Serial relaying communications over generalized-gamma fading channels

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ABSTRACT

In this paper, a study on the end-to-end performance of multi-hop non-regenerative relaying networks over independent generalized-gamma (GG) fading channels is presented. Using an upper bound for the end-to-end signal-to-noise ratio (SNR), novel closed-form expressions for the probability density function, the moments, and the moments-generating function of the end-to-end SNR are presented. Based on these derived formulas, lower bounds for the outage and the average bit error probability (ABEP) are derived in closed form. Special attention is given to the low- and high-SNR regions having practical interest as well as to the Nakagami fading scenario. Moreover, the performance of the considered system when employing adaptive square-quadrature amplitude modulation is further analyzed in terms of the average spectral efficiency, the bit error outage, and the ABEP. Computer simulation results verify the tightness and the accuracy of the proposed bounds. Copyright © 2010 John Wiley & Sons, Ltd.

KEYWORDS

Bit error outage (BEO); multi-hop relaying; generalized-gamma fading; outage probability; average bit error probability (ABEP); Nakagami; Weibull; amplify-and-forward; adaptive square-QAM

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1. INTRODUCTION

Recently, multi-hop networks technology has attracted great interest as it is a promising solution to achieve high data rate coverage required in future cellular wireless local area and hybrid networks, as well as to mitigate wireless channels impairment. In a multi-hop system, intermediate nodes are used to relay signals between the source and the destination terminal. Relaying techniques can achieve network connectivity when the direct transmission is difficult for practical reasons, such as large path-loss or power constraints. As a result, signals from the source to the destination propagate through different hops/links. A special class of multi-hop networks is the serial relaying networks for which various works have shown that they are able to achieve high performance gains [1, 2].

In the open technical literature, there are several works dealing with the performance analysis of multi-hop systems. In [3], the end-to-end outage probability and the average error rate for multi-hop wireless systems with non-regenerative relaying operating over Weibull fading channels were evaluated. In [4, 5, 6, 7], the end-to-end outage probability as well as the average error rate for dual-hop wireless systems with non-regenerative relaying operating over Rayleigh and Nakagami-$m$ fading channels were presented. In [8, 9], performance bounds for multi-hop relaying transmissions with fixed-gain relays over Rice, Hoyt, and Nakagami-$m$ fading channels were given using the moments-based approach. Moreover, in [10], an extensive performance analysis for dual-hop non-regenerative relaying communication systems over generalized-gamma (GG) fading was presented. It is noted that the GG distribution is quite general as it includes the Rayleigh, the Nakagami-$m$, and the Weibull distribution as special cases, as well as the lognormal one as a limiting case. Furthermore, it is considered to be mathematically tractable, as compared to lognormal-based models, and recently has gained increased interest in the field of digital communications over fading channels [11].

Adaptive modulation techniques [12, 13, 14], where the modulation index is chosen according to the instantaneous channel conditions, are considered as efficient means to cope with channel variations, while keeping an acceptable quality of service (QoS). When channel conditions are favorable, the transmitter can use higher power, larger symbol constellations, and reduced coding and error correction schemes, otherwise the transmitter switches to low power, smaller symbol constellations, and includes improved coding and error correction schemes. Although adaptive modulation techniques have been extensively applied for direct-link systems, corresponding techniques for multi-hop systems have only recently received some special attention. In [15], the performance of adaptive modulation in a single relay network was studied, while [16], generalizes [15] for multi-hop systems.

In this paper, we analyze the statistics and the end-to-end performance of non-regenerative (amplify-and-forward) multi-hop systems, with relay nodes in series,
operating over independent GG fading channels. By considering a union bound for the end-to-end SNR, closed-form expressions for its cumulative distribution function (CDF), probability density function (PDF), moments, as well as its moments-generating function (MGF) are derived. Using the CDF expression, lower bounds of the end-to-end outage probability (OP) are derived. Moreover, using the well-known MGF-based approach, lower bounds for the average bit error probability (ABEP) of binary differential phase shift keying (BDPSK), binary phase shift keying (BPSK) and binary frequency shift keying (BFSK) are also presented. For identically distributed hops, the ABEP expressions are in terms of a finite sum of Meijer G-functions. An accurate analytical method for the computation of the ABEP is given for arbitrarily distributed hops. For the high- and the low-SNR region, having special practical interest for multi-hop networks, our derived ABEP expressions are significantly simplified. The same also occurs when Nakagami fading is being considered as a special case of the GG distribution. Finally, the performance of the considered system with fast adaptive square-QAM and fixed switching levels is analyzed in terms of the average spectral efficiency (ASE), the bit error outage (BEO), and the corresponding ABEP. Theoretical expressions for the outage probability the achievable spectral efficiency and ABEP are derived, while Monte Carlo simulation is used in order to verify the proposed analysis.

