $Y$ are Hilbert spaces, $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$. We define the operators $\Phi_{\tau}$ (for $\tau > 0$) by

$$\Phi_{\tau}u = \int_0^\tau \mathbb{T}_{\tau-t}Bu(t)dt.$$  \hfill (2.1)

where $u(t) \in L^2([0, \infty); U)$. Clearly $\Phi_{\tau}u \in X_{-1}$.

Definition 2.1: $B$ is said to be an admissible control operator for the semigroup $\mathbb{T}$ if $\text{Ran} \Phi_{\tau} \subset X$ for some $\tau > 0$.

The admissibility of $B$ implies that the solutions $z(\cdot)$ of

$$\dot{z}(t) = Az(t) + Bu(t),$$  \hfill (2.2)
with initial state \( z(0) = z_0 \in X \) and with \( u \in L^2_{\text{loc}}([0,\infty); U) \) remain in \( X \). Moreover it follows that \( z(\cdot) \) is a continuous \( X \)-valued function of \( t \) and
\[
z(t) = T_t z_0 + \Phi_t u. \tag{2.3}
\]
The operator \( B \) is said to be bounded if \( B \in \mathcal{L}(U, X) \) and unbounded otherwise.

In the sequel, we denote by \( \omega_0 \) the growth bound of \( T \). We also use the notation \( C_{\alpha} \) for the right half-plane determined by the real number \( \alpha \):
\[
C_{\alpha} = \{ s \in \mathbb{C} \mid \Re s > \alpha \}. \tag{2.4}
\]

**Proposition 2.2:** If \( B \) is admissible then for every \( \alpha > \omega_0 \) there exists a constant \( K_{\alpha} > 0 \) such that
\[
\| (sI - A)^{-1} B \|_{\mathcal{L}(U, X)} \leq \frac{K_{\alpha}}{\sqrt{\Re s - \alpha}} \quad \forall s \in C_{\alpha}.
\]

We define the operators \( \Psi_{\tau} \) (for \( \tau > 0 \)) by
\[
(\Psi_{\tau} z_0)(t) = \left\{ \begin{array}{ll}
C T_t z_0 & \text{for } t \in [0, \tau], \\
0 & \text{for } t > \tau.
\end{array} \right.
\]
It is clear that \( \Psi_{\tau} \in \mathcal{L}(X, L^2([0, \infty); Y)) \). \( C \) is said to be bounded if it can be extended such that \( C \in \mathcal{L}(X, Y) \) and unbounded otherwise.

**Definition 2.3:** \( C \) is said to be an admissible observation operator for the semigroup \( T \), if \( \Psi_{\tau} \) has a continuous extension to \( X \) for some \( \tau > 0 \).

The admissibility of \( C \) means that there exists a continuous operator \( \Psi : X \to L^2_{\text{loc}}([0, \infty); Y) \) such that
\[
(\Psi z_0)(t) = C T_t z_0 \quad \forall z_0 \in D(A). \tag{2.5}
\]
The \( \Lambda \)-extension of \( C \), denoted by \( C_{\Lambda} \), is defined by
\[
C_{\Lambda} z_0 = \lim_{\lambda \to +\infty} C(\lambda I - A)^{-1} z_0,
\]
and its domain \( D(C_{\Lambda}) \) consists of all \( z_0 \in X \) for which the limit exists. If we replace \( C \) by \( C_{\Lambda} \), formula (2.4) becomes true for all \( z_0 \in X \) and for almost every \( t \geq 0 \).

A well-posed linear system with input space \( U \), state space \( X \) and output space \( Y \) is a family of bounded linear operators (parametrized by \( \tau \geq 0 \)) that associates to every initial state \( z_0 \in X \) and every input signal \( u \in L^2([0, \tau); U) \) a final state \( z(\tau) \) and an output signal \( y \in L^2([0, \tau); Y) \). These operators have to satisfy certain natural functional equations, for the formal definition we refer to Salamon [11] and Weiss [17]. We recall some facts about well-posed linear systems from [17].

