Competition and Regulation in Wireless Services Markets

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Abstract

We consider a wireless services market where a set of operators compete for a large common pool of users. The latter have a reservation utility of U_0 units or, equivalently, an alternative option to satisfy their communication needs. The operators must satisfy these minimum requirements in order to attract the users. We model the users decisions and interaction as an evolutionary game and the competition among the operators as a non cooperative price game which is proved to be a potential game. For each set of prices selected by the operators, the evolutionary game attains a different stationary point. We show that the outcome of both games depend on the reservation utility of the users and the amount of spectrum W the operators have at their disposal. We express the market welfare and the revenue of the operators as functions of these two parameters. Accordingly, we consider the scenario where a regulating agency is able to intervene and change the outcome of the market by tuning W and/or U_0. Different regulators may have different objectives and criteria according to which they intervene. We analyze the various possible regulation methods and discuss their requirements, implications and impact on the market.

Index Terms: Spectrum, Evolutionary Game Theory, Pricing, Regulation, Resource Allocation, Mechanism Design.

I. Introduction

Consider a city where 3 commercial operators (companies) and one municipal operator offer WiFi Internet access to the citizens (users). The companies charge for their services and offer better rates than the municipal WiFi service which however is given gratis. Users with high needs will select one of the companies. However, if they are charged with high prices, or served with low rates, a portion of them will eventually migrate to the municipal network. In other words, the municipal service constitutes an alternative choice for the users and therefore sets the minimum requirements which the commercial providers should satisfy. Apparently, the existence of the municipal network affects both the user decisions and the operators pricing policy. In different settings, the minimum requirement can be an inherent characteristic of the users as for example a lower bound on transmission rate for a particular application, an upper bound on the price they are willing to pay or certain combinations of both of these parameters. Again, the operators can attract the users only if they offer more appealing services and prices.

In this paper, we consider a general wireless communication services market where a set of *operators*, compete to sell their services to a common large pool of *users*. We assume that users have minimum requirements or alternative options to satisfy their needs which we model by introducing the reservation utility U_0, [1]. Users select an operator only if the offered service and the charged price ensure utility higher than U_0. We analyze the users strategy for selecting operator and the price competition among the operators under this constraint. We find that the market outcome depends on U_0 and on the amount of spectrum each operator has at his disposal W. Accordingly, we consider the existence of a regulating agency who is interested in affecting the market and enforcing a more desirable outcome, by tuning either W or U_0. For example, consider the municipal WiFi provider who is actually able to set U_0 and bias the competition among the commercial providers. This is of crucial importance since in many cases the competition of operators may yield inefficient allocation of the network resources, [1] or even reduced revenue for them, [2]. We introduce a rigorous framework that allows us to analyze the various methods through which the regulator can intervene and affect the market

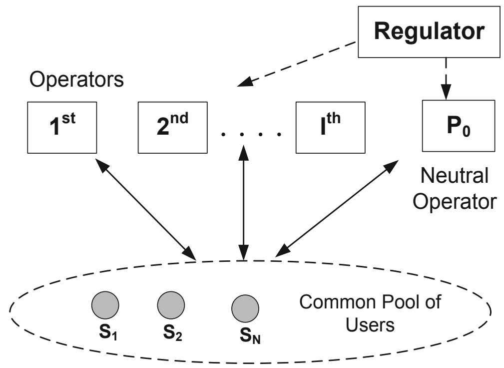

Fig. 1. The market consists of I operators and N users (S). Each user is associated with one operator at each specific time slot. Every operator $i = 1, 2, \ldots, I$ can serve more than one users at a certain time slot. The users that fail to satisfy their minimum requirements, $U_i \leq U_0$, $\forall i \in \mathcal{I}$, abstain from the market and select the neutral operator P_0.

outcome according to his objective.

Our model captures many different settings such as a WiFi market in a city, a mobile/cell-phone market in a country or even a secondary spectrum market where primary users lease their spectrum to secondary users. In order to make our study more realistic, we adopt a macroscopic perspective and analyze the interaction of the operators and users in a large time scale, for large population of users, and under limited information. The operators are not aware of the users specific needs and the latter cannot predict in advance the exact level of service they will receive. Each operator has a total resource at his disposal (e.g. the aggregate service rate) which is on average equally allocated to his subscribers, [1], [3]. This is due to the various network management and load balancing techniques that the operators employ, or because of the specific protocol that is used, [4]. Each user selects the operator that will provide the optimal combination of service quality and price. Apparently, the decision of each user affects the utility of the other users. We model this interdependency as an evolutionary game, [5] the stationary point of which represents the users distribution among the operators and depends on the charged prices. This gives rise to a non cooperative price competition game among the operators who strive to maximize their profits.

Central to our analysis is the concept or the *neutral operator* P_0 which provides to the users a constant and given utility of U_0 units. The P_0 can be a special kind of operator, like the municipal WiFi provider in the example above, or it can simply model the user choice to abstain from the market. This way, we can directly calculate how many users are served by the market and how many abstain from it and select P_0. Moreover, P_0 allows us to introduce the role of a regulating agency who can intervene and bias the market outcome through the service U_0. We show that P_0 can be used to increase the revenue of the operators or the efficiency of the market. In some cases, both of these metrics can be simultaneously improved at a cost which is incurred by the regulator. Alternatively, the outcome of the market can be regulated by changing the amount of spectrum each operator has at his disposal. Different regulating methods give different results and entail different cost for the regulator.

