Well-posedness and controllability of a wind turbine tower model

XIAOWEI ZHAO*

Department of Engineering Science, University of Oxford, Parks Road,
Oxford OX1 3PJ, UK
*Corresponding author: xiaowei.zhao@eng.ox.ac.uk

AND

GEORGE WEISS

Department of Electrical Engineering-Systems, Tel Aviv University,
Ramat Aviv 69978, Israel
gweiss@eng.tau.ac.il

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We derive a model for a wind turbine tower in the plane of the turbine blades, which comprises an Euler–Bernoulli beam coupled with a nacelle (rigid body) and a two-mass drive-train model (with gearbox). This model has two possible control inputs: the torque created by the electrical generator and the force created by an electrically driven mass located in the nacelle. First, we consider the case of only torque control and a possibly non-uniform tower. Using the theory of coupled linear systems (one infinite dimensional and one finite dimensional) developed by us recently, we show that this wind turbine tower model is well-posed and regular on either the energy state space $X^c$ or the domain of its generator on $X^c$, denoted by $X^c_1$. We also show that generically, this model is exactly controllable on $X^c_1$ in arbitrarily short time. More precisely, for every $T > 0$, we show that if we vary a certain parameter in the model, then exact controllability in time $T$ holds for all except three values of the parameter. In the case of using both force and torque control, we derive similar well-posedness, regularity and generic exact controllability results on a state space that is larger than $X^c_1$ but smaller than $X^c$. In this second case, we assume that the tower is uniform.

Keywords: wind turbine tower; coupled system; SCOLE model; exact controllability; well-posed linear system.

1. Introduction

The aim of this paper is to develop a wind turbine model in the plane of the turbine blades and investigate its well-posedness, regularity and exact controllability in various state spaces with two kind of control inputs: only torque control as well as both force and torque control.

Large offshore turbines are subjected to severe weather causing vibrations leading to fatigue. Their life expectancy is difficult to evaluate, but it is important to keep it high. This implies a need for a control system to suppress the vibrations of the turbine tower without affecting the reliability of the power supply.
To design a vibration suppression controller, we need to find a suitable model for the wind turbine tower and then to study its controllability. The current wind turbine tower models are mostly based on finite-element approximations, which divide the tower into many slices and consider these slices as interconnected finite-dimensional systems. These models are very complex and are used mainly for simulation. Here, we develop a distributed parameter model of the tower. We know that for systems with an essentially skew-adjoint generator (such as our model below), exact controllability implies the possibility to exponentially stabilize the system by output feedback (see Curtain & Weiss, 2006). This is the reason why examining the controllability of this system is of practical significance.

Think of a wind turbine tower clamped in the ocean floor that carries at its top the nacelle (with a weight of several hundred tons). The tower is partially submerged in water and its diameter decreases with height. The NASA Spacecraft Control Laboratory Experiment (SCOLE) system is a well-known model for a flexible beam with one end clamped and the other end linked to a rigid body. It has been originally developed to understand the dynamics of a mast carrying an antenna on a satellite (see Littman & Markus, 1988a,b). In the vertical plane of the turbine axis, it is sensible to model the wind turbine tower as a non-uniform SCOLE system with only force control (this control can be obtained by modulating the turbine pitch angle). The exact controllability of the non-uniform SCOLE model on certain smoother spaces (contained in the natural energy space) and its approximate controllability on the energy state space using either force control or torque control has been shown in Guo (2002) and in Guo & Ivanov (2005). An exponential stabilizing high order feedback for the non-uniform SCOLE model on the energy state space has been found in Guo & Ivanov (2005). The strong stabilization of this system on the energy state space via feedback from either the velocity or the angular velocity of the rigid body has been shown in Zhao & Weiss (2008).

In the plane of the turbine blades, we have to use a more complex model. The gearbox transfers most of the torque caused by the wind to the nacelle. Our model must take the gearbox and the flexible (low-speed) turbine shaft into account. We use a two-mass drive-train model (see Fig. 1) studied in Hansen et al. (2002), Lubosny (2003) and Wang & Weiss (2009) to analyse the mechanical system within the nacelle.

For a wind turbine tower moving in the plane of the turbine blades, we denote by $l$ the height of the tower, while $EI$ and $\rho$ are its flexural rigidity function and mass density function. $m > 0$ and $J > 0$ are the mass and the moment of inertia of the nacelle. Denoting by $w$ the transverse displacement

![Fig. 1. The two-mass drive-train model with gearbox. $T_a$ is the active torque from the turbine and $T_e$ is the electric torque of the generator, which acts between the rotor (connected to the high speed shaft) and the stator (connected to the nacelle).](image)
The model we have taken into account is

\[
\rho(x)w_{tt}(x, t) + (EI(x)w_{xx}(x, t))_{xx} = 0, \quad (x, t) \in (0, l) \times [0, \infty),
\]  
\[
w(0, t) = 0, \quad w(x, 0, t) = 0,
\]
\[
mw_{tt}(l, t) - (EIw_{xx})(l, t) = f(t),
\]
\[
Jw_{xtt}(l, t) + EI(l)w_{xx}(l, t) = T_{lss}(t) + T_{hss}(t) - b_m(\theta_m)(t) - T_c(t),
\]
\[
(\theta_T)_{tt}(t) = \frac{1}{J_T}(T_a(t) - T_{lss}(t)) - w_{xtt}(l, t),
\]
\[
(\theta_m)_{tt}(t) = \frac{1}{J_G}(T_{hss}(t) - b_m(\theta_m)(t) - T_c(t)) + w_{xtt}(l, t),
\]
\[
\theta_k(t) = (\theta_T) - \frac{\theta_m(t)}{n_g},
\]
\[
T_{lss}(t) = K_s\theta_k(t) + C_s(\theta_k)(t) = n_gT_{hss}(t),
\]

where the subscripts \( t \) and \( x \) denote derivatives with respect to the time \( t \) and the position \( x \), respectively. Equations (1.1)–(1.4) are a non-uniform SCOLE model. Equations (1.5)–(1.8) are a two-mass drive-train model.

In this model, \(-(EI(x)w_{xx}(x, t))_{xx} dx\) is the total lateral force acting on a slice of the tower of length \( dx \), located at the position \( x \) and the time \( t \). \((EIw_{xx})(l, t)\) and \(-EI(l)w_{xx}(l, t)\) are the force and the torque acting on the nacelle from the tower at the time \( t \). \( T_{hss} \) and \( T_{lss} \) are the torque acting on the gearbox from the high-speed shaft and the low-speed shaft, respectively. \( f \) is the force input that can be set to be zero or it can be produced by an electrically driven mass located in the nacelle. \( T_c \) is the torque control created by the electrical generator. We assume that \( \rho, EI \in C^4[0, l] \) are strictly positive functions.