The paper is organized as follows: In Section 2 the system and channel models are described in details. In Section 3 closed-form expressions for the CDF, PDF, moments, and MGF of a bounded end-to-end SNR are derived. Based on these formulas, in Section 4, an end-to-end performance analysis is presented for both fixed modulation and adaptive QAM schemes. In Section 5 numerical and computer simulation results are presented, demonstrating the tightness of the proposed bound, while the paper concludes with a summary given in Section 6.

2. SYSTEM AND CHANNEL MODEL

We consider a multi-hop system, as shown in Fig. 1, with a source node communicating with a destination node via \(N - 1\) relay nodes in series. The fading channel coefficients \(h_i\) between source-to-relay (\(S \rightarrow R\)), relay-to-relay (\(R \rightarrow R\)) and relay-to-destination (\(R \rightarrow D\)) are considered as independent GG random variables. By defining as \(\gamma_i = h_i^2 E_s / N_0\) the instantaneous receive SNR of the \(i\)-th hop, with \(i = 1, 2, \ldots, N\), \(E_s\) being the symbols energy, and \(N_0\) being the single sided power spectral density, the PDF of \(\gamma_i\) can be written as

\[
f_{\gamma_i}(\gamma) = \frac{\beta_i \gamma_i^{m_i - 1} \exp\left(-\frac{\gamma_i}{\tau_i}\right)}{2 \Gamma(m_i)} \left(\frac{\gamma_i}{\tau_i}\right)^{m_i - 1/2} \times \exp\left[-\left(\frac{\gamma_i}{\tau_i}\right)^{\beta_i/2}\right]
\]

(1)

where \(\beta_i > 0\) and \(m_i > 1/2\) are parameters related to fading severity, \(\tau_i = E(\gamma_i)\) with \(E(\cdot)\) denoting
expectation, and $\tau_i = \Gamma(m_i)/\Gamma(m_i + 2/\beta_i)$ where $\Gamma(\cdot)$ is the gamma function. For $\beta_i = 2$, (1) reduces to the square Nakagami-$m$ fading distribution, whereas for $m_i = 1$, the Weibull distribution is obtained. Also the CDF of $\gamma_i$ is given by

$$F_{\gamma_i}(\gamma) = 1 - \frac{1}{\Gamma(m_i)} \Gamma\left[ m_i, \left( \frac{\gamma}{\tau_i \gamma_i} \right)^{\beta_i/2} \right]. \quad (2)$$

Note that for integer $m_i$ and using [17, eq. (8.352.2)], the CDF of $\gamma_i$ can be rewritten as

$$F_{\gamma_i}(\gamma) = 1 - \exp \left[ - \left( \frac{\gamma}{\tau_i \gamma_i} \right)^{\beta_i/2} \right] \times \sum_{i=0}^{m_i-1} \frac{1}{i!} \left( \frac{\gamma}{\tau_i \gamma_i} \right)^{i \beta_i/2}. \quad (3)$$

Mathematically, this bound is obtained by observing that the following inequalities are valid [18]

$$\gamma_{\text{equ}} \leq \left( \sum_{i=1}^{N} \frac{1}{\gamma_i} \right)^{-1} \leq \min \{ \gamma_1, \gamma_2, \ldots, \gamma_N \}. \quad (6)$$

A physical interpretation of this bound is that at high SNR region, the hop with the weakest SNR determines the end-to-end system performance. This approximation, is adopted in many recent papers e.g. [18, 19, 20] and is shown to be accurate enough, especially at medium and high SNR values.

3. STATISTICS OF THE END-TO-END SNR

In this section bounds for the end-to-end SNR are proposed, while associated closed-form lower bounds for the CDF, the PDF, the moments, and the MGF are derived.

For the system in Fig. 1 employing the amplify-and-forward relaying protocol, the end-to-end SNR at the destination node can be written as [5]

$$\gamma_{\text{equ}} = \left[ \prod_{i=1}^{N} \left( 1 + \frac{1}{\gamma_i} \right) - 1 \right]^{-1}. \quad (4)$$

The above SNR expression is not mathematically tractable in its current form due to the difficulty in finding the statistics associated with it. Similarly to [10, 18], this form can be upper bounded by

$$\gamma_{\text{equ}} \leq \gamma_b = \min \{ \gamma_1, \gamma_2, \ldots, \gamma_N \}. \quad (5)$$

Figure 1. The serial relaying communication system under consideration.
3.1. Cumulative Distribution Function

Using (2) and (5), the CDF of $\gamma_b$ can be expressed as

$$F_{\gamma_b}(\gamma) = 1 - \prod_{i=1}^{N} [1 - F_{\gamma_i}(\gamma)]$$

$$= 1 - \prod_{i=1}^{N} \frac{1}{\Gamma(m_i)} \Gamma \left( m_i \left( \frac{\gamma - \tau_i}{\gamma} \right)^{\beta_i/2} \right). \quad (7)$$