A. Related Work and Contribution

The competition of sellers for attracting buyers has been studied extensively in the context of network economics, [6], [7], both for the Internet and more recently for wireless systems. In many cases, the competition results in undesirable outcome. For example, in [1] the authors consider an oligopoly communication market and show that it yields inefficient resource allocation for the users. From a different perspective it is explained in [2], that selfish pricing strategies may also decrease the revenue of the sellers-providers. In these cases, the strategy of each node (buyer) affects the performance of the other nodes by increasing the delay of the services they receive, [1] (*effective cost*) or, equivalently, decreasing the resource the provider allocates to them, [3] (*delivered price*). This equal-resource sharing assumption represents many different access schemes and protocols (TDMA, CSMA/CA, etc), [4].

More recently, the competition of operators in wireless services markets has been studied in [3], [8], [9], [10], [11]. The users can be charged either with a usage-based pricing scheme, [8], or on a per-subscription basis, [9], [11]. We adopt the latter approach since it is more representative of the current wireless communication systems. We assume that users may migrate (churn) from one operator to the other, [11], and we use evolutionary game theory (EVGT) to model this process, [12]. This allows us to capture many realistic aspects and to analyze the interaction of very large population of users under limited information. The motivation for using EVGT in such systems is very nicely discussed in [10]. Due to the existence of the *neutral operator*, the user strategy is updated through a hybrid scheme based on imitation and direct selection of P_0. We define a new revision protocol to capture this aspect and we derive the respective system dynamic equations.

Although the regulation has been discussed in context of networks, [6], it remains largely unexplored. Some recent work [13], [14] study how a regulator or an *intervention device* may affect a non-cooperative game among a set of players (e.g. operators). However, these works do not consider hierarchical systems, with large populations and limited information. Our contribution can be summarized as follows: **(i)** we model the wireless service market using an evolutionary game where the users use a hybrid revision protocol, based both on imitation and direct selection of P_0. We derive the differential equations that describe the evolution of this system and find the stationary points, **(ii)** we define the price competition game for I operators and the particular case that users have minimum requirements, or equivalently, alternative choices/offers, **(iii)** we prove that this is a Potential game and we analytically find the Nash equilibria, **(iv)** we introduce the concept of the neutral operator who represents the system/state regulator or the minimum users requirement, and **(iv)** we discuss different regulation methods and analyze their efficacy, implications and the resources that are required for their implementation.

The rest of this paper is organized as follows. In Section II we introduce the system model and in Section III we analyze the dynamics of the users interaction and find the stationary point of the market. In Section IV we define and solve the price competition game among the operators and in Section V we discuss the relation between the revenue of the operators and the efficiency of the market and their dependency on the system parameters. Accordingly, we analyze various regulation methods for different regulation objectives and give related numerical examples. We conclude in Section VI.

II. SYSTEM MODEL

We consider a wireless service market (hereafter referred to a *market*) with a very large set of users $\mathcal{N} = (1, 2, \ldots, N)$ and a set of operators $\mathcal{I} = (1, 2, \ldots, I)$, which is depicted in Figure 1. Each user cannot be served by more than one operators simultaneously. However, users can switch between the operators or even they can opt to refrain and not purchage services from anyone of the I operators. The net utility perceived by each user who is served by operator i is:

$$U_i(W_i, n_i(t), \lambda_i(t)) = V_i(W_i, n_i(t)) - \lambda_i(t) \tag{1}$$

where n_i is the number of the users served by this specific operator, W_i the total spectrum at his disposal, and λ_i the charged price. In order to describe the market operation we introduce the users vector $\mathbf{x}(\mathbf{t}) = (x_1(t), x_2(t), \ldots, x_I(t), x_0(t))$, where the i^{th} component $x_i(t) = n_i(t)/N$ represents the portion of the users that have selected operator $i \in \mathcal{I}$. Additionally, with $x_0(t) = n_0(t)/N$ we denote the portion of users that have selected neither of the I operators.

Valuation function: The function $V_i(\cdot)$ represents the value of the offered service for each user associated with operator $i \in \mathcal{I}$. Notice that we do not distinguish the different users: all the users served by a certain operator are charged the same price and perceive the same utility. We assume that this valuation decreases with the total number of served users and increases with the amount of available spectrum W_i:

$$V_i(W_i, x_i(t)) = \log\left(\frac{W_i}{N x_i(t)}\right),\ x_i(t) = \frac{1}{N}, \frac{2}{N}, \ldots, 1 \tag{2}$$

Since N is given, we can use $x_i(t)$ instead of $n_i(t)$. Apparently, from a macroscopic perspective and in a large time scale, all the users that are associated with a certain operator will utilize on average the same amount of his resource. Spatiotemporal variations in the quality of the offered services will be smoothed out due to load balancing and other similar network management techniques that the operators employ. We want to stress that this assumption holds independently of the employed technology and captures many different settings in wireline, [1], or wireless networks, [3], [4], [9], [10], [15]. Finally, we use the logarithmic function in order to capture the effect of *diminishing marginal returns*, [6].