Let \( \Sigma \) be a well-posed system with input space \( U \), state space \( X \) and output space \( Y \). Then \( \Sigma \) is completely determined by its generating triple \((A, B, C)\) and its transfer function \( G \). Here, \( A \) is the semigroup generator of \( \Sigma \), which generates a strongly continuous semigroup \( T \) on \( X \), \( B \in \mathcal{L}(U, X - I) \) is the control operator of \( \Sigma \) and \( C \in \mathcal{L}(X, Y) \) is its observation operator. The transfer function \( G \) satisfies
\[
G(s) - G(\beta) = C((sI - A)^{-1} - (\beta I - A)^{-1})B \tag{2.5}
\]
for all \( s, \beta \in \rho(A) \). The state trajectories of \( \Sigma \) satisfy (2.2), hence also (2.3). If \( u \in L^2_{\text{loc}}([0, \infty); U) \) is the input function of \( \Sigma \), \( z_0 \in X \) is its initial state and \( y \in L^2_{\text{loc}}([0, \infty); Y) \) is the corresponding output function, then
\[
y = \Psi z_0 + F u. \tag{2.6}
\]
Here, \( \Psi \) is as in (2.4), while \( F : L^2_{\text{loc}}([0, \infty); U) \to L^2_{\text{loc}}([0, \infty); Y) \) is easiest to represent using Laplace transforms, as follows: if \( u \in L^2([0, \infty); U) \) and \( y = Fu \), then \( y \) has a Laplace transform \( \hat{y} \) and
\[
\hat{y}(s) = G(s) \hat{u}(s) \tag{2.7}
\]
for all \( s \in \mathbb{C} \) with \( \Re s \) sufficiently large. \( G \) is proper, which means that its domain contains a right half-plane \( C_{\alpha} \) such that \( G \) is uniformly bounded on \( C_{\alpha} \).

**Definition 2.4:** The well-posed linear system \( \Sigma \) is called regular if the limit
\[
\lim_{s \to +\infty} G(s)v = Dv
\]
e exists for every \( v \in U \), where \( s \) is real. In this case, the operator \( D \in \mathcal{L}(U, Y) \) is called the feedthrough operator of \( \Sigma \).

We mention a few facts about regular systems, following [17]. Regularity is equivalent to the fact that the product \( C_{\alpha}(sI - A)^{-1} B \) makes sense, for some (hence for every) \( s \in \rho(A) \). If \( \Sigma \) is regular then for every initial state \( z_0 \in X \) and every \( u \in L^2_{\text{loc}}([0, \infty); U) \) with \( z(0) = z_0 \) satisfies \( z(t) \in D(C_{\Lambda}) \) for almost every \( t \geq 0 \) and the corresponding output from (2.6) is given by
\[
y(t) = C_{\Lambda} z(t) + Du(t) \quad \text{for almost every } t \geq 0. \tag{2.8}
\]
The transfer function of the regular system \( \Sigma \) is given by
\[
G(s) = C_{\alpha}(sI - A)^{-1} B + D \quad \forall s \in \rho(A). \tag{2.9}
\]
The operators \( A, B, C, D \) are called the generating operators of \( \Sigma \). This is because they determine \( \Sigma \) via (2.2) and (2.8). If \( C \) is bounded (i.e. \( C \in \mathcal{L}(X, Y) \)), then \( C \) replaces \( C_{\Lambda} \) in (2.8) and (2.9), and (2.8) holds for every \( t \geq 0 \).

Let \( p \) be a function defined on some domain in \( \mathbb{C} \) that contains a right half-plane, with values in a normed space. We say that \( p \) is strictly proper if
\[
\lim_{\Re s \to +\infty} \|p(s)\| = 0, \quad \text{uniformly with respect to } \Im s.
\]
A linear system is called strictly proper if its transfer function is strictly proper.

**Proposition 2.5:** Let the infinite-dimensional system \( \Sigma \) with input space \( U \), state space \( X \) and output space \( Y \) be described by the equations (2.2) and
\[
y(t) = C z(t) + Du(t).
\]
Here \( A \) generates a semigroup \( T \) on \( X \), \( B \) is an admissible control operator for \( T \), \( C \in \mathcal{L}(X, Y) \) and \( D \in \mathcal{L}(U, Y) \). Then \( \Sigma \) is well-posed and regular, with feedthrough operator \( D \). If \( D = 0 \), then \( \Sigma \) is strictly proper.

This is an easy consequence of the main result in Curtain and Weiss [2] together with Proposition 2.2.
B. Controllability and observability

Let $U, X, Y, T, A, B, C, \Phi_\tau$ and $\Psi_\tau$ be as at Section 2.1. We assume that $B$ and $C$ are admissible for $T$.