For independent and identically distributed (i.i.d) hops, $(m_i = m, \beta_i = \beta, \tau_i = \tau)$, and $\gamma = \tau \forall i$ and for integer values of $m_i$, the CDF of $\gamma_b$ can be expressed as

$$F_{\gamma_b}(\gamma) = 1 - \exp \left[-N \left( \frac{\gamma}{\tau} \right)^{\beta/2} \right]$$

$$\times \left[ \sum_{i=0}^{m-1} \frac{1}{i!} \left( \frac{\gamma}{\tau} \right)^{i \beta/2} \right]^N. \quad (8)$$

3.2. Probability Density Function

Using the multinomial identity [21, eq. (24.1.2)], (8) can be reexpressed as

$$F_{\gamma_b}(\gamma) = 1 - N! \exp \left[-N \left( \frac{\gamma}{\tau} \right)^{\beta/2} \right]$$

$$\times \sum_{n_0, n_1, \ldots, n_{m-1} = 0}^{N} \sum_{n_0 + n_1 + \cdots + n_{m-1} = N} A_{n_0, n_1, \ldots, n_{m-1}} \gamma^{\sum_{i=1}^{m-1} i n_i/2}$$

$$\times \gamma^{-1+\beta} \gamma^{m-1} \prod_{i=1}^{m-1} i n_i. \quad (11)$$

The PDF of $\gamma_b$ can be found by taking the derivative of (9) with respect to $\gamma$. After some straightforward algebraic manipulations, the PDF of $\gamma_b$ can be finally expressed as

$$f_{\gamma_b}(\gamma) = \frac{\beta N^{-\beta/2} N!}{2(\gamma \tau)^{\beta/2}} \exp \left[-N \left( \frac{\gamma}{\tau} \right)^{\beta/2} \right]$$

$$\times \sum_{n_0, n_1, \ldots, n_{m-1} = 0}^{N} \sum_{n_0 + n_1 + \cdots + n_{m-1} = N} A_{n_0, n_1, \ldots, n_{m-1}} \gamma^{\sum_{i=1}^{m-1} i n_i/2}$$

$$\times \gamma^{-1+\beta} \gamma^{m-1} \prod_{i=1}^{m-1} i n_i. \quad (11)$$

3.3. Moments

The $\nu$-th order moment of $\gamma_b$ is defined as

$$\mu_{\gamma_b}(\nu) \triangleq \mathbb{E} (\gamma_b^\nu) = \int_{0}^{\infty} \gamma^\nu f_{\gamma_b}(\gamma) d\gamma. \quad (12)$$

By substituting (11) in (12), making a change of variables $t = N [(\gamma/\tau)^{\beta/2}]$, and using the definition of the gamma function, the $\nu$-th order moment of $\gamma_b$ can be expressed in

$$A_{n_0, n_1, \ldots, n_{m-1}} = \prod_{i=0}^{m-1} \left[ (d)^{n_i} n_i! (\tau \tau)^{i n_i/2} \right]^{-1}. \quad (10)$$
closed form as
\[
\mu_{\gamma_b}(\nu) = N! \sum_{n_0, n_1, \ldots, n_{m-1} = 0}^{N} A_{n_0, n_1, \ldots, n_{m-1}} \times N^{1-2n_1/\beta} (\tau \gamma)_{\tfrac{n_1}{2}} \beta/2 \Gamma \left( \tfrac{2n_1}{\beta} \right)
\]
\[
- N! \sum_{n_0, n_1, \ldots, n_{m-1} = 0}^{N} A_{n_0, n_1, \ldots, n_{m-1}} \times (\tau \gamma)_{\tfrac{n_1}{2}} \beta/2 \Gamma \left( \tfrac{2n_1}{\beta} \right) \sum_{i=1}^{m-1} i \eta_i
\]
(13)

where
\[
\eta_1 = \nu + \frac{\beta}{2} \sum_{i=1}^{m-1} i \eta_i + \frac{\beta}{2}
\] (14a)
and
\[
\eta_2 = \eta_1 - \frac{\beta}{2}.
\] (14b)