Neutral Operator: The variable $x_0(t)$ represents the users that do not select anyone of the I operators. Namely, a user in each time slot t is willing to pay operator $i \in \mathcal{I}$ only if the offered utility $U_i(W_i, x_i(t), \lambda_i(t))$ is greater than a threshold $U_0 \geq 0$. If all the operators fail to satisfy this minimum requirement then the user abstains from the market and is associated with the *Neutral Operator* P_0, Figure 1. In other words, P_0 represents the choice of selecting neither of the I operators and receiving utility of U_0 units. The latter is considered identical for all the users and constant through time. Technically, as it will be shown in the sequel, the inclusion of P_0 affects both the user decision process for selecting operator and the competition among the operators.

From a modeling perspective, the neutral operator may be used to represent different realistic aspects of the wireless service market. First, P_0 can be an actual operator owned by the state, as the public/municipal WiFi provider we considered in the introductory example. In this case, through the gratis U_0 service, the state intervenes and regulates the market as we will explain in Section V. Additionally, U_0 can be indirectly imposed by the state (the regulator) through certain rules such as the minimum amount of spectrum/rate per user. Finally, it can represent the users reluctancy to pay very high prices for poor QoS, similar to the individual rationality constraint in mechanism design. We take these realistic aspects into account and moreover, by using $x_0(t)$, we find precisely how many users are not satisfied by the market of the I operators.

Revenue: Each operator $i \in \mathcal{I}$ determines the price $\lambda_i \in R$ that he will charge to his clients for a certain time period of $T >> 1$ time slots. Let us define the price vector $\lambda = (\lambda_i\,:\, i = 1, 2, \ldots, I)$ and the vector of the $I-1$ prices of operators other than i as $\lambda_{-i} = (\lambda_j\,:\, j \in \mathcal{I} \backslash i)$. The decisions of the operators are realized in a different time scale than the user strategy update. That is, we assume that for a given set of operator prices, the market of the users reaches a stationary point - if such a point is attainable - before any operator changes his price. Since it is, $\lambda_i(t) = \lambda_i,\ t = 1, 2, \ldots, T$, we omit the time index for operators prices when we study the interaction of users. The objective of each operator $i \in \mathcal{I}$, is to maximize his revenue:

$$R_i(x_i(t), \lambda_i) = \lambda_i x_i(t) N \tag{3}$$

In these markets there are no service level agreements (SLAs) or any other type of QoS guarantees and hence the operators are willing to admit and serve as many users as possible.

III. User Strategy and Market Dynamics

A. Evolutionary Game among Users

In order to select the optimal operator that maximizes eq. (1), each user must be aware of all the system parameters, i.e. the spectrum W_i, the number of served users n_i and the charged price λ_i for each $i \in \mathcal{I}$. However, in realistic settings this information will not be available in advance. Given these restrictions and the large number of users, we model their interaction and the operator selection process by defining an evolutionary game, $\mathcal{G}_{\mathcal{U}}$, as follows:

- Players: the set of the N users, $\mathcal{N} = (1, 2, \ldots, N)$.
- Strategies: each user selects a certain provider/operator $i \in \mathcal{I}$ or the neutral operator P_0.
- Population State: the users distribution over the I operators and the neutral operator, $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_I(t), x_0(t))$.
- Payoff: the user's net utility $U_i(W_i, x_i(t), \lambda_i(t))$ when he selects operator $i \in \mathcal{I}$, or U_0 when he selects P_0.

In the sequel we explain how each user selects his strategy under this limited information and what is the outcome of this game.

B. User Strategy Update

The users update their strategy and select operators according to the so-called *revision protocol*. To account for the alternative choice of selecting the neutral operator, we consider a *hybrid* revision protocol which is based partially on imitation and partially on direct selection of P_0. In particular, the probability that a user will migrate from operator i to operator $j \in \mathcal{I} \setminus i$

in time slot t, is:

$$p_{ij}(t) = x_j(t)[U_j(t) - U_i(t)]_+ \tag{4}$$

where $x_j(t)$ is the portion of users already associated with the j^{th} operator. Notice that, for simplicity, we express the user utilities as a function with a single argument, the time t.

However, since U_0 is given, known in advance and independent of the population, users select operator P_0 through a different process. Namely, the probability that a user will switch from operator i to neutral operator P_0, is:

$$p_{i0}(t) = \gamma[U_0 - U_i(t)]_+ \tag{5}$$

The difference between imitation and direct selection, [5], is that instead of multiplying with the population $x_0(t)$, we use a constant multiplier $\gamma \in R$. Notice that we use a probabilistic model in order to capture the bounded rationality, the inertia of the users and other similar realistic aspects of these markets. Finally, the user leaves P_0 and returns to the market again through an imitation process:

$$p_{0i}(t) = x_i(t)[U_i(t) - U_0]_+ \tag{6}$$

C. Market Stationary Point

Given the above *hybrid* revision protocol for the users, we prove in Appendix A that the dynamics of the system are given by the following equations:

$$\begin{aligned} \frac{\mathrm{dx_i(t)}}{\mathrm{dt}} &= x_i(t)[U_i(t) - U_{avg}(t) - x_0(t)(U_i(t) - U_0) \\ &- \gamma(U_0 - U_i(t))_+ + x_0(t)(U_i(t) - U_0)_+], \forall i \in \mathcal{I} \end{aligned} \tag{7}$$

where $U_{avg}(t) = \sum_{i\in\mathcal{I}} x_i(t)U_i(t)$ is the average utility of the market in each slot t. The user population associated with P_0 is:

$$\frac{\mathrm{dx_0(t)}}{\mathrm{dt}} = x_0 \sum_{i\in\mathcal{I}^+} x_i(U_0 - U_i) + \gamma \sum_{j\in\mathcal{I}^-} x_j(U_0 - U_j) \tag{8}$$

where $\mathcal{I}^+$ is the subset of operators offering utility $U_i(t) > U_0$, and $\mathcal{I}^-$ is the subset of operators offering utility $U_i(t) < U_0$, at slot t.

Apparently, the inclusion of the neutral operator changes the dynamics of the system. Nevertheless, despite its different evolution, this system has the same stationary points as the systems that are described by the classical replicator dynamic equations, (see Appendix B for detailed proof):

$$\dot{x}_i(t) = 0 \Rightarrow x_i(t)[U_i(t) - U_{avg}(t)] = 0,\; i \in \mathcal{I} \tag{9}$$

and

$$\dot{x}_0(t) = 0 \Rightarrow x_0(t)[U_0 - U_{avg}(t)] = 0 \tag{10}$$

The user state vector $\mathbf{x}^*$ and the respective user utility U_i^*, $i \in \mathcal{I}$, that satisfy these stationary conditions can be summarized in the following 3 cases:

- **Case A**: x_i^*, $x_0^* > 0$ and $U_i^* = U_0$, $i \in \mathcal{I}$.
- **Case B**: x_i^*, $x_j^* > 0$, $x_0^* = 0$ and $U_i^* = U_j^*$, with $U_i^*, U_j^* > U_0, \forall\, i,\, j \in \mathcal{I}$.
- **Case C**: x_i^*, $x_j^* > 0$, $x_0^* = 0$ and $U_i^* = U_j^* = U_0,\; \forall\, i,\, j \in \mathcal{I}$.

Before calculating the stationary point $\mathbf{x}^*$ for each case, we need to define for every operator $i \in \mathcal{I}$ the scalar parameter $\alpha_i = W_i/Ne^{U_0}$ and the respective vector $\alpha = (\alpha_i \,:\, i \in \mathcal{I})$. As it will be explained in the sequel, these parameters determine the operators and users interaction and bring to the fore the role of the regulator. We can find the stationary points for **Case A** by using the equation $U_i(W_i, x_i^*, \lambda_i) = U_0$ and imposing the constraint $x_0^* > 0$. Apparently, the state vector $\mathbf{x}^*$ depends on the price vector λ. Therefore, we define the set of all possible **Case A** stationary points, X_A, as follows (see Appendix B for details):

$$X_A = \left\{ x_i^* = \frac{\alpha_i}{e^{\lambda_i}}, \forall i \in \mathcal{I}, x_0^* = 1 - \sum_{i=1}^{I} \frac{\alpha_i}{e^{\lambda_i}} \,:\, \lambda \in \Lambda_A \right\} \tag{11}$$

where Λ_A is the set of prices for which a stationary point in X_A is attainable, i.e. $x_0^* > 0$:

$$\Lambda_A = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_I) \,:\, \sum_{i=1}^{I} \frac{\alpha_i}{e^{\lambda_i}} < 1 \right\} \tag{12}$$

TABLE I
WIRELESS SERVICE MARKET STATIONARY POINTS.

	X_A	X_B	X_C
x_i^*	α_i/e^{λ_i}	$\frac{\alpha_i}{e^{\lambda_i}\sum_{j=1}^{I}\alpha_j/e^{\lambda_j}}$	α_i/e^{λ_i}
x_0^*	$1-\sum_{i=1}^{I}\alpha_i/e^{\lambda_i}$	0	0
Cond.	$\lambda\in\Lambda_A$	$\lambda\in\Lambda_B$	$\lambda\in\Lambda_C$

Similarly, for **Case B**, we calculate the stationary points by using the set of equations $U_i(W_i, x_i^*, \lambda_i) = U_j(W_j, x_j^*, \lambda_j)$, $\forall\, i, j \in \mathcal{I}$:

$$X_B = \left\{ x_i^* = \frac{\alpha_i}{e^{\lambda_i}\sum_{j=1}^{I}\alpha_j/e^{\lambda_j}}, \forall i \in \mathcal{I}, x_0^* = 0 :\ \lambda \in \Lambda_B \right\} \tag{13}$$

where Λ_B is the set of prices for which a stationary point in X_B is feasible, i.e. $U_i^* > U_0$:

$$\Lambda_B = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_I)\ :\ \sum_{i=1}^{I}\frac{\alpha_i}{e^{\lambda_i}} > 1 \right\} \tag{14}$$

Finally, the stationary points for the **Case C** solution must satisfy the constraint $\sum_{i=1}^{I}\alpha_i/e^{\lambda_i} = 1$ which yields:

$$X_C = \left\{ x_i^* = \frac{\alpha_i}{e^{\lambda_i}}, \forall\, i \in \mathcal{I}, x_0^* = 0 :\ \lambda \in \Lambda_C \right\} \tag{15}$$

with

$$\Lambda_C = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_I)\ :\ \sum_{i=1}^{I}\frac{\alpha_i}{e^{\lambda_i}} = 1 \right\} \tag{16}$$

Notice that the stationary point sets X_A, X_B and X_C and the respective price sets, Λ_A, Λ_B, and Λ_C depend on the vector α. These results are summarized in Table I. The utility of the users is equal to U_0 for the **Case A** and **Case C**, while for **Case B** it depends on λ.