Definition 2.6: The pair $(A, B)$ is said to be exactly controllable in time $\tau > 0$ if $\text{Ran} \, \Phi_\tau = X$; $(A, B)$ is said to be approximately controllable in time $\tau > 0$ if $\ker \, \Psi_\tau$ is dense in $X$.

Definition 2.7: The pair $(A, C)$ is said to be exactly observable in time $\tau > 0$ if $\Psi_\tau$ is bounded from below. $(A, C)$ is said to be approximately observable in time $\tau > 0$ if $\ker \, \Psi_\tau = \{0\}$.

We often need the controllability concepts without specifying a time $\tau$. Therefore the following definition is introduced.

Definition 2.8: The pair $(A, B)$ is said to be exactly controllable if it is exactly controllable in some finite time $\tau > 0$. $(A, B)$ is said to be approximately controllable if it is approximately controllable in some finite time.

Observability concepts without a specified time are introduced in a similar way.

C. Passive and conservative linear systems

Passive systems are a class of dynamical systems that can only dissipate energy and cannot produce energy. In passive systems, the energy dissipated by some components in the system equals the difference between the absorbed energy and the increased stored energy. Conservative systems are a special case of passive systems. A passive system is conservative if neither this system nor its dual have any energy dissipation. There are many types of passive and conservative systems. We focus on impedance passive and conservative systems. For more details about passive and conservative systems we refer to [13].

Let $H$ be a Hilbert space, $P \in \mathcal{L}(H)$ and $P > 0$. We define an inner product by $\langle \xi, \nu \rangle_p = (P \xi, \nu)$ (for any $\xi, \nu \in H$) which induces the norm $\|\xi\|_p = \sqrt{\langle \xi, \xi \rangle_p}$. $P$ is said to be impedance preserving if $P$ is impedance preserving and its dual system $\Sigma^*$ is impedance preserving. $P$ is said to be impedance conservative if it is impedance preserving and its dual system $\Sigma^*$ is impedance preserving.

If $P = I$, we say "impedance passive" instead of "impedance $I$-passive". The concepts of impedance energy preserving and impedance conservative are defined similarly. From the energy point of view, (2.10) is an energy balance inequality. $E(t) = \frac{1}{2} \|z(t)\|_p^2 = \frac{1}{2} \langle Pz(t), z(t) \rangle$ is the energy stored in the system at the time $t$, and $\text{Re} \langle u(t), y(t) \rangle$ is the incoming power of the system at the time $t$.

It is well known that for finite-dimensional systems, the Kalman-Yakubovich-Popov (KYP) lemma states the equivalence between the positive real property of the transfer function of a system, the passivity of a realization of the transfer function in the time domain, and the existence of a solution to a linear matrix inequality that depends on the operators of a systems state space representation (see [3], [9]). In the infinite-dimensional case, we have a similar result. The following lemma is an extension of the KYP lemma to regular systems, and it follows from a more general result in Staffans [12].

Lemma 2.10: A regular system with generating operator $(A, B, C)$ and feedthrough operator $D$ is impedance passive if and only if the operator

$$N = \begin{bmatrix} A & B \\ -C_A & -D \end{bmatrix},$$

$$\mathcal{D}(N) = \left\{ \begin{bmatrix} z \\ u \end{bmatrix} \in X \times U \mid Az + Bu \in X \right\},$$

is maximal dissipative.

The fact that $N$ is dissipative means that

$$\text{Re} \left\langle \begin{bmatrix} A & B \\ -C_A & -D \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix}, \begin{bmatrix} z \\ u \end{bmatrix} \right\} \leq 0 \quad \forall \begin{bmatrix} z \\ u \end{bmatrix} \in \mathcal{D}(C_A).$$

Now we define almost impedance passivity following [18].

Definition 2.11: Let $\Sigma$ be a well-posed system with generating triple $(A, B, C)$, transfer function $G$, state space $X$, input space $U$, and output space $Y$. Let $E = \Sigma^* \in \mathcal{L}(U)$. If we replace $G$ with $G + E$ (and keep $A, B, C$ unchanged), we get a modified well-posed $\Sigma_E$. If there exists $E$ such that $\Sigma_E$ is impedance passive, we call the original well-posed system $\Sigma$ almost impedance passive.

D. Stability and stabilization

Unlike the finite-dimensional linear systems, there are at least three kinds of different asymptotic stability of state space in infinite-dimensional linear systems: weak, strong and exponential stability. The stability of a system is equivalent to the stability of the semiflows of this system.