### 3.4. Moments-Generating Function

The MGF of \( \gamma_b \) defined as \( M_{\gamma_b}(s) \triangleq \mathbb{E} \{ \exp(-s \gamma_b) \} \), can be extracted from the CDF of \( \gamma_b \) as
\[
M_{\gamma_b}(s) = s \mathcal{L} \{ F_{\gamma_b}(\gamma); s \}
\] (15)

where \( \mathcal{L} \{ \cdot; s \} \) denotes the Laplace transform. By substituting (11) in the above equation and using [22, eq. (2.2.1.22)], the MGF of \( \gamma_b \) can be expressed in closed form as
\[
M_{\gamma_b}(s) = 1 - N! \sum_{n_0, n_1, \ldots, n_{m-1} = 0}^{N} A_{n_0, n_1, \ldots, n_{m-1}} \times \sqrt{k} \left[ l_1 + \frac{1}{8} \Delta_{l_1} \right] \left[ \frac{N^{1-2n_1/\beta} (\tau \gamma)_{\tfrac{n_1}{2}} \beta/2 \Gamma \left( \tfrac{2n_1}{\beta} \right)}{\eta_1} \right]^{\alpha_1/2}
\] (16)

where \( G_{\alpha,k}^k [ \cdot ] \) is the Meijer’s G-function [17, eq. (9.301)], \( k \) and \( l \) are two minimum integers that satisfy \( \beta = 2l/k \),
\[
\Delta(k, \alpha) = \alpha_k \alpha_{k-1} \ldots \alpha_0 \alpha_{k-1} \ldots \alpha_0
\]

It is noted that for Weibull fading channels \( (m = 1) \), (16) numerically coincides with a previously known result [3, eq. (11)]. Also, for Nakagami-\( m \) fading channels, \( (\beta = 2, k = 1, l = 1) \), using [23, eq. 07.34.03.0271.01], (16) simplifies to
\[
M_{\gamma_b}(s) = 1 - N! \sum_{n_0, n_1, \ldots, n_{m-1} = 0}^{N} B_{n_0, n_1, \ldots, n_{m-1}} \times s^{-\nu_2} \nu_2! \left( 1 + \frac{N m}{\tau s} \right)^{-\nu_2 - 1}
\] (18)

where
\[
B_{n_0, n_1, \ldots, n_{m-1}} = \prod_{i=0}^{m-1} \left( i \right)^{n_i} n_i! \left( \frac{\tau s}{m} \right)^{i n_i}^{-1}
\] (19a)
and
\[
\nu_2 = 2 \nu_1.
\] (19b)

For non-identically distributed hops, the MGF of \( \gamma_b \) can be obtained using (15) and (7) as
\[
M_{\gamma_b}(s) = 1 - \int_0^{\infty} s \exp(-s \gamma) \times \frac{1}{\Gamma(m_i)} \left[ m_i \left( \frac{\gamma}{\tau_{i1}} \right)^{\beta_i/2} \right].
\] (20)
Using the Gauss-Laguerre quadrature rule [21, pp. 890 and 923], an accurate approximation of the MGF is obtained as

$$\mathcal{M}_{\gamma_b}(s) \simeq 1 - \sum_{i=0}^{N} \frac{w_i}{\Gamma(m_i)} \Gamma \left[ m_i, \left( \frac{x_i}{s \cdot \gamma_i} \right)^{\beta_i/2} \right]$$

(21)

where $x_i$ are the roots of the Laguerre polynomial $L_n(x)$ and $w_i$ are the corresponding weights with

$$w_i = \frac{x_i}{(N+1)^2 L_{n+1}^2(x_i)}.$$  

(22)

Next, based on (16), asymptotic MGF expressions for high- and low-SNR regions are presented. We show that the MGF of $\gamma_b$ can be significantly simplified expressing it only in the form of elementary functions.

### 3.4.2. High-SNR Region

A simplified expression for the MGF of $\gamma_{eq}$ at high SNR, defined as $\mathcal{M}_{\gamma_{eq}}(s) \triangleq \mathbb{E}(\exp(-s \gamma_{eq}))$, is obtained by applying the analysis presented in [24]. In that work it has been proved that if $X_i$'s, $i = 1, 2, \ldots, N$, are $N$ independent and non-negative random variables and the series expansion of the PDF of $X_i$ in the neighborhood zero can be expressed as$^1$

$$P_{X_i}(x) = a_i x^{t_i} + o(x^{t_i+\epsilon})$$

(24)

where $a_i > 0$, $t_i \geq 0$, $\epsilon > 0$, then the series expansion of the PDF of $Z = (\sum_{i=1}^{N} 1/X_i)^{-1}$ is given by

$$f_Z(z) = \sum_{i=1}^{K} a_i z^{t_{min} + \epsilon} + o(z^{t_{min} + \epsilon})$$

(25)

where $t_{min} = \min_i t_i$ and $\zeta$ is a constant that depends on $t_i$ and $\epsilon$. Using $\exp(z) \simeq 1$ when $z \to 0$ along with the previously cited theorem, the PDF of $\gamma_b$ can be approximated as

$$f_{\gamma_b}(\gamma) \simeq \frac{\beta_i}{2 \Gamma(m_i) (\tau_i \gamma)^{m_i} \beta_i/2} \gamma^{m_i - 1/2}.$$  

(26)

$^1$It is noted that for two functions $f$ and $g$ of a real variable $x$, we write $f(x) = o(g(x))$ when $x \to x_0$ if $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$. 