\( \theta_T \) and \( \theta_m \) are the angles of the turbine rotor and generator rotor with respect to the nacelle. \( \theta_k \) is the angular difference between the endpoints of the low-speed shaft. \( J_T > 0 \) is the rotational inertia of the turbine blades and other low-speed components (e.g. the hub), while \( J_G > 0 \) is the rotational inertia of the generator rotor. \( K_s > 0 \) and \( C_s \geq 0 \) are the torsional stiffness and torsional damping coefficient of the low-speed shaft. \( b_m \geq 0 \) is the damping coefficient of the high-speed shaft. \( T_a \) is the active torque from the turbine, which is a disturbance from our point of view. \( n_g \) is the gearbox ratio. To derive this model, we have taken into account \( w_{xtt}, \) the angular acceleration of the nacelle: \((\theta_T)_{tt}(t) + w_{xtt}\) and \((\theta_m)_{tt}(t) - w_{xtt}\) are the angular accelerations of the turbine rotor and generator rotor with respect to the earth. We assume that \( J_T > J_Gn_g \), which is always true in reality (e.g. from Wang, 2009, Table 3.1, we can see that \( J_T \) is much bigger than \( J_Gn_g \)).

Note that the damping terms in the wind turbine tower model are very small in reality, e.g. from the nominal physical parameters of a 6 MW wind turbine in Wang (2009, Table 3.1) we can see that their damping coefficients are \( C_s = 100 \text{ Nms/rad} \) and \( b_m = 0 \text{ kg m}^2/\text{s} \), respectively. We allow \( b_m \geq 0 \) and \( C_s \geq 0 \), so that our results do not rely on damping (they hold also without damping).

We introduce the following auxiliary functions: \( z_1(x, t) = w(x, t), z_2(x, t) = w_t(x, t), z_3(t) = w_x(l, t), z_4(t) = w_{xt}(l, t), q_1(t) = (\theta_T)(t) + w_{xt}(l, t), q_2(t) = (\theta_m)(t) - w_{xt}(l, t) \) and \( q_3(t) = \theta_k(t) \).

The natural state at the time \( t \) of this wind turbine tower model \( \Sigma_c \) is

\[
z^c(t) = [z(c, t) \quad q(t)]^T,
\]
where \( z(t) = [z_1(t), z_2(t), z_3(t), z_4(t)]^T \) (the superscript \( T \) means transpose) is the state of the SCOLE model at the time \( t \), while \( q(t) = [q_1(t), q_2(t), q_3(t)]^T \) is the state of the drive train at the time \( t \). (This decomposition into subsystems will be explained in detail in the later sections.) The energy state space of \( \Sigma_c \) is

\[
X^c = H^2(0, l) \times L^2[0, l] \times C^5.
\]  

(1.10)

Here,

\[
H^2(0, l) = \{ h \in H^2(0, l) \mid h(0) = 0, \ h_x(0) = 0 \},
\]

(1.11)

where \( H^n (n \in \mathbb{N}) \) denote the usual Sobolev spaces. The natural norm on \( X^c \) is

\[
\| z^c(t) \|^2 = \int_0^l EI(x) |z_{1xx}(x, t)|^2 \, dx + \int_0^l \rho(x) |z_2(x, t)|^2 \, dx + m |z_3(t)|^2 + J |z_4(t)|^2
\]

\[
+ J_T |q_1(t)|^2 + J_G |q_2(t)|^2 + K_s |q_3(t)|^2,
\]

(1.12)

which represents twice the physical energy. The input space \( U \) is either \( \mathbb{C} \) (if we use \( T_e \) as the control input) or \( \mathbb{C}^2 \) (if we use both the force \( f \) and the torque \( T_e \) as control inputs). This wind turbine tower model \( \Sigma_c \) is not exactly controllable on \( X^c \) since its control operator is bounded from \( U \) to \( X^c \) and hence compact.

In this paper, we show that the wind turbine tower model \( \Sigma_c \) with input \( u_e = T_e \), state \( \begin{bmatrix} z \\ q \end{bmatrix} \) and output \( w_{tx}(l, \cdot) \) is well-posed and regular on the energy state space \( X^c \). If we denote by \( A^c \) the generator of \( \Sigma_c \) on \( X^c \), then

\[
\mathcal{D}(A^c) = \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in [H^4(0, l) \cap H^2(0, l)] \times H^2(0, l) \times C^5 \mid z_3 = z_2(l), \ z_4 = z_{2x}(l) \right\}.
\]

(1.13)

We denote \( X^c_1 = \mathcal{D}(A^c) \), which is a Hilbert space with the graph norm of \( A^c \). We show that \( \Sigma_c \) with the same input, output and state remains well-posed and regular if we chose \( X^c_1 \) as the state space. Moreover, we have the following generic exact controllability result: for every time \( T > 0 \) and for every choice of the strictly positive functions \( \rho_n, EI_n \in C^4[0, l] \) and of the parameters \( l > 0, m_n > 0, J_n > 0, K_s > 0, J_T > 0, J_G > 0, n_g > 0, C_s \geq 0 \) and \( b_m \geq 0 \), there are at most three values \( \mu > 0 \) such that the system \( \Sigma_c \) with

\[
\rho = \mu \rho_n, \quad EI = \mu EI_n, \quad m = \mu m_n, \quad J = \mu J_n
\]

is not exactly controllable on \( X^c_1 \) in time \( T \), see Theorem 4.1. (Clearly, exact controllability on \( X^c_1 \) implies approximate controllability on \( X^c \).)

Now, consider the case when we use both force and torque control for \( \Sigma_c \) and the tower is uniform, i.e. \( EI \) and \( \rho \) are constant. We show that \( \Sigma_c \) with input \( u_e = \begin{bmatrix} f_e \\ T_e \end{bmatrix} \), state \( \begin{bmatrix} z \\ q \end{bmatrix} \) and output \( \begin{bmatrix} w_{tx}(l, \cdot) \end{bmatrix} \) is well-posed and regular on the state space

\[
\hat{X}^c = \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in [H^3(0, l) \cap H^2(0, l)] \times H^1(0, l) \times C^5 \mid z_2(l) = q_1 \right\}.
\]

Here,

\[
H^1(0, l) = \{ h \in H^1(0, l) \mid h(0) = 0 \}.
\]

(1.14)
We also get similar generic exact controllability result on $\hat{X}^c$ as in the only torque control case, see Theorem 5.2.

The structure of the paper is as follows: Section 2 is dedicated to the background on well-posed systems. Here, we give the necessary preliminaries about admissible control and observation operators, well-posed and regular linear systems.

In Section 3, we recall result about a class of coupled systems from our recent paper (Zhao & Weiss, 2010b). A coupled system from this class consists of a well-posed and strictly proper (hence regular) subsystem and a finite-dimensional subsystem connected in feedback. Under several assumptions, we can show the well-posedness, regularity and generic exact controllability of such coupled systems.

In Section 4, we consider the case when the tower $\Sigma_c$ described by (1.1)–(1.8) has only torque control, i.e. $u_c = T_c$. To apply the results from Section 3, we decompose $\Sigma_c$ into a non-uniform SCOLE model $\Sigma_d$ with only torque input coupled with a two-mass drive-train model $\Sigma_f$. We get the well-posedness and regularity of $\Sigma_c$ on both the energy state space $X^c$ from (1.10) and on the space $X^c_1$ from (1.13). We also obtain the generic exact controllability of $\Sigma_c$ on $X^c_1$ (hence, its generic approximate controllability on $X^c$), see Theorem 4.1.