Definition 2.12: Let $\Sigma$ be an almost well-posed linear system with strongly continuous semiflow $T$ and state space $X$.

1. The system $\Sigma$ (or the semiflaw $T$) is called weakly stable if $\langle T_t z, y \rangle > 0$ for $t \to \infty$, for all $z, y \in X$.

2. The system $\Sigma$ (or the semiflaw $T$) is called strongly stable if $T_t z \to 0$ as $t \to \infty$, for all $z \in X$.

3. The system $\Sigma$ (or the semiflaw $T$) is called exponentially stable if its growth bound is negative.

Output feedback stabilization of a system means finding a feedback operator $K$, to make the system stable with the input $u = Ky$. The closed-loop system must have a strongly continuous semiflow describing the evolution of its state. If the resulted closed-loop system is exponentially, or strongly, or weakly stable, then we call the original system exponentially, or strongly, or weakly stabilizable (by output feedback) respectively.

The following is a simple but useful proposition, taken from Benchimol [1].

Proposition 2.13: If a system is weakly stable and its generator has compact resolvents, this system is strongly stable.
The theorem below follows from Weiss and Curtain [18, Theorems 5.2, 5.3].

**Theorem 2.14:** If an almost impedance passive well-posed system is either approximately controllable or approximately observable in infinite time, then this system is weakly stabilizable by static output feedback for sufficiently small gain $k$. The resulting closed-loop system is well-posed. Furthermore if the intersection of the spectrum of the semigroup generator of the open-loop system and the imaginary axis is countable, then the closed-loop system and its dual are both strongly stable. If the well-posed system is impedance passive (i.e., $E = 0$ in Definition 2.11) then the gain $k$ can be taken to be any positive number.

## III. STRONG STABILIZATION OF THE WIND TURBINE TOWER MODEL

In this section, we analyze the strong stabilization of the wind turbine tower model in the plane of the turbine blades $\Sigma$ described by (1.1)–(1.8). As mentioned in Section 1, the state of $\Sigma$ at the time $t$ is

$$z(t) = [z_1(t) \ z_2(t) \ z_3(t) \ z_4(t) \ z_5(t) \ z_6(t) \ z_7(t)]^T.$$

Here $z_1(t)$ and $z_2(t)$ are the states of the tower, among which $z_1(x, t)$ is the transverse displacement of the beam at the position $x$ and the time $t$, and $z_2(x, t)$ is the transverse movement velocity of the beam at the position $x$ and the time $t$. $z_3(t)$ and $z_4(t)$ are the states of the nacelle, among which $z_3(t)$ is the velocity of the nacelle at the time $t$ and $z_4(t)$ is the angular velocity of the nacelle at the time $t$. $z_5(t)$, $z_6(t)$ and $z_7(t)$ are the states of the two-mass drive-train model. $z_5(t) = (\theta_1 x)(t) + w_{xt}(t)$ and $z_6(t) = (\theta_1 x)(t) - w_{xt}(t)$ are the angular velocities of the turbine rotor and generator rotor with respect to the earth and $z_7(t)$ is the angular difference between the endpoints of the low-speed shaft. The energy state space of $\Sigma$ is

$$X = \mathcal{H}_1^2(0, l) \times L^2(0, l) \times \mathbb{C}^5.$$  (3.1)

The norm on $X$ is

$$||\xi||^2 = \int_0^l \left( EI(x) |\xi_{1xx}(x)|^2 + \rho(x) |\xi_2(x)|^2 \right) dx + m |\xi_3|^2 + J|\xi_4|^2 + J_G|\xi_5|^2 + K_s|\xi_6|^2 + K_t|\xi_7|^2$$

for all $\xi = [\xi_1 \ \xi_2 \ \xi_3 \ \xi_4 \ \xi_5 \ \xi_6 \ \xi_7]^T \in X$. For every $\xi \in \mathcal{D}(A)$ we have

$$A\xi = \begin{bmatrix}
\xi_2 \\
-\rho^{-1}(x) (EI(x) |\xi_{1xx}(x)|^2)_{xx} - m^{-1}(EI_f_{1xx}(l))_x \\
-\frac{EI_1}{J_{1g}} |\xi_{1xx}(l)| + \frac{K_s(1+n_s)}{J_{ng}} \xi_7 - \frac{b_T}{J_G} (\xi_6 + \xi_4) + \frac{C_s(1+n_s)}{J_{ng}} \kappa \\
-\frac{b_T}{J_G} \xi_7 - \frac{b_T}{J_G} (\xi_6 + \xi_4) + \frac{C_s}{J_{ng}} \kappa \\
\frac{K_t}{J_G} \xi_7 - \frac{b_T}{J_G} (\xi_6 + \xi_4) + \frac{C_s}{J_{ng}} \kappa \\
\end{bmatrix}$$