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Using the definition of the gamma function, the MGF of \( \gamma_{equ} \) may be easily expressed as

\[
\mathcal{M}_{\gamma_{equ}}(s) \simeq \frac{\Gamma(t_{\text{min}})}{2^{t_{\text{min}}}} \sum_{i=1}^{\infty} \frac{\beta_i}{\Gamma(m_i, \tau_i, \gamma_i)} m_i \beta_i / 2
\]  

(27)

where \( t_{\text{min}} = \min_i \{ \beta_i / 2 \} \). Note once again that the above expression is given in terms of elementary functions only.

4. END-TO-END PERFORMANCE ANALYSIS

Using the previous analysis, lower bounds for the OP of the end-to-end SNR are derived in this section. Also, using the MGF approach, lower bounds for the ABEP for a variety of modulation schemes are presented, while the performance is further investigated when employing fast adaptive QAM.

4.1. Fixed Modulation Scheme

Here we assume a fixed modulation scheme with a constant modulation index.

4.1.1. Outage Probability

The outage probability is defined as the probability that the end-to-end output SNR, falls below a specified threshold \( \gamma_{th} \). This threshold is a minimum value of the SNR above which the quality of service is satisfactory.

For the considered multi-hop system the use of upper bound \( \gamma_{th} \) leads to lower bounds for the outage probability at the destination terminal \( D \) expressed as

\[
P_{\text{out}}(\gamma_{th}) \geq F_{\gamma_{th}}(\gamma_{th}).
\]

(28)

It is noted that for identical parameters \( m_i = m, \beta_i = \beta, \tau_i = \tau, \) and \( \gamma_i = \gamma \) \( \forall i \) and \( m \) integer, the outage probability can be extracted based on (8).

4.1.2. Average Bit Error Probability

The MGF of \( \gamma_b \) can be efficiently used to evaluate lower bounds for the ABEP of BDPSK, BPSK and BFSK. The ABEP of BDPSK can be readily obtained from (16) or (21) as

\[
P_{\text{be}} = 0.5 M_{\gamma_b}(1),
\]

while for coherent binary signals as

\[
P_{\text{be}} = 1 \int_0^{\pi/2} M_{\gamma_b} \left[ \frac{\psi}{\sin^2(\theta)} \right] d\theta
\]

(29)

where \( \psi = 1 \) for coherent BPSK, \( \psi = 1/2 \) for coherent BFSK and \( \psi = 0.715 \) for coherent BFSK with minimum correlation. An alternative ABEP expression can be obtained following a technique described in [26] as

\[
P_{\text{be}} = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} F_{\gamma_{th}} \left( \frac{t}{2\psi} \right) \exp \left( -\frac{t^2}{2} \right) dt.
\]

(30)

For non-identically distributed hops, the ABEP for coherent signals can be evaluated using (21) and (29) or (7) and (30) by means of numerical integration.

For identically distributed hops, substituting (16) to (29) and by making the change of variables, \( \sin^2(\theta) = x, \)
integrals of the form

\[ I = \int_0^1 x^{v_1-1/2}(1-x)^{-1/2} \times G_{l,t}^{k,1} \left[ \frac{N^k t^l x^l}{(7\pi)^{k/2} k^k \psi^l} \right] \Delta(k,0) \Delta(l-v_1) \Delta(l+\nu_1) \delta(k,l) dx \]

need to be evaluated. Using [27, eq. (2.24.2.2)], \( I \) can be evaluated in closed form as

\[ I = l^{-1/2} \sqrt{\pi} \times G_{2l,k+l}^{k,2l} \left[ \frac{N^k t^l}{(7\pi)^{k/2} k^k \psi^l} \right] \Delta(k,0) \Delta(l-v_1) \Delta(l+\nu_1) \delta(k,l) . \]

Therefore, the ABEP at the destination node is lower bounded as

\[ \overline{P}_{be} = \frac{1}{2} - N! \sqrt{\pi} \times \sum_{n_0,n_1,\ldots,n_m-1=0}^{N} A_{n_0,n_1,\ldots,n_m-1} \cdot \frac{\sqrt{I} \psi_{-\nu_1}}{(2\pi)^{l} \nu_{1}!!} \times G_{2l,k+l}^{k,2l} \left[ \frac{N^k t^l}{(7\pi)^{k/2} k^k \psi^l} \right] \Delta(k,0) \Delta(l-v_1) \Delta(l+\nu_1) \delta(k,l) , \]

(33)

For Nakagami-\( m \) fading channels, (\( \beta = 2, k = 1, l = 1 \)), using [23, eq. 07.34.03.0400.01] and [27, eq. (7.3.1.27)], (33) simplifies to

\[ \overline{P}_{be} = \frac{1}{2} - \frac{N!}{2} \times \sum_{n_0,n_1,\ldots,n_m-1=0}^{N} B_{n_0,n_1,\ldots,n_m-1} \times \frac{\sqrt{2} \psi_{-\nu_1}}{(2\psi)^{\nu_1} \nu_{1}!!} \times (1 + \frac{N m}{\psi})^{-\nu_2-\frac{1}{2}} \]

(34)

where \((2\nu_2 - 1)!! \equiv 1 \cdot 3 \cdot \ldots \cdot (2\nu_2 - 1)\) is the double factorial.