In Section 5, we consider the case when $\Sigma_c$ has both force and torque control, i.e. $u_c = \begin{bmatrix} f_c \\ T_c \end{bmatrix}$. To apply the results from Section 3, again we decompose $\Sigma_c$ into a non-uniform SCOLE model $\Sigma_d$ with both force and torque inputs, coupled with a two-mass drive-train model $\Sigma_f$. We derive well-posedness, regularity and generic exact controllability results for $\Sigma_c$ on a state space which is larger than $X^c_1$ but smaller than $X^c$, see Theorem 5.2.

2. Some background on well-posed systems

In this section, we introduce some concepts and results concerning well-posed linear systems, without proof. For details, we refer to the literature. The material about admissible control and observation operators can be found (in much greater detail and with many references) in Tucsnak & Weiss (2009, Chapter 4).

Let $A$ be the generator of a strongly continuous semigroup $\mathbb{T}$ on a Hilbert space $X$. Then, $\mathbb{T}$ and $X$ determine two additional Hilbert spaces: $X_1$ is $\mathcal{D}(A)$ with the norm $\|z\|_1 = \|\beta I - A\|_1 z$ and $X_{-1}$ is the completion of $X$ with respect to the norm $\|z\|_{-1} = \|(\beta I - A)^{-1} z\|$, where $\beta \in \rho(A)$ is fixed. These spaces are independent of the choice of $\beta$ since different values of $\beta$ lead to equivalent norms on $X_1$ and $X$. The norm $\|z\|_1$ is equivalent to the graph norm of $A$. We have $X_1 \subset X \subset X_{-1}$ densely and with continuous embeddings. We can extend $A$ to a bounded operator from $X$ to $X_{-1}$, still denoted by $A$. The semigroup generated by this extended $A$ is the extension of $\mathbb{T}$ to $X_{-1}$, which is still denoted by $\mathbb{T}$.

In this section, $U$ and $Y$ are Hilbert spaces, $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$. We define the operators $\Phi_\tau$ (for $\tau > 0$) by

$$
\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau-t} B u(t) \, dt,
$$

where $u(t) \in L^2_{loc}([0, \infty), U)$. Clearly, $\Phi_\tau u \in X_{-1}$.

**Definition 2.1** $B$ is said to be an ‘admissible control operator’ for the semigroup $\mathbb{T}$ if $\text{Ran} \, \Phi_\tau \subset X$ for some $\tau > 0$.

The admissibility of $B$ implies that the solutions $z(\cdot)$ of

$$
\dot{z}(t) = Az(t) + Bu(t),
$$

are well-posed.
with initial state \( z(0) = z_0 \in X \) and with \( u \in L^2_{\text{loc}}((0, \infty), U) \) remain in \( X \). Moreover, it follows that \( z(\cdot) \) is a continuous \( X \)-valued function of \( t \) and
\[
z(t) = T_t z_0 + \Phi_t u.
\] The operator \( B \) is said to be ‘bounded’ if \( B \in \mathcal{L}(U, X) \) and ‘unbounded’ otherwise.

In the sequel, we denote by \( \omega_0 \) the growth bound of \( \mathbb{T} \). We also use the notation \( C_\alpha \) for the right half-plane determined by the real number \( \alpha \):
\[
C_\alpha = \{ s \in \mathbb{C} | \Re s > \alpha \}.
\]

**Proposition 2.1** If \( B \) is admissible, then for every \( \alpha > \omega_0 \), there exists a constant \( K_\alpha > 0 \) such that
\[
\|(sI - A)^{-1}B\|_{\mathcal{L}(U, X)} \leq \frac{K_\alpha}{\sqrt{\Re s - \alpha}} \quad \forall s \in C_\alpha.
\]

We define the operators \( \Psi_\tau \) (for \( \tau > 0 \)) by
\[
(\Psi_\tau z_0)(t) = \begin{cases} 
C T_\tau z_0 & \text{for } t \in [0, \tau], \\
0 & \text{for } t > \tau.
\end{cases}
\]
It is clear that \( \Psi_\tau \in \mathcal{L}(X_1, L^2((0, \infty), Y)) \). \( C \) is said to be bounded if it can be extended such that \( C \in \mathcal{L}(X, Y) \) and unbounded otherwise.

**Definition 2.2** \( C \) is said to be an admissible observation operator for the semigroup \( \mathbb{T} \) if \( \Psi_\tau \) has a continuous extension to \( X \) for some \( \tau > 0 \).

The admissibility of \( C \) means that there exists a continuous operator \( \Psi : X \to L^2_{\text{loc}}((0, \infty), Y) \) such that
\[
(\Psi z_0)(t) = C T_\tau z_0 \quad \forall z_0 \in \mathcal{D}(A).
\] The \( A \)-extension of \( C \), denoted by \( C_A \), is defined by
\[
C_A z_0 = \lim_{\lambda \to +\infty} C \lambda (\lambda I - A)^{-1} z_0,
\]
and its domain \( \mathcal{D}(C_A) \) consists of all \( z_0 \in X \) for which the limit exists. If we replace \( C \) by \( C_A \), formula (2.4) becomes true for all \( z_0 \in X \) and for almost every \( t \geq 0 \).

A ‘well-posed linear system’ with ‘input space’ \( U \), ‘state space’ \( X \) and ‘output space’ \( Y \) is a family of bounded linear operators \( \tau \geq 0 \) that associates to every initial state \( z_0 \in X \) and every input signal \( u \in L^2((0, \tau]; U) \) a final state \( z(\tau) \) and an output signal \( y \in L^2((0, \tau]; Y) \). These operators have to satisfy certain natural functional equations, for the formal definition we refer to Salamon (1987) and Weiss (1994). We recall some facts about well-posed linear systems from Weiss (1994).

Let \( \Sigma \) be a well-posed system with input space \( U \), state space \( X \) and output space \( Y \). Then \( \Sigma \) is completely determined by its ‘generating triple’ \( (A, B, C) \) and its ‘transfer function’ \( G \). Here, \( A \) is the ‘semigroup generator’ of \( \Sigma \), which generates a strongly continuous semigroup \( \mathbb{T} \) on \( X \), \( B \in \mathcal{L}(U, X_1) \) is the ‘control operator’ of \( \Sigma \) and \( C \in \mathcal{L}(X_1, Y) \) is its ‘observation operator’. The transfer function \( G \) satisfies
\[
G(s) - G(\beta) = C ((sI - A)^{-1} - (\beta I - A)^{-1}) B \quad \forall s, \beta \in \rho(A).
\]
The state trajectories of $\Sigma$ satisfy (2.2), hence also (2.3). If $u \in L^2_{\text{loc}}((0, \infty), U)$ is the input function of $\Sigma$, $z_0 \in X$ is its initial state and $y \in L^2_{\text{loc}}((0, \infty), Y)$ is the corresponding output function, then

$$y = \Psi z_0 + \mathcal{F} u. \quad (2.6)$$

Here, $\Psi$ is as in (2.4), while $\mathcal{F}: L^2_{\text{loc}}((0, \infty), U) \rightarrow L^2_{\text{loc}}((0, \infty), Y)$ is easiest to represent using Laplace transforms as follows: if $u \in L^2((0, \infty), U)$ and $y = \mathcal{F} u$, then $y$ has a Laplace transform $\hat{y}$ and

$$\hat{y}(s) = G(s)\hat{u}(s) \quad (2.7)$$

for all $s \in \mathbb{C}$ with $\Re s$ sufficiently large. $G$ is ‘proper’, which means that its domain contains a right half-plane $\mathbb{C}_\alpha$ such that $G$ is uniformly bounded on $\mathbb{C}_\alpha$.