where

$$\kappa = \xi_5 - \frac{1}{n_g} \xi_6 - \frac{1+n_g}{n_g} \xi_4,$$

$$\mathcal{D}(A) = \left\{ \xi \in [\mathcal{H}_1^2 \cap \mathcal{H}_1^2] \times \mathcal{H}_1^2 \times \mathbb{C}^5 \middle| \xi_3 = \xi_2(l), \xi_4 = \xi_2(e) \right\}.$$

We use colocated sensors and actuators,

$$B = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{J_G} & 0 & -\frac{1}{J_G} & 0 \end{bmatrix}^T, \quad B^* = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{J_G} & 0 & -\frac{1}{J_G} & 0 \end{bmatrix}.$$

Then the wind turbine tower model $\Sigma$ described by (1.1)–(1.8) can be rewritten as

$$\begin{cases}
\dot{z}(t) = Az(t) + Bu_c(t), \\
y(t) = B^* z(t),
\end{cases}$$

(3.2)

where $u_c = T_e$ is the electric torque control. To prove our main result, we need a proposition from the operator theory:

**Proposition 3.1:** Let $X$ be a Hilbert space and let $A : \mathcal{D}(A) \to X$ be an operator with non-empty resolvent set $\rho(A)$ and with compact resolvents. Then $\sigma(A)$ (the spectrum of $A$) is countable.

**Proposition 3.2:** Let $A_b : \mathcal{D}(A_b) \to X$ be a maximal dissipative operator with compact resolvents. Let $Q \in \mathcal{L}(X)$ and $A = A_b + Q$ (with $\mathcal{D}(A) = \mathcal{D}(A_b)$). Then $A$ has compact resolvents. If $A$ is dissipative, then $A$ is maximal dissipative on $X$. Furthermore if $A$ is skew-symmetric, then $A$ is skew-adjoint on $X$.

**Proof.** $A$ is a bounded perturbation of $A_b$, hence it is a semigroup generator. Therefore, for sufficiently large $\beta > 0$, $\beta I - A$ has a bounded inverse (hence it is onto). Hence, if $A$ is dissipative then $A$ is maximal dissipative on $X$ (see for example Tucsnak and Weiss [14, Definition 3.1.8]).

Since $A_b$ has compact resolvents, and $A$ has a non-empty resolvent set, also $A$ has compact resolvents. Hence $\sigma(A)$ consists of isolated points, so that $sI - A$ is invertible for almost every $s < 0$. The facts that $A$ is skew-symmetric and both $\beta I - A$ and $\beta I + A$ are onto for some $\beta > 0$ implies that $A$ is skew-adjoint on $X$, see for example Tucsnak and Weiss [14, Proposition 3.7.3].

**Proposition 3.3:** The generator $A$ of the wind turbine tower model $\Sigma$ described by (3.2) is maximal dissipative with compact resolvents on the state space $X$. $\Sigma$ is well-posed and strictly proper (hence regular) on $X$. If the dampings $C_s$ and $b_m$ are zero, then $A$ is skew-adjoint.

**Proof.** It can be verified that for all $\xi \in \mathcal{D}(A)$,

$$\Re \langle A\xi, \xi \rangle = -b_m |\xi_4| + \xi_6^2 - C_s |\xi_5| - \frac{1}{n_g} \xi_6 - \frac{1+n_g}{n_g} |\xi_4|^2 \leq 0.$$  (3.3)

Thus $A$ is dissipative. The operator $A$ in the particular case when $C_s = 0$ and $b_m > 0$ is denoted $A_b$. It is not difficult to show that $A_b$ is onto (based on the well-known result that the generator of the SCOLE model is onto) and has compact resolvents. As

$$A = A_b + Q$$

where $Q \in \mathcal{L}(X)$ is a function of $b_m$ and $C_s$. Then by Proposition 3.2 it follows that $A$ is maximal dissipative with compact resolvents. This means that $A$ is the generator of a contraction semigroup with isolated eigenvalues in the closed
left-half plane. Clearly $B$ and $B^*$ are bounded on $X$. By Proposition 2.5 $\Sigma$ is well-posed, strictly proper and regular on $X$.