Finally, at the high-SNR region and using (27) and (29), a simplified asymptotic expression for \( \overline{P}_{be} \) is obtained as

\[ \overline{P}_{be} \simeq \frac{\Gamma \left( \frac{4}{\nu_{1}} + \nu_{\min} \right)}{4 \nu_{\min} \psi_{\nu_{\min}} \sum_{i=1}^{N} \beta_{i}} \]

(35)

### 4.2. Fast Adaptive Square-QAM

According to the fast adaptive QAM technique, the constellation size is selected based on the instantaneous received SNR at the destination. In particular, for a fixed target bit error probability, SNR thresholds for different constellation sizes are calculated. By comparing the instantaneous SNR at the destination with these thresholds, the size of the constellation \( M \) is adapted to provide the best possible throughput while satisfying QoS requirements. Information on which value of \( M \) to be set is determined after communicating the source with the destination through a reliable and low-delay feedback link.

Note that an error-free feedback from the destination to source is being assumed.

#### 4.2.1. Average Spectral Efficiency

Let \( M_j \) and \( \gamma_j^*, j = 0 \cdots J \), be the \( j \)th element from the set of possible constellation sizes and corresponding SNR threshold respectively, to achieve a target bit error probability \( P_b^* \). The ASE is obtained using [14, Eq. (19)]
as
\[
\eta = \sum_{j=0}^{J-1} \tilde{M}_j \mathbb{P}\{\gamma_j^* < \gamma_{equ} \leq \gamma_{j+1}^*\}
\]
\[
+ \tilde{M}_J \mathbb{P}\{\gamma_J^* < \gamma_{equ}\}
\]
\[
= \sum_{j=0}^{J-1} \tilde{M}_j \left[ F_{\gamma_{equ}}(\gamma_{j+1}^*) - F_{\gamma_{equ}}(\gamma_j^*) \right]
\]
\[
+ \tilde{M}_J \left[ 1 - F_{\gamma_{equ}}(\gamma_J^*) \right]
\]
(36)

where \(\tilde{M}_j = \log_2(M_j)\) and \(\mathbb{P}\{\}\) denotes the probability operator. To evaluate the corresponding thresholds \(\gamma_j^*\), the analytical expression for the instantaneous bit error probability is required. This expression is given in [28] as

\[
P_b(\gamma|\gamma_{equ}) = \frac{2}{\sqrt{M} \log_2(\sqrt{M})} \sum_{h=1}^{\log_2(\sqrt{M})} \sum_{i=0}^{\frac{1}{2}(2^h - 1)} (-1)^{i} \left[ 2^{h-1} - \left[ 2^{h-1} \frac{1}{\sqrt{M}} + \frac{i}{2} \right] \right]
\]
\[
\times Q\left( (2i + 1) \sqrt{\frac{\gamma_{equ}}{M - 1}} \right)
\]
(37)

where \([x]\) denotes the largest integer less than or equal to \(x\) and \(Q(\cdot)\) is the well-known Gaussian Q-function. For a given target bit error probability \(P_b^*\), \(\gamma_j^*\) is obtained as the solution of the following equation

\[
P_b(\gamma|\gamma_j^*) = P_b^*.
\]
(38)

Since the roots of (38) cannot be obtained in closed form, any of the well-known root-finding techniques may be used for numerical evaluation.

### 4.2.2. Bit Error Outage

An important performance measure in adaptive modulation systems is the BEO, defined as the outage probability based on bit error probability, that is [14]

\[
P_o(P_b^*) = \mathbb{P}\{P_b(\gamma_{equ}) \geq P_b^*\},
\]
(39)

where \(P_b(\gamma_{equ})\) is the bit error probability as a function of the instantaneous SNR \(\gamma_{equ}\) and \(P_b^*\) is the target bit error probability defined above. Since the event \(P_b(\gamma_{equ}) \geq P_b^*\) is equivalent to the event \(\gamma_{equ} \leq \gamma_j^*, \forall j = 0, 1, \ldots, J - 1\), a lower bound for BEO can be readily obtained as

\[
P_o(P_b^*) \leq F_{\gamma_{equ}}(\gamma_j^*).
\]
(40)

### 4.2.3. Average Bit Error Probability

The ABEP of an adaptive \(M\)-QAM system is given by [29, eq. (11)]

\[
\overline{P}_{be} = \frac{\sum_{j=0}^{J-1} \tilde{M}_j P_j}{\sum_{j=0}^{J-1} \tilde{M}_j \delta_j}.
\]
(41)