**Definition 2.3** The well-posed linear system $\Sigma$ is called ‘regular’ if the limit

$$\lim_{s \rightarrow +\infty} G(s)v = Dv$$

exists for every $v \in U$, where $s$ is real. In this case, the operator $D \in \mathcal{L}(U, Y)$ is called the ‘feedthrough operator’ of $\Sigma$.

We mention a few facts about regular systems, following Weiss (1994). Regularity is equivalent to the fact that the product $C_A(sI - A)^{-1}B$ makes sense for some (hence for every) $s \in \rho(A)$. If $\Sigma$ is regular, then for every initial state $z_0 \in X$ and every $u \in L^2_{\text{loc}}((0, \infty), U)$, the solution of $\dot{z} = Az + Bu$ with $z(0) = z_0$ satisfies $z(t) \in \mathcal{D}(C_A)$ for almost every $t \geq 0$ and the corresponding output from (2.6) is given by

$$y(t) = C_A z(t) + D u(t) \quad \text{for almost every } t \geq 0. \quad (2.8)$$

The transfer function of the regular system $\Sigma$ is given by

$$G(s) = C_A(sI - A)^{-1}B + D \quad \forall s \in \rho(A). \quad (2.9)$$

The operators $A$, $B$, $C$, $D$ are called the ‘generating operators’ of $\Sigma$. This is because they determine $\Sigma$ via (2.2) and (2.8). If $C$ is bounded (i.e. $C \in \mathcal{L}(X, Y)$), then $C$ replaces $C_A$ in (2.8) and (2.9), and (2.8) holds for every $t \geq 0$.

Let $p$ be a function defined on some domain in $\mathbb{C}$ that contains a right half-plane, with values in a normed space. We say that $p$ is ‘strictly proper’ if

$$\lim_{\Re s \rightarrow \infty} \|p(s)\| = 0, \quad \text{uniformly with respect to } \Im s.$$  

A linear system is called strictly proper if its transfer function is strictly proper.

**Proposition 2.2** Let the infinite-dimensional system $\Sigma$ with input space $U$, state space $X$ and output space $Y$ be described by (2.2) and

$$y(t) = C z(t) + D u(t).$$

Here, $A$ generates a semigroup $\mathbb{T}$ on $X$, $B$ is an admissible control operator for $\mathbb{T}$, $C \in \mathcal{L}(X, Y)$ and $D \in \mathcal{L}(U, Y)$. Then $\Sigma$ is well-posed and regular, with feedthrough operator $D$. If $D = 0$, then $\Sigma$ is strictly proper.

This is an easy consequence of the main result in Curtain & Weiss (1989) together with Proposition 2.1.
3. Background on a class of coupled systems

In this section, we recall one of the main well-posedness and exact controllability results from Zhao & Weiss (2010b). First, we recall the generally accepted definitions of exact controllability and approximate controllability. Let $U$, $X$, $T$, $A$, $B$ and $\Phi_t$ be as at the beginning of Section 2. We assume that $B$ is admissible for $T$.

**Definition 3.1** The pair $(A,B)$ is said to be 'exactly controllable in time $\tau > 0$' if $\text{Ran} \Phi_\tau = X$; $(A,B)$ is said to be 'approximately controllable in time $\tau > 0$' if $\text{Ran} \Phi_\tau$ is dense in $X$.

Consider a coupled system $\Sigma_c$ in which a well-posed system $\Sigma_d$ is connected to a finite-dimensional system $\Sigma_f$ as shown in Fig. 2. The finite-dimensional part $\Sigma_f$ receives the inputs $u_e$ and $y$, where $u_e$ is the input of $\Sigma_c$ and the signal $y$ comes from $\Sigma_d$. The equations of $\Sigma_f$ are

$$\dot{q}(t) = aq(t) + bu_e(t) - bf y(t),$$

$$u(t) = cq(t) + du_e(t) - df y(t),$$

where $q(t) \in \mathbb{C}^n$ is the state of $\Sigma_f$ at the time $t$, $a \in \mathbb{C}^{n \times n}$, $b \in \mathbb{C}^{n \times m}$, $bf \in \mathbb{C}^{n \times p}$, $c \in \mathbb{C}^{m \times n}$, $d \in \mathbb{C}^{m \times m}$ and $df \in \mathbb{C}^{m \times p}$.

The well-posed linear system $\Sigma_d$, with input function $u$, input space $\mathbb{C}^m$, state trajectory $z$, state space $X$ (a Hilbert space), output function $y$ and output space $\mathbb{C}^p$ is assumed to be strictly proper (hence regular with feedthrough operator zero, see Section 2). $\Sigma_d$ is determined by its generating triple $(A,B,C)$. Here, $A$ is the semigroup generator of $\Sigma_d$, which generates a strongly continuous semigroup $\mathbb{T}$ on $X$, $B \in \mathcal{L}(\mathbb{C}^m,X)$ is the control operator of $\Sigma_d$ and $C \in \mathcal{L}(X_1,\mathbb{C}^p)$ is its observation operator. According to (2.9), the transfer function of $\Sigma_d$ is given by $G(s) = C_A(sI - A)^{-1}B$ for all $s \in \rho(A)$.

We can consider the coupled system $\Sigma_c$ as a cascaded system $\Sigma_{c\text{asc}}$ (the open loop system in Fig. 3) with a feedback. The inputs of $\Sigma_{c\text{asc}}$ are $u_e$ and $y$, (the latter corresponds to the input of $\Sigma_f$ coming from $y$ in Fig. 2), and its outputs are $u$ and $y$. The system $\Sigma_{c\text{asc}}$ is described by

$$\dot{q}(t) = aq(t) + bu_e(t) - bf y(t),$$

$$u(t) = cq(t) + du_e(t) - df y(t),$$

$$\dot{z}(t) = Az(t) + Bu(t),$$

$$y(t) = C_Az(t).$$

![Diagram](image.png)

**Fig. 2.** A coupled system $\Sigma_c$ consisting of a well-posed and strictly proper system $\Sigma_d$ and a finite-dimensional system $\Sigma_f = (a,b,bf,c,d,df)$, connected in feedback.
FIG. 3. The cascaded system $\Sigma_{\text{casc}}$ corresponding to $\Sigma_c$, consisting of a well-posed and strictly proper system $\Sigma_d$ and a finite-dimensional system $\Sigma_f = (a, b, b_f, c, d, d_f)$. To obtain from here $\Sigma_c$, we have to close the feedback $u_y = y$.

For the well-posedness and exact controllability of the coupled system $\Sigma_c$, we have the following theorem, which implies the exact controllability of $\Sigma_c$ for a very large (open and dense) set of pairs $(b_f \in \mathbb{C}^{n \times p}, d_f \in \mathbb{C}^{m \times p})$ but not for all. Thus, exact controllability is a generic property with respect to $b_f$ and $d_f$.