IV. PRICE COMPETITION AMONG OPERATORS

In the previous section we analyzed the stationary points of the users interaction and showed that they depend on the prices selected by the operators. Each operator anticipates the users strategy and chooses accordingly the price that maximizes his revenue. This gives rise to a non-cooperative price competition game among the operators.

A. Price Competition Game $\mathcal{G}_{\mathcal{P}}$

Before analyzing this game, it is important to emphasize that each operator has different revenue function depending on the set that the price vector λ belongs to. In particular, using equation (3), we can calculate the revenue of operator i when $\lambda \in \Lambda_A$ and when $\lambda \in \Lambda_B$:

$$R_i^A(\lambda_i) = \frac{\alpha_i \lambda_i N}{e^{\lambda_i}},\ R_i^B(\lambda_i, \lambda_{-i}) = \frac{\alpha_i \lambda_i N}{e^{\lambda_i} \sum_{i=1}^{I} \alpha_i / e^{\lambda_i}} \tag{17}$$

When $\lambda \in \Lambda_C$, it is $R_i^C(\lambda_i) = \alpha_i \lambda_i N / e^{\lambda_i}$. Notice that $R_i^A(\cdot)$ and $R_i^C(\cdot)$ depend only on the price selected by operator i, while $R_i^B(\cdot)$ depends on the entire price vector λ. However, in all cases, the price set (Λ_A, Λ_B or Λ_C) to which the price vector $\lambda = (\lambda_i, \lambda_{-i})$ belongs, is determined jointly by all the I operators.

Let us now define the **Pricing Game** among the I operators, $\mathcal{G}_{\mathcal{P}} = (\mathcal{I}, \{\lambda_i\}, \{R_i\})$:

- The set of *Players* is the set of the I operators $\mathcal{I} = (1, 2, \ldots, I)$.
- The *strategy space* of each player i is its price $\lambda_i \in [0, \lambda_{max}]$, $\lambda_{max} \in \mathcal{R}^+$, and the strategy profile is the price vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_I)$ of the operators.
- The payoff function of each player is his revenue $R_i : (\lambda_i, \lambda_{-i}) \rightarrow \mathcal{R}$, where $R_i = R_i^A$ or R_i^B or R_i^C.

The particular characteristic of this game is that each operator has 3 different payoff functions depending on the stationary point area of the market. However, still the payoff function is continuous and quasi-concave (see Appendix C). In the sequel, we analyze the best response of each operator and we find accordingly the equilibrium of the game $\mathcal{G}_{\mathcal{P}}$.

B. Best Response Strategy of Operators

The best response of each operator i, λ_i^*, to the prices selected by the other $I-1$ operators, λ_{-i}, depends on the users market stationary point. Notice that for certain λ_{-i}, operator i may be able to select a price such that $(\lambda_i, \lambda_{-i})$ belongs to any price set (Λ_A, Λ_B or Λ_C) while for some λ_{-i} the operator choice will be restricted in two or even a single price set.

Best Response when $\lambda \in \Lambda_A$**:** If the $I-1$ operators $j \in \mathcal{I} \setminus i$ select such prices, λ_{-i}, that the market stationary point can be $\mathbf{x}^* \in X_A$, then operator i finds the price λ_i^* that maximizes his revenue $R_i^A(\cdot)$ by solving the following constrained optimization problem ($\mathbf{P}_i^A$):

$$\max_{\lambda_i \geq 0} \frac{\alpha_i \lambda_i N}{e^{\lambda_i}} \tag{18}$$

s.t.

$$\sum_{j=1}^{I} \frac{\alpha_j}{e^{\lambda_j}} < 1 \;\Rightarrow\; \lambda_i > \log\left(\frac{\alpha_i}{1 - \sum_{j \neq i} \alpha_j / e^{\lambda_j}}\right) \tag{19}$$

This problem is quasi-concave, [16], and hence it has a unique optimal solution.

Best Response when $\lambda \in \Lambda_B$**:** Similarly, when λ_{-i} is such that operator i can select a price λ_i^* such that $(\lambda_i^*, \lambda_{-i}) \in \Lambda_B$, then his revenue is given by the function $R_i^B(\cdot)$ and is maximized by the solution of the prolbem ($\mathbf{P}_i^B$):

$$\max_{\lambda_i \geq 0} \frac{\lambda_i \alpha_i N}{e^{\lambda_i} \sum_{j \in \mathcal{I}} \frac{\alpha_j}{e^{\lambda_j}}} \tag{20}$$

s.t.

$$\sum_{j \in \mathcal{I}} \frac{\alpha_j}{e^{\lambda_j}} > 1 \;\Rightarrow\; \lambda_i < \log\left(\frac{\alpha_i}{1 - \sum_{j \neq i} \alpha_j / e^{\lambda_j}}\right) \tag{21}$$

This is also a concave problem which has unique solution. The latter is the same as the optimal solution of the respective unconstrained problem, denoted μ_i^*, if $(\mu_i^*, \lambda_{-i}) \in \Lambda_B$ (see Appendix D).