In the above equation

\[
\delta_j = \mathbb{P}\{\gamma_j^* \leq \gamma_{equ} \leq \gamma_{j+1}^*\} = F_{\gamma_{equ}}(\gamma_{j+1}^*) - F_{\gamma_{equ}}(\gamma_j^*)
\]
and \(P_j\) is the bit error probability of the \(j\)th transmission mode, given by

\[
P_j = \int_{\gamma_j^*}^{\gamma_{j+1}^*} P_{M_j}(\gamma) f_{\gamma_{equ}}(\gamma) d\gamma,
\]
(42)

where \(P_{M_j}(\gamma)\) is the bit error probability of \(M_j\)-QAM given by (37). Since \(\gamma_j^* = \infty\), it can be observed that the denominator of (41) is equal to the ASE, \(\eta\). Also, by
performing partial integration, (42) can be written as

\[ P_j = F_{\gamma_{eq}}(\gamma_{j}^*) P_{M_j}(\gamma_{j}^*) - F_{\gamma_{eq}}(\gamma_{j}^*) P_{M_j}(\gamma_{j}^*) - \int_{\gamma_{j}^*}^{\gamma_{j+1}^*} \frac{dP_{M_j}(\gamma)}{d\gamma} F_{\gamma_{eq}}(\gamma) d\gamma. \]  

(43)

In the above equation the probabilities \( P_{M_j}(\gamma) \) are linear combinations of Gaussian \( Q \)-functions, the derivatives of which with respect to \( \gamma \) can be easily derived using the identity \( \frac{dQ(x)}{dx} = -\exp(-x^2/2)/\sqrt{2\pi} \). Consequently, \( P_j \) can be easily evaluated using numerical integration.

5. NUMERICAL AND COMPUTER SIMULATION RESULTS

In this section, numerical results for the derived lower bounds and computer simulation results for the corresponding exact OP and ABEP performance criteria are presented. In Fig. 2, lower bounds curves for the OP of a three-hop system \((N = 3)\) are plotted as a function of the first hop normalized outage threshold \( \gamma_{1h}/\gamma_1 \), for non-identically distributed hops, having \( \beta = 2, \gamma_2 = 2\gamma_1, \gamma_3 = 3\gamma_1 \) and for different values of \( m \). As expected, OP improves as \( \gamma_{1h}/\gamma_1 \) decreases and/or \( m \) increases. Dashed curves for the exact outage performance, obtained via Monte Carlo simulations based on (4), are also included for comparison purposes. As it can be observed the difference between the exact value of the OP and the obtained bound gets tighter as \( \gamma_{1h}/\gamma_1 \) decreases. However, at very high values of \( \gamma_{1h}/\gamma_1 \), the bounds get loose.

Moreover, in Fig. 3 lower bounds for the OP for i.i.d. hops, are plotted as a function of the normalized outage threshold \( \delta \), for different values of \( m \) and \( N \). As expected, the bounds get tighter as \( \delta \) increases.
threshold $\gamma_{th}/\overline{\gamma}$, for $\beta = 1$, $m = 3$ and for different values of $N$. As it is evident, OP improves as $\gamma_{th}/\overline{\gamma}$ and/or $N$ decrease. Similarly to the Fig. 2 the analytically obtained lower bound results are compared to corresponding exact outage performance results, obtained via Monte Carlo simulations. It is obvious that the difference between the exact value of the OP and the obtained bound gets tighter with the decrease of $\gamma_{th}/\overline{\gamma}$ and/or $N$. Also, it can be observed that the bounds get less tight at high values of $\gamma_{th}/\overline{\gamma}$ and as $N$ increases.

In Figs. 4 and 5 lower bound curves for the ABEP of BDPSK and BPSK modulations are plotted, respectively, for i.i.d. hops, as a function of the average input SNR per bit $\overline{\gamma}$, for various values of $N$, $\beta = 1$ and $m = 3$. It is obvious that ABEP improves as $N$ decreases. Also, in Fig. 6 the ABEP of BDPSK is plotted as a function of $\overline{\gamma}$ for various values of $\beta$, $m = 3$ and $N = 3$. As expected, the ABEP improves as $\beta$ increases. For the low ($\overline{\gamma} \leq 2\text{dB}$) and high ($\overline{\gamma} \geq 20\text{dB}$) SNR regions, the simplified expression given by (23) and (27) can be used.
Figure 7. End-to-end ABEP of BDPSK and BPSK of a three-hop wireless communication system operating over non identical GG fading channels as a function of the first hop average input SNR per bit ($m = [2.8, 1.3, 0.4]$, $N = 3$, $\overline{\gamma} = [1.75, 3.2, 4.25]$, $\overline{\gamma}_2 = 2 \overline{\gamma}_1$, $\overline{\gamma}_3 = 3 \overline{\gamma}_1$).