**Theorem 3.1** Let $\Sigma_d$ be a well-posed and strictly proper (hence regular) system with input space $\mathbb{C}^m$, state space $X$ (a Hilbert space), output space $\mathbb{C}^p$, semigroup $\mathbb{T}$, generating triple $(A, B, C)$ and transfer function $G$. Let $a, b, b_f, c, d, d_f$ be matrices as in (3.1)–(3.2). Then the coupled system $\Sigma_c$ from Fig. 2 described by (3.1), (3.2), (3.5) and (3.6), with input $u_e$, state $[z_q]$, and output $[u_y]$, is well-posed and regular with the state space $X \times \mathbb{C}^n$. The generating operators $(A^c, B^c, C^c, D^c)$ of $\Sigma_c$ and its transfer function $G^c$ are

$$A^c = \begin{bmatrix} A - B d_f C_A & B c \\ -b_f C_A & a \end{bmatrix}, \quad \mathcal{D}(A^c) = \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in X \times \mathbb{C}^n \mid A^c \begin{bmatrix} z \\ q \end{bmatrix} \in X \times \mathbb{C}^n \right\},$$

$$B^c = \begin{bmatrix} B d \\ b \end{bmatrix}, \quad C^c = \begin{bmatrix} -d_f C_A \\ C_A \end{bmatrix}, \quad D^c = \begin{bmatrix} d \\ 0 \end{bmatrix},$$

$$G^c = \begin{bmatrix} I \\ G \end{bmatrix} (I + g_f G)^{-1} g,$$

where

$$g(s) = c(sI - a)^{-1} b + d, \quad g_f(s) = c(sI - a)^{-1} b_f + d_f.$$

If $B$ is bounded (i.e. $B \in \mathcal{L}(\mathbb{C}^m, X)$), then $\mathcal{D}(A^c) = \mathcal{D}(A) \times \mathbb{C}^n$.

Now assume additionally the following:

(i) $(A, B)$ is exactly controllable in time $T_0$.

(ii) $(a, b)$ is controllable.

(iii) $d \in \mathbb{C}^{m \times m}$ is invertible.

(iv) Denote $a^* = a - b d^{-1} c$. Then $A^*$ and $a^*$ have no common eigenvalue.

Then for every $T > T_0$, $b_1 \in \mathbb{C}^{n \times p}$ and $d_1 \in \mathbb{C}^{m \times p}$ let $F_T$ be the set of those $\lambda \in \mathbb{C}$ such that the coupled system $\Sigma_c$ with $b_f = \lambda b_1$ and $d_f = \lambda d_1$ is not exactly controllable in time $T$. Then $F_T$ has at most $n$ elements.

We mention the obvious fact that $F_T$ is non-increasing (because for any linear system, the reachable space in time $T$ is non-decreasing with respect to $T$).

We also mention the well-posedness and controllability results for another related class of coupled systems were derived in Weiss & Zhao (2009). In Weiss & Zhao (2009), we consider a finite-dimensional
system coupled with an infinite-dimensional system which becomes well-posed and strictly proper when connected in cascade with an integrator. There is a certain analogy but the result in this section cannot be derived from those in Weiss & Zhao (2009) (or the other way round) because of different assumptions on the subsystems.

4. Exact controllability of the wind turbine tower model with only torque control

In this section, we show the well-posedness, exact controllability and approximate controllability results for the wind turbine tower model $\Sigma_e$ described by (1.1)–(1.8) using only electric torque control $u_e = T_e$, based on Theorem 3.1. We decompose $\Sigma_e$ into a non-uniform SCOLE model $\Sigma_d$ with torque input $u$ and angular velocity output $y$, and a two-mass drive-train model $\Sigma_f$, as in Fig. 2.

As mentioned in Section 1, the state of $\Sigma_d$ at the time $t$ is

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{bmatrix} = \begin{bmatrix} w(\cdot, t) \\ w_t(\cdot, t) \\ w_t(l, t) \\ w_{xt}(l, t) \end{bmatrix}.$$  \hspace{1cm} (4.1)

The natural energy state space of $\Sigma_d$ (described by (1.1)–(1.4) with $f = 0$) is

$$X = \mathcal{H}^2_4(0, l) \times L^2[0, l] \times \mathbb{C}^2,$$ \hspace{1cm} (4.2)

where $\mathcal{H}^2_4(0, l)$ is defined as in (1.11). The natural norm on $X$ is as in (1.12), but without the last three terms, which represents twice the physical energy of $\Sigma_d$. Of course, the formulas $z_3(t) = w_t(l, t)$ and $z_4(t) = w_{xt}(l, t)$ do not make sense for $z(t) \in X$, only for smoother $z(t)$ (e.g. for $z(t) \in \mathcal{D}(A)$, defined below). We define the generator $A$ as follows:

$$A \xi = \begin{bmatrix} \xi_2 \\ -\rho^{-1}(x)(E\xi_2(x))_{xx} \\ m^{-1}(E\xi_1(x))_x(l) \\ -J^{-1}E(l)\xi_1_x(l) \end{bmatrix} \hspace{1cm} \forall \xi \in \mathcal{D}(A),$$ \hspace{1cm} (4.3)

$$\mathcal{D}(A) = \left\{ \xi \in [\mathcal{H}^4(0, l) \cap \mathcal{H}^2_4(0, l)] \times \mathcal{H}^2_4(0, l) \times \mathbb{C}^2 \mid \xi_3 = \xi_2(l), \xi_4 = \xi_{2x}(l) \right\}.$$

As in Section 2, we denote $X_1 = \mathcal{D}(A)$, with a suitable norm. Let

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T, \hspace{0.5cm} C = [0 \hspace{0.2cm} 0 \hspace{0.2cm} 0 \hspace{0.2cm} 1].$$

Then (1.1)–(1.4) can be rewritten as

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), \\ y(t) = Cz(t), \end{cases}$$ \hspace{1cm} (4.4)

where $u(t) = T_{hs}(t) + T_{hss}(t) - b_m(\theta_m)\dot{\gamma}(t) - T_e(t)$ is the input of $\Sigma_d$ (and the output of $\Sigma_f$). $u(t)$ is the total torque acting on the nacelle from the gearbox (the first two terms) and from the electrical
generator (the remaining two terms). \( y(t) = w_{xt}(l, t) \) is the output of \( \Sigma_d \) (the angular velocity of the nacelle).

The non-uniform SCOLE model described by (4.4) has been studied in Guo (2002) and Guo & Ivanov (2005). We recall here several results from these papers. The following proposition is taken from Guo & Ivanov (2005, Propositions 1.1 and 1.2).

**Proposition 4.1**  \( A \) is skew-adjoint on \( X \). Its spectrum \( \sigma(A) \) consists of simple eigenvalues that are isolated, purely imaginary and non-zero.

The following result is proven in Guo (2002, Proposition 4.2 and Theorem 4.3).

**Proposition 4.2**  \( B \) is admissible on \( X_1 \) and \((A, B)\) is exactly controllable in any time \( T_0 > 0 \) on \( X_1 \).

**Proposition 4.3**  \( \Sigma_d \) described by (4.4) is well-posed, regular and strictly proper with the state space either \( X \) or \( X_1 \).