Best Response when $\lambda \in \Lambda_C$**:** In this special case, the price of each operator i is directly determined by the prices that the other operators have selected:

$$\lambda_i^* = \log\left(\frac{\alpha_i}{1 - \sum_{j \neq i} \alpha_j / e^{\lambda_j}}\right) \tag{22}$$

Whether each operator i will agree and adopt this price or not, depends on the respective accrued revenue, $R_i^C(\lambda_i^*, \lambda_{-i})$.

We can summarize the best response price strategy of each operator $i \in \mathcal{I}$, by defining his revenue function as follows:

$$R_i(\lambda_i, \lambda_{-i}, \alpha) = \begin{cases} \frac{\alpha_i \lambda_i N}{\sum_{i=1}^{I} \alpha_j e^{\lambda_i}/e^{\lambda_j}} & \text{if } \lambda_i < l_0, \\ \frac{\alpha_i \lambda_i N}{e^{\lambda_i}} & \text{if } \lambda_i \geq l_0. \end{cases} \tag{23}$$

where $l_0 = \log(\alpha_i/(1 - \sum_{j \neq i} \alpha_j/e^{\lambda_j}))$. Clearly, the optimal price λ_i^* depends both on the prices of the other operators λ_{-i} and on parameters α_i, $i \in \mathcal{I}$:

$$\lambda_i^* = arg \max_{\lambda_i} R_i(\lambda_i, \lambda_{-i}, \alpha) \tag{24}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_I)$.

Apparently, for each possible price vector of the $\mathcal{I} \setminus i$ operators, λ_{-i}, operator i will solve all the above optimization problems and find the solution that yields the highest revenue. In Appendix D we prove that this results in the following best response strategy:

$$\lambda_i^*(\lambda_{-i}, \alpha) = \begin{cases} 1 & \text{if } (1, \lambda_{-i}) \in \Lambda_A, \\ \mu_i^* & \text{if } (\mu_i^*, \lambda_{-i}) \in \Lambda_B, \\ l_0 & \text{otherwise}. \end{cases} \tag{25}$$

Notice also that if $\sum_{j \neq i} \alpha_j/e^{\lambda_j} \geq 1$, the only feasible response is $\lambda_i^* = \mu_i^*$. Despite its particular structure, the revenue function $R_i(\cdot)$ is still continuous and quasi-concave (see Appendix C). The additional dependence of eq. (25) on parameters $\alpha_i = W_i/Ne^{U_0}$, $i \in \mathcal{I}$, has interesting implications and brings into the fore the role of the regulator.

C. $\mathcal{G}_{\mathcal{P}}$ Equilibrium Analysis

The price competition game $\mathcal{G}_{\mathcal{P}}$ is a finite ordinal potential game and therefore not only has pure Nash equilibria but also the convergence to them is ensured under any finite improvement path (FIP), [17]. The potential function is:

$$\mathcal{P}(\lambda) = \begin{cases} \sum_{j=1}^{I} (\log \lambda_j - \lambda_j), \text{ if } \sum_{j=1}^{I} \alpha_j / e^{\lambda_j} \leq 1, \\ \sum_{j=1}^{I} (\log \lambda_j - \lambda_j) - \log (\sum_{j=1}^{I} \alpha_j / e^{\lambda_j}), \text{ else.} \end{cases} \tag{26}$$

The detailed proof is given in Appendix E. In order to find the NE we solve the system of equations (25), $i = 1, 2, \ldots, I$ and actually specifically we use the iterated dominance method (see Appendix F).

The outcome of the game $\mathcal{G}_{\mathcal{U}}$ affects the strategy of the operators and therefore the outcome of the game $\mathcal{G}_{\mathcal{P}}$. A price vector $(\lambda_i^*, \lambda_{-i}^*)$ is an equilibrium of the game $\mathcal{G}_{\mathcal{P}}$, parameterized by the vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_I)$, if it is:

$$R_i(\lambda_i^*, \lambda_{-i}^*, \alpha) \geq R_i(\lambda_i, \lambda_{-i}^*, \alpha), \forall\, i \in \mathcal{I},\ \forall \lambda_i \geq 0, \forall\, \mathbf{x}^* \in\ X_A \cup X_B \cup X_C \tag{27}$$

In order to simplify our study and focus on the results and implications of our analysis, we assume that all operators have the same amount of available spectrum $W_i = W$, $\forall i \in \mathcal{I}$ and therefore it is also $\alpha_i = \alpha$.