respectively, for the numerical evaluation of the proposed ABEP bounds. Moreover, in Fig. 7 lower bound curves for the ABEP of BDPSK and BPSK modulations are plotted as a function of the first hop average input SNR per bit, $\overline{\gamma}_1$, for non-identically distributed hops, $N = 3$, $\overline{\gamma} = [2.8, 1.3, 0.4]$, $\overline{\gamma}_2 = 2 \overline{\gamma}_1$, and $\overline{\gamma}_3 = 3 \overline{\gamma}_1$. Asymptotic ABEP curves are also plotted and as it can be observed, (27) and (35) correctly predict the behavior of the ABEP at high SNR region.

For all the ABEP results in Figs. 4–7, associate curves for the exact error performance, obtained via Monte Carlo simulations based on (4) are also depicted for comparison purposes. From all comparisons one can verify similar findings to that mentioned in Figs. 2 and 3.

In Fig. 8 lower bound curves for the BEO of a three-hop system using fast adaptive $M$-QAM are plotted versus $\overline{\gamma}_1$, assuming $P_b^* = 10^{-2}$, $m = 1.4$, $\beta = 2.25$, $\overline{\gamma}_2 = 2 \overline{\gamma}_1$, $\overline{\gamma}_3 = 3 \overline{\gamma}_1$, and different constellation sizes. Associate curves for the exact BEO, obtained via Monte Carlo simulations based on (4) are also presented and as it can be observed, similar findings to that mentioned in Figs. 2 and 3 may be verified.

In Fig. 9 lower bound curves for the ASE of a three-hop system using fast adaptive $M$-QAM are plotted versus $\overline{\gamma}_1$, assuming $P_b^* = 10^{-2}$, $m = 1.4$, $\beta = 2.25$, $\overline{\gamma}_2 = 2 \overline{\gamma}_1$, $\overline{\gamma}_3 = 3 \overline{\gamma}_1$. We assume $M_0 = 4$, $M_1 = 16$, $M_2 = 64$ and $M_3 = 256$. Different maximum constellation sizes are considered. Curves for the exact ASE based on (4) are also presented and as one can observe the ASE bounds are tight at high values of $\overline{\gamma}_1$.

Finally, in Fig. 10 curves for the ABEP of the same system using non adaptive 256-QAM as well fast adaptive 256-QAM are plotted versus $\overline{\gamma}_1$. Clearly the adaptive system outperforms the non-adaptive one and maintains the target BER requirement, namely $P_b^* = 10^{-2}$. Curves for the exact ABEP based on (4) are also plotted and as it is obvious the ABEP bounds are tight at high values of $\overline{\gamma}_1$ for both the adaptive and non-adaptive systems.

6. CONCLUSIONS

In this paper, we provided union tight bounds for multi-hop transmissions with non-regenerative relays in series, operating over independent GG fading channels. Using a
Figure 8. End-to-end BEO of a multi-hop wireless communication system operating over non identical GG fading channels versus $\gamma_1$, for an adaptive QAM scheme with $P_b^* = 10^{-2}$, $N = 3$, $m = 1.4$, $\beta = 2.25$, $\tau_2 = 2 \tau_1$, $\tau_3 = 3 \tau_1$ and different constellation sizes $M$.

Figure 9. End-to-end ASE of a multi-hop wireless communication system operating over non identical GG fading channels versus $\gamma_1$, for an adaptive QAM scheme with $P_b^* = 10^{-2}$, $N = 3$, $m = 1.4$, $\beta = 2.25$, $\tau_2 = 2 \tau_1$ and $\tau_3 = 3 \tau_1$.

Figure 10. End-to-end ABEP of a multi-hop wireless communication system operating over non identical GG fading channels versus $\gamma_1$, for non-adaptive and adaptive 256-QAM schemes with $P_b^* = 10^{-2}$, $N = 3$, $m = 1.4$, $\beta = 2.25$, $\tau_2 = 2 \tau_1$ and $\tau_3 = 3 \tau_1$.

tight upper bound of the end-to-end SNR, novel closed-form expressions for the MGF, PDF, and CDF of this upper bounded SNR were derived. Additionally, tight lower bounds for the OP and the ABEP were presented. Moreover the performance of fast adaptive QAM was addressed, deriving tight bounds for the ASE, the BEO and the ABEP. Also, it is obvious that all the derived bounds gets tighter as the number of relays decreases. Numerical results were presented and demonstrated the accuracy and the tightness of the proposed bounds. The obtained results show that the proposed bounds gets tighter with the increase of the SNR corresponding to computer simulation results which are also included and verify the accuracy and the correctness of the proposed analysis.
REFERENCES