**Proof.** From Proposition 4.1, we know that \( A \) is skew-adjoint, so that it is the generator of a unitary group \( \mathbb{T} \) on \( X \). Clearly, \( B \) and \( C \) are bounded (i.e. \( B \in \mathcal{L}(\mathbb{C}, X) \), \( C \in \mathcal{L}(X, \mathbb{C}) \)). By Proposition 2.2, \( \Sigma_d \) is well-posed, regular and strictly proper on \( X \). The restriction of \( A \) to \( \mathcal{D}(A^2) \) generates the restriction of \( \mathbb{T} \) to \( X_1 \). From Proposition 4.2, we know that \( B \) is admissible for \( \mathbb{T} \) restricted \( X_1 \). Clearly, \( C \in \mathcal{L}(X_1, \mathbb{C}) \), so that \( \Sigma_d \) is well-posed, regular and strictly proper on \( X_1 \) as well according to Proposition 2.2. \( \square \)

Doing some computations, we can extract the following two-mass drive-train subsystem \( \Sigma_f \) from the wind turbine tower model (1.1)–(1.8): the state vector is

\[
q(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} = \begin{bmatrix} (\theta_T)_r(t) + w_{xt}(l, t) \\ (\theta_m)_r(t) - w_{xt}(l, t) \\ \theta_k(t) \end{bmatrix}, \tag{4.5}
\]

and the system equations are

\[
\begin{align*}
\dot{q}_1 &= -\frac{C_s}{J_T} q_1 + \frac{C_s}{J_T n_g} q_2 - \frac{K_s}{J_T} q_3 + \frac{1}{J_T} T_a + \frac{C_s(1 + n_g)}{J_T n_g} y, \\
\dot{q}_2 &= \frac{C_s}{J_G n_g} q_1 - \left( \frac{C_s}{J_G n_g^2} + \frac{b_m}{J_G} \right) q_2 + \frac{K_s}{J_G n_g} q_3 - \frac{1}{J_G} T_e \\
&- \frac{C_s(1 + n_g)}{J_G n_g^2} y - \frac{b_m}{J_G} y, \\
\dot{q}_3 &= q_1 - \frac{1}{n_g} q_2 - \frac{1 + n_g}{n_g} y, \\
u &= \frac{C_s(1 + n_g)}{n_g} q_1 - \left( \frac{C_s(1 + n_g)}{n_g^2} + b_m \right) q_2 \\
&+ \frac{K_s(1 + n_g)}{n_g} q_3 - T_e - \frac{C_s(1 + n_g)^2}{n_g^2} y - b_m y. \tag{4.9}
\end{align*}
\]

Here \( u_e = T_e \) is the control input and \( T_a \) is a disturbance input. In the sequel we put \( T_a = 0 \), since this input is irrelevant for controllability. \( y \) is the feedback input of \( \Sigma_f \) (the output of \( \Sigma_d \)) while \( u \) is the
output of $\Sigma_f$ (the input of $\Sigma_d$). We can write $\Sigma_f$ as in (3.1)–(3.2), with the matrices

$$a = \begin{bmatrix}
-\frac{C_s}{J_T} & -\frac{C_s}{J_G n_g} & -\frac{K_s}{J_T} \\
\frac{C_s}{J_G n_g^2} & -\frac{K_s}{J_G n_g} & 0 \\
1 & -\frac{1}{n_g} & 0
\end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ -\frac{1}{J_G} \\ 0 \end{bmatrix}, \quad b_f = \begin{bmatrix}
-\frac{C_s(1+n_g)}{J_T n_g} \\
\frac{C_s(1+n_g)}{J_G n_g} + \frac{b_m}{J_G} \\
(1+n_g) n_g
\end{bmatrix}.$$  

THEOREM 4.1 The wind turbine tower model $\Sigma_c$ described by (1.1)–(1.8), with input $u_c = T_c$, state $\begin{bmatrix} \dot{z} \\ \dot{y} \end{bmatrix}$ (as defined in (4.1) and (4.5)) and output $\begin{bmatrix} u \end{bmatrix}$ is well-posed and regular with the state space either

$$X^c = \mathcal{H}_l^2(0,l) \times L^2[0,l] \times \mathbb{C}^5$$

or

$$X^c_1 = \mathcal{D}(A^c) = \left\{ \begin{bmatrix} z \\ y \end{bmatrix} \in \left[ \mathcal{H}^4(0, l) \cap \mathcal{H}_l^2(0, l) \right] \times \mathcal{H}_l^2(0, l) \times \mathbb{C}^5 \middle| \begin{array}{c} z_3 = z_2(l) \\ z_4 = z_2(l) \end{array} \right\}.$$  

Here $u = T_{\text{iss}} + T_{\text{hss}} - b_m (\theta_m)_{\text{f}} - T_c$, $y = \omega_x (l, \cdot)$ and $A^c$ is the semigroup generator of $\Sigma_c$ with the state space $X^c$.

Moreover, we have the following generic exact controllability result: for every time $T > 0$ and for every choice of the strictly positive functions $\rho_n, EI_n \in C^4[0,l]$ and of the parameters $l > 0, m_n > 0, J_n > 0, K_s > 0, J_T > 0, J_G > 0, n_g > 0, C_s \geq 0$ and $b_m \geq 0$, there are at most three values $\mu > 0$ such that the system $\Sigma_c$ with

$$\rho = \mu \rho_n, \quad EI = \mu EI_n, \quad m = \mu m_n, \quad J = \mu J_n$$

is not exactly controllable on $X^c_1$ in time $T$.

Proof. First we prove the well-posedness part. We have seen earlier that $\Sigma_c$ can be decomposed into $\Sigma_d$ described by (4.4) and $\Sigma_f$ described by (4.6)–(4.9), interconnected as in Fig 2. These subsystems fit into the framework of Theorem 3.1: according to Proposition 4.3, $\Sigma_d$ is well-posed and strictly proper with the state space either $X$ from (4.2) or $X_1$. Therefore, by Theorem 3.1, $\Sigma_c$ (with input $u_c = T_c$, state $\begin{bmatrix} \dot{z} \\ \dot{y} \end{bmatrix}$ and output $\begin{bmatrix} u \end{bmatrix}$) is well-posed and regular with the state space either

$$X^c = X \times \mathbb{C}^3 = \mathcal{H}_l^2(0,l) \times L^2[0,l] \times \mathbb{C}^5 \quad (4.10)$$

or $X_1 \times \mathbb{C}^3$, which is exactly $X^c_1$ defined in the theorem. To see that $\mathcal{D}(A^c) = X_1 \times \mathbb{C}^3$, we use Theorem 3.1 and the fact that the control operator $B$ is bounded when we use the state space $X$ for $\Sigma_d$.

Now, we show the generic exact controllability of $\Sigma_c$ on $X^c_1$. We consider the time $T$ and the functions $\rho_n$ and $EI_n$ fixed. The parameters $l, m_n, J_n, K_s, J_T, J_G, n_g, C_s$ and $b_m$ are also fixed ($C_s$ and $b_m$ are $\geq 0$, the others are strictly positive). Define $\rho = \mu \rho_n, EI = \mu EI_n, m = \mu m_n$ and $J = \mu J_n$, where $\mu > 0$. We want to show that the resulting system $\Sigma_c$ is exactly controllable in time $T$ for almost every $\mu > 0$, with the exception of at most 3 ‘bad’ values for $\mu$.