Apparently, the equilibrium of the price competition game - and subsequently, the market stationary point $\mathbf{x}^*$, depend on the value of α. These results are summarized in Table II and stem from the following Theorem:

Theorem IV.1. *The game $\mathcal{G}_{\mathcal{P}}$ converges to one of the following pure Nash equilibria:*

- *If $\alpha \in A_1 = (0, e/I)$, there is a unique Nash Equilibrium $\lambda^* \in \Lambda_A$, with $\lambda^* = (\lambda_i^* = 1 : i \in \mathcal{I})$ and respective market stationary point $\mathbf{x}^* \in X_A$.*
- *If $\alpha \in A_3 = (e^{\frac{I}{I-1}}/I, \infty)$, there is a unique Nash Equilibrium $\lambda^* \in \Lambda_B$, with $\lambda^* = (\lambda_i^* = \frac{I}{I-1} : i \in \mathcal{I})$, which induces a respective market stationary point $\mathbf{x}^* \in X_B$.*
- *If $\alpha \in A_2 = [e/I, e^{\frac{I}{I-1}}/I]$, there exist infinitely many equilibria, $\lambda^* \in \Lambda_C$, and each one of them yields a respective market stationary point $\mathbf{x}^* \in X_C$.*

Proof: In Appendix E, we provide the detailed proof according to which $\mathcal{G}_{\mathcal{P}}$ is a potential

TABLE II
EQUILIBRIA OF I OPERATORS COMPETITION FOR DIFFERENT VALUES OF α.

Prices/Rev.	$\alpha \in A_1$	$\alpha \in A_2$	$\alpha \in A_3$
λ_i^*	1	$\lambda_i \neq \lambda_j$ or $\lambda_i = \lambda_j = \log(I\alpha)$	$\frac{I}{I-1}$
R_i^*	$\frac{\alpha N}{e}$	$R_i \neq R_j$ or $R_i = R_j = \frac{N}{I}\log(I\alpha)$	$\frac{N}{I-1}$
$\mathbf{x}^*$	X_A	X_C	X_B

game and in Appendix F, we use iterated strict dominance to find the Nash equilibrium λ^* which depends on parameter α. ■

V. MARKET OUTCOME AND REGULATION

The outcome of the users and operators interaction can be characterized by the following two fundamental criteria: the efficiency of the users market and the total revenue the operators accrue. We show that both of them depend on parameter α. Accordingly, we analyze the problem from a mechanism design perspective and explain how a regulator, as the municipal WiFi provider in the introductory example, can bias the market operation (outcome) by adjusting the value of α. We consider different regulation methods and discuss their implications.

A. Market Outcome and Regulation Criteria

1) Market Efficiency: A market is efficient if the users enjoy high utilities in the stationary point. However, in certain scenarios, the services provided by the P_0 may impose an additional cost to the system (e.g. the cost of the municipal WiFi provider is borne by the citizens) and hence it would be preferable to have all the users served by the I operators. Therefore, we use the following two metrics to characterize the efficiency of the market: **(i)** the aggregate utility (U_{agg}) of users in the stationary point $\mathbf{x}^*$, and **(ii)** the portion of users (x_s^*) that are served by any of the I operators, $x_s^* = \sum_{i=1}^{I} x_i^* = 1 - x_0^*$. Both of these metrics depend on parameter α.

Namely, when $\alpha \in A_1 = (0, e/I)$ it is $\mathbf{x}^* \in X_A$, and hence a portion of the users is not served by any of the I operators, while all of them receive utility of U_0 units. On the other hand, when

$\alpha \in A_2 = [e/I, e^{I/(I-1)}/I]$, it is $\mathbf{x}^* \in X_B$. In this case, all users are served by the I operators but receive only marginal utility, $U_i = U_0$, $i \in \mathcal{I}$. Finally, if $\alpha \in A_3 = (e^{I/(I-1)}/I, \infty)$, it is $\mathbf{x}^* \in X_C$, all users are served by the I operators, $x_s^* = 1$, and receive high utilities $U_i^* > U_0$, $\forall i \in \mathcal{I}$. In summary, the aggregate utility of the users changes with α as follows:

$$U_{agg} = \begin{cases} NU_0 & \text{if } \alpha \in A_1 \cup A_2, \\ N(log(\frac{WI}{N}) - \frac{I}{I-1}) & \text{if } \alpha \in A_3. \end{cases} \tag{28}$$

Apparently, when $\alpha \in A_1 \cup A_2$, the U_{agg} increases with U_0 and is independent of the spectrum W. On the contrary, when $\alpha \in A_3$, U_{agg} increases with W and is independent of U_0. Notice also that when the value of α changes from A_1 to interval A_2, U_{agg} remains the same but the other metric of efficiency, x_s^* is improved:

$$x_s^* = \begin{cases} 1 - \frac{\alpha I}{e} & \text{if } \alpha \in A_1, \\ 1 & \text{if } \alpha \in A_2 \cup A_3. \end{cases} \tag{29}$$

2) Revenue of Operators: When α lies in the interval A_1, the optimal prices are $\lambda_i^* = 1$, $\forall i \in \mathcal{I}$ and all the operators accrue the same revenue $R_i^* = \alpha N/e$, which is proportional to α. In Figure 3 we depict the revenue of each operator (normalized with the number of users) for different values of α, in a duopoly market. Notice that the revenue increases linearly with $\alpha \in (0, e/2)$.