If there are no damping terms i.e., $b_m = 0$ and $C_s = 0$, then from (3.3)
\[
\text{Re} \left( \langle A\xi, \xi \rangle \right) = 0 \quad \forall \xi \in D(A),
\]
which means that $A$ is skew-symmetric on $X$. Using Proposition 3.2 again, we conclude that $A$ is skew-adjoint.

Proposition 3.4: The wind turbine tower model $\Sigma$ described by (3.2) is impedance passive on the state space $X$.

Proof. From Propositions 3.3 we know that $A$ is maximal dissipative with compact resolvents. We decompose
\[
\begin{bmatrix} -A & B \\ -B^* & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} + \begin{bmatrix} 0 & B \\ -B^* & 0 \end{bmatrix}.
\]
Clearly $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is maximal dissipative and $\begin{bmatrix} 0 & B \\ -B^* & 0 \end{bmatrix}$ is bounded (since $B$ and $B^*$ are bounded). Thus by Proposition 3.2, $\begin{bmatrix} -A & B \\ -B^* & 0 \end{bmatrix}$ is maximal dissipative if it is dissipative. Combining this fact with Lemma 2.10, we know that $\Sigma$ is impedance passive if and only if
\[
\text{Re} \left( \begin{bmatrix} A & B \\ -B^* & 0 \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix}, \begin{bmatrix} z \\ u \end{bmatrix} \right) \leq 0.
\]
As $A$ is maximal dissipative, it follows that $\text{Re} \left< Az, z \right> \leq 0$. Now we compute
\[
\begin{align*}
\text{Re} \left( \begin{bmatrix} A & B \\ -B^* & 0 \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix}, \begin{bmatrix} z \\ u \end{bmatrix} \right) &= \text{Re} \left( \begin{bmatrix} Az + Bu \\ -B^*z \end{bmatrix}, \begin{bmatrix} z \\ u \end{bmatrix} \right) \\
&= \text{Re} < Az, z > + \text{Re} < Bu, z > - \text{Re} < B^*z, u > \\
&= \text{Re} < Az, z > + \text{Re} < Bu, z > - \text{Re} < z, Bu > \\
&= \text{Re} < Az, z > + \text{Re} \left( B^*u \right) - \text{Re} \left( Bu \right) \\
&= \text{Re} < Az, z > + \text{Re} \left( z^*Bu \right) - \text{Re} \left( Bu \right) \\
&= \text{Re} < Az, z > \leq 0. \quad (3.4)
\end{align*}
\]
Thus $\Sigma$ is impedance passive.

Theorem 3.5: For every time $T > 0$ and for every choice of the strictly positive functions $\rho_n, E_I_n \subset C^4[0, l]$ and of the parameters $l > 0$, $m_n > 0$, $J_n > 0$, $K_s > 0$, $J_T > 0$, $J_G > 0$, $n_g > 0$, $C_s > 0$ and $b_m > 0$, there are at most 3 values $\mu > 0$ such that the wind turbine tower model $\Sigma$ described by (3.2) with
\[
\rho = \mu \rho_n, \quad E_I = \mu E_I_n, \quad m = \mu m_n, \quad J = \mu J_n
\]

is not strongly stabilizable on the state space $X$ from (3.1) by a static output feedback $u_e = -ky + v$ where
\[
y(t) = B^*z(t) = -\frac{1}{J_G}w_{zt}(l, t) - \frac{1}{J_G} \left( (\theta_m)_l(t) - w_{zt}(l, t) \right)
\]

and $v$ is the new input function. The feedback gain $k$ can be any positive number and the closed-loop system is well-posed.

Proof. From our paper [19] we know that $\Sigma$ described by (3.2) is well-posed and approximately controllable on the state space $X$ subject to the conditions in Theorem 3.5. From Proposition 3.4 we know that $\Sigma$ is impedance passive. By Theorem 2.14, it follows that $\Sigma$ is weakly stabilizable subject to the conditions in Theorem 3.5 by the static output feedback $u_e = -ky + v$. From Propositions 3.3 and 3.1, it follows that $\sigma(A)$ consists of at most countably many imaginary eigenvalues. Using Theorem 2.14 again, we get Theorem 3.5.

References