From Proposition 4.2, we know that $(A, B)$ is exactly controllable in any time $T_0 > 0$ with the state space $X_1$. Thus, assumption (i) of Theorem 3.1 is satisfied.
It is easy to check that

\[
[ b \ ab \ a^2 b] = \begin{bmatrix}
0 & -\frac{C_s}{J_G J_T n_g} & \left( \frac{C_s^2}{J_G^2 J_T n_g^2} + \frac{C_s}{J_G} \frac{b_m}{J_G} - \frac{K_s}{J_G^2 J_T n_g^2} \right) \\
-\frac{1}{J_G} & \left( \frac{C_s}{J_G^2 n_g^2} + \frac{b_m}{J_G^2} \right) & -\left( \frac{C_s^2}{J_G^2 J_T n_g^2} + \frac{C_s}{J_G} \frac{b_m}{J_G^2} + \frac{1}{J_G^2} \right) \\
0 & \frac{1}{J_G^2 n_g^2} & -\left( \frac{C_s}{J_G J_T n_g^2} + \frac{1}{J_G^2} \left( \frac{C_s}{J_G^2 n_g^2} + \frac{b_m}{J_G^2} \right) \right)
\end{bmatrix}.
\]

The determinant of the above matrix is

\[
\det [ b \ ab \ a^2 b] = \frac{K_s}{J_T J_G^2 n_g^2} \neq 0.
\] (4.11)

Thus, \((a, b)\) is controllable, which is assumption (ii) of Theorem 3.1. Clearly, \(d = -1\) is invertible, so that assumption (iii) is also satisfied. For assumption (iv), by computation, we have

\[
a^\times = \begin{bmatrix}
-\frac{c_1}{J_T} & \frac{c_s}{J_T n_g} & -\frac{k_s}{J_T} \\
-\frac{c_1}{J_G} & \frac{c_s}{J_G n_g} & -\frac{k_s}{J_G} \\
1 & -\frac{1}{n_g} & 0
\end{bmatrix}.
\]

It is easy to verify that \(a^\times\) has three eigenvalues:

\[
0, \quad \frac{1}{2} \left( C_s J_d + \sqrt{C_s^2 J_d^2 + 4 K_s J_d} \right), \quad \frac{1}{2} \left( C_s J_d - \sqrt{C_s^2 J_d^2 + 4 K_s J_d} \right),
\]

where \(J_d = \frac{1}{J_G n_g} - \frac{1}{J_T} > 0\) (recall from Section 1 that we assume that \(J_T > J_G n_g\)). From Proposition 4.1, we know that \(\sigma(A)\) consists of simple eigenvalues that are isolated, purely imaginary and non-zero. Therefore, \(A^*\) and \(a^\times\) have no common eigenvalues and so assumption (iv) is satisfied.

So far all the assumptions of Theorem 3.1 are satisfied. Note that \(A\) from (4.3) as well as \(C\) and the system \(\Sigma_f\) are independent of \(\mu\), while \(B\) is proportional to \(1/\mu\). We obtain an equivalent system (from the point of view of exact controllability) if we move the factor \(1/\mu\) to the output of \(\Sigma_d\), i.e. we consider \(B = \begin{bmatrix} 0 & 0 & 0 & 1/n \end{bmatrix}\) and replace \(b_f\) and \(d_f\) with \(\frac{1}{\mu} b_f\) and \(\frac{1}{\mu} d_f\), respectively. According to Theorem 3.1 (with \(\lambda = 1/\mu\)), \(\Sigma_c\) is exactly controllable in time \(T\) (with the state space \(X_1^c\)) for all except at most three bad values for \(\mu\). \(\square\)

**Remark 4.1** Suppose that we can somehow prove that the set \(Q\) of quadruples \((\mu_1, \mu_2, \mu_3, \mu_4) \in (0, \infty)^4\), for which \(\Sigma_c\) determined by

\[
\rho = \mu_1 \rho_n, \quad EI = \mu_2 EI_n, \quad m = \mu_3 m_n, \quad J = \mu_4 J_n
\]

is not exactly controllable on \(X_1^c\) in time \(T\), is Lebesgue measurable. In this case, the generic controllability part of the last theorem implies that \(Q\) has Lebesgue measure zero since on every ray starting from zero, there are at most three points of \(Q\).
5. Exact controllability of the wind turbine tower model with both force and torque control

In this section, we consider the situation when the wind turbine tower model $\Sigma_c$ described by (1.1)–(1.8) has two control inputs: the force $f$ created by an electrically driven mass located in the nacelle, and the electric torque $T_e$ created by the generator, so that $u_c = \begin{bmatrix} f \\ T_e \end{bmatrix}$. We show the well-posedness and generic exact controllability of this model, based again on Theorem 3.1. As we did in Section 4, we decompose $\Sigma_c$ into a SCOLE model $\Sigma_d$ and a two-mass drive-train model $\Sigma_f$ as in Fig. 2. In this case, the input $u$ of $\Sigma_d$ has two components: the force $f$ and the total torque acting on the nacelle, and its output $y$ also has two components: the velocity and the angular velocity of the nacelle. We assume that $\Sigma_c$ is uniform, i.e. EI and $\rho$ in (1.1), (1.3) and (1.4) are strictly positive constants. If we use the state space $X$ from (4.2), we get the following state space formulation for $\Sigma_d$:

$$
\begin{align*}
\dot{z}(t) &= Az(t) + \hat{B}u(t), \\
y(t) &= \hat{C}z(t),
\end{align*}
$$

(5.1)

where $A$ is a special case of the one in (4.4), with EI and $\rho$ being constant. The control operator and observation operator become

$$
\hat{B} = \begin{bmatrix} 0 & 0 & 1/m & 0 \\
0 & 0 & 0 & 1/T \end{bmatrix}^\top, \quad \hat{C} = \begin{bmatrix} 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{bmatrix},
$$

while

$$
u(t) = \begin{bmatrix} f(t) \\
T_{lss}(t) + T_{hss}(t) - b_m(\theta_m)(t) - T_e(t) \end{bmatrix}, \quad y(t) = \begin{bmatrix} w_l(l, t) \\
w_{xt}(l, t) \end{bmatrix}.
$$

(5.2)

Here, $u$ is the input of $\Sigma_d$ (and the output of $\Sigma_f$). Its first component is the external force control acting on the nacelle, while its second component is the total torque acting on the nacelle from the gearbox and from the electrical generator. $y$ is the output of $\Sigma_d$ (the velocity and angular velocity of the nacelle). Note that, to fit the framework of Theorem 3.1, it is enough to take only $w_{xt}(l, t)$ as the output. The reason why we also include $w_l(l, t)$ into $y$ is to show that $\Sigma_c$ is well-posed and regular even considering this additional output. This is important because, when using colocated control for stabilization, we would use this variable to drive the force input of $\Sigma_c$. (We shall not discuss colocated control for this system here.)

The model (5.1) has been analysed in detail in our paper (Zhao & Weiss, 2010a). We take the following result from Zhao & Weiss (2010a).