When $\alpha \in A_2$, the competition of the operators may attain different equilibria, $\lambda^* \in \Lambda_C$, depending on the initial prices and on the sequence the operators update their prices. In Figure 4 we present the revenue of two operators (duopoly) at the equilibrium, for various initial prices and for $\alpha = e \in A_2$. Here we assume that the 1^{st} operator is able to set his price $\lambda_1(0)$ before the 2^{nd} operator. This gives him a strategic advantage. Also, in Figure 3 we illustrate the dependence of the revenue of the operators on the value of α when it lies in A_2, given that $\lambda_1(0) = 1.1$. For certain prices, e.g. when $\lambda_1(0) = \log(2\alpha)$, both operators accrue the same revenue at the equilibrium, $R_1^* = R_2^* = \log(2\alpha)N/2$.

If $\alpha \in A_3 = (e^{I/(I-1)}, \infty)$ all operators set their prices to $\lambda_i^* = I/(I-1)$ and get $R_i^* = N/(I-1)$ units, as shown in Table II. Figure 5 depicts the competition of two operators and the convergence to the respective Nash equilibria for $\alpha = e^3 \in A_3$. We assume that both operators have selected prices $\lambda_1(0) = \lambda_2(0) = \log(2\alpha) \approx 3.7$. However, this price vector does not constitute a NE and hence an operator (e.g. the 1^{st}) can temporarily increase his revenue by decreasing his price to $\lambda_1 = 3$. Accordingly, the other operator (2^{nd}) will react by reducing his price to $\lambda_2 = 2.5$. Gradually, the competition of the operators will converge to the NE where both of them will set $\lambda_1^* = \lambda_2^* = 2/(2-1) = 2$ and will have revenue $R_1^* = R_2^* = 1$. Interestingly, the revenue of both operators in the equilibrium is lower than their initial revenue when they did not compete. Finally, notice that, unlike the social welfare, the revenue of the operators depends only on $\alpha = NW/e^{U_0}$ and not the specific values of W and U_0.

B. Regulation of the Wireless Service Market

Since both the market efficiency and the operator revenue depend on α, a regulating agency can act as a *mechanism designer* and steer the outcome of the market in a more desirable equilibrium according to his objective. This can be achieved by determining directly or indirectly (e.g. through pricing) the amount of spectrum W each operator has at his disposal, or by intervening in the market and setting the value U_0 as the example with the municipal WiFi Internet provider. This process is depicted in Figure 2.

1) Regulating to Increase Market Efficiency: Assume that initially it is $\alpha \in A_1 = (0, e/I)$. Hence, a portion of users is not served by anyone of the I operators, $x_s^* < 1$ and all the users receive utility equal to U_0. The regulator can improve the market efficiency and increase both x_s^* and U_{agg} by increasing the value of α. This can be achieved either by increasing W or decreasing U_0. Let us assume that the regulator selects the first method. For example, he can change the price of W and allow the operators to acquire more spectrum. If W is increased until $\alpha = e/I$, then the market stationary point $\mathbf{x}^*$ switches to X_B. In this case, all the users are served by the market $x_s^* = 1$ but still they receive only marginal utility, $U_{agg} = NU_0$. If the regulator provides even more spectrum W to the operators and ensures that $\alpha > e^{I/(I-1)/I}$, then

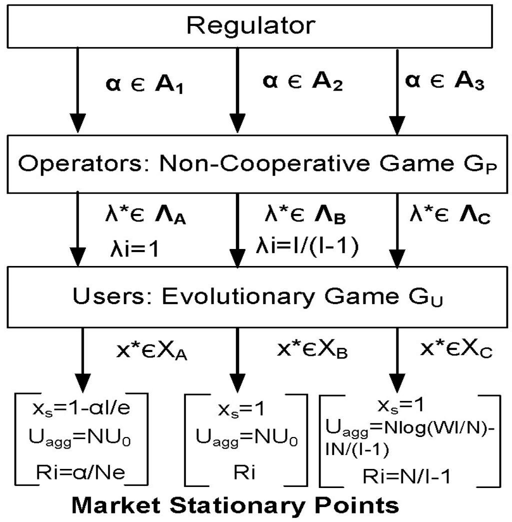

Fig. 2. The regulator selects parameter α, the operators compete and select the respective optimal prices λ_i^*, and then, the users are divided among the operators.

it is $x_s^* = 1$ and moreover the users perceive higher utility since U_{agg} increases proportional to $\log(W)$, eq. (28).

Apparently, the improvement in market efficiency comes at the cost of the additional spectrum the regulator must provide to operators. This introduces some additional cost (or *opportunity cost*) that depends on the specific setting and must be taken into account. On the other hand, the regulator may prefer to directly intervene in the market through P_0 and change U_0. If U_0 decreases, users move to the market operators $\mathcal{I}$ and x_s^* increases and attains its maximum value $x_s^* = 1$ when $\alpha = e/I$. This way, the cost of the regulator may decrease (since P_0 serves less users) but the aggregate utility, $U_{agg} = NU_0$, is reduced. Namely, U_{agg} decreases linearly with U_0 until $\alpha = e^{I/(I-1)/I}$ but remains constant for larger values of U_0, eq. (28). Again, the decision of the regulator depends on