**Theorem 5.1** The SCOLE model $\Sigma_d$ described by (5.1) is well-posed, regular and exactly controllable in any time $T_0 > 0$ with the state space

$$
\mathcal{X} = \left\{ \begin{bmatrix} \tilde{z} \\ \tilde{q} \end{bmatrix} \in [\mathcal{H}^2(0, l) \cap \mathcal{H}^2_i(0, l)] \times \mathcal{H}^1_i(0, l) \times \mathbb{C}^2 | \tilde{z}_2(l) = q_1 \right\}
$$

(using both force and torque control). Its feedthrough operator is 0.

In the case that the wind turbine tower model $\Sigma_c$ has two control inputs, the two-mass drive-train subsystem $\Sigma_f$ (after setting the disturbance $T_a = 0$) is

$$
\begin{align*}
\dot{q}(t) &= aq(t) + \hat{b}u_c(t) - \hat{b}_f y(t), \\
u(t) &= \hat{c}q(t) + \hat{d}u_c(t) - \hat{d}_f y(t),
\end{align*}
$$

(5.3)

(5.4)
where \( a \) is the same as before Theorem 4.1 and

\[
\hat{b} = \begin{bmatrix} 0 & 0 \\ 0 & -1/J_G \\ 0 & 0 \end{bmatrix}, \quad \hat{b}_f = \begin{bmatrix} 0 & -\frac{C_s(1+n_g)}{J_G n_g} \\ 0 & \frac{C_s(1+n_g)}{J_G n_g} + \frac{b_m}{J_G} \\ 0 & (1+n_g) \frac{1}{n_g} \end{bmatrix},
\]

\[
\hat{c} = \begin{bmatrix} 0 & 0 \\ \frac{C_s(1+n_g)}{n_g} - \left( \frac{C_s(1+n_g)}{n_g^2} + b_m \right) & 0 \\ 0 & \frac{K_s(1+n_g)}{n_g} \end{bmatrix},
\]

\[
\hat{d} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \hat{d}_f = \begin{bmatrix} 0 & 0 \\ 0 & \frac{C_s(1+n_g)^2}{n_g^2} + b_m \end{bmatrix}.
\]

**Theorem 5.2** Assume that the wind turbine tower model \( \Sigma_c \) described by (1.1)–(1.8) is uniform, i.e. \( \text{EI} \) and \( \rho \) are strictly positive constants. Then \( \Sigma_c \) with input \( u_c = \begin{bmatrix} f_c^T \end{bmatrix} \), state \( \begin{bmatrix} z \end{bmatrix} \) (as defined in (4.1) and (4.5)) and output \( \begin{bmatrix} u \end{bmatrix} \) (as defined in (5.2)) is well-posed and regular with the state space

\[
\hat{X}^c = \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in [\mathcal{H}^3(0,1) \cap \mathcal{H}^2(0,1)] \times \mathcal{H}^1(0,1) \times \mathbb{C}^5 \mid z_2(l) = q_1 \right\}.
\]

Moreover, we have the following generic exact controllability result: for every time \( T > 0 \) and for every choice of the strictly positive constants \( l, \rho_n, \text{EI}_n, m_n, J_n, \text{K}_s, J_T, J_G, n_g \) and of the constants \( C_s \geq 0 \) and \( b_m \geq 0 \), there are at most three values \( \mu > 0 \) such that the system \( \Sigma_c \) with

\[
\rho = \mu \rho_n, \quad \text{EI} = \mu \text{EI}_n, \quad m = \mu m_n, \quad J = \mu J_n
\]

is not exactly controllable on \( \hat{X}^c \) in time \( T \).

**Proof.** First we prove the well-posedness part. Recall that we decompose \( \Sigma_c \) into \( \Sigma_d \) described by (5.1) and \( \Sigma_f \) described by (5.3)–(5.4). From Proposition 5.1, we know that \( \Sigma_d \) is well-posed with the state space

\[
\mathcal{X} = \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in [\mathcal{H}^3(0,1) \cap \mathcal{H}^2(0,1)] \times \mathcal{H}^1(0,1) \times \mathbb{C}^2 \mid z_2(l) = q_1 \right\}.
\]

We also know that its feedthrough operator is 0. Thus, by Proposition 2.2, \( \Sigma_d \) is strictly proper on \( \mathcal{X} \). From the descriptions of \( \Sigma_c \), \( \Sigma_d \) and \( \Sigma_f \), clearly they fit into the framework of Theorem 3.1. Therefore, \( \Sigma_c \left( \text{with input } u_c = \begin{bmatrix} f_c^T \end{bmatrix} \right) \), output \( \begin{bmatrix} u \end{bmatrix} \) and state \( \begin{bmatrix} z \end{bmatrix} \) is well-posed and regular with the state space

\[
\hat{X}^c = \mathcal{X} \times \mathbb{C}^3 = \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in [\mathcal{H}^3(0,1) \cap \mathcal{H}^2(0,1)] \times \mathcal{H}^1(0,1) \times \mathbb{C}^5 \mid z_2(l) = q_1 \right\}.
\]

Now, we show the generic exact controllability of \( \Sigma_c \) on \( \hat{X}^c \). As in the proof of Theorem 4.1, we consider the time \( T \) and the parameters \( l, \rho_n, \text{EI}_n, m_n, J_n, \text{K}_s, J_T, J_G, n_g, C_s \) and \( b_m \) to be fixed.
Define $\rho$, $EI$, $m$ and $J$ as in the theorem, with $\mu > 0$. We want to show that the resulting system $\Sigma_c$ is exactly controllable in time $T$ for almost every $\mu > 0$, with the exception of at most three bad values for $\mu$.

From Proposition 5.1, we know that $(A, \hat{B})$ is exactly controllable in any time $T_0 > 0$ with the state space $X$. To see that $(a, \hat{b})$ is controllable, we use only the second column of $\hat{b}$ and then we have the same computation as at (4.11). Thus, assumptions (i) and (ii) of Theorem 3.1 are satisfied. Clearly, $\hat{d}$ is invertible, so that assumption (iii) is satisfied. For assumption (iv), it is easy to verify that we get the same $a^\times$ as in the proof of Theorem 4.1, while $A$ is a special case of the one in Section 4 (with $EI$ and $\rho$ constant). By the same argument as in the proof of Theorem 4.1, assumption (iv) is also satisfied.

Note that $A$ as well as $\hat{C}$ and the system $\Sigma_f$ are independent of $\mu$, while $\hat{B}$ is proportional to $1/\mu$.

We obtain an equivalent system (from the point of view of exact controllability) if we move the factor $1/\mu$ to the output of $\Sigma_d$, i.e. we consider $\hat{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\mu} \hat{b}_f & \frac{1}{\mu} \hat{d}_f \end{bmatrix}^\top$ and replace $\hat{b}_f$ and $\hat{d}_f$ with $\frac{1}{\mu} \hat{b}_f$ and $\frac{1}{\mu} \hat{d}_f$, respectively. According to Theorem 3.1, $\Sigma_c$ is exactly controllable in time $T$ (with the state space $\hat{X}^c$) for all except at most three values of $\mu > 0$. □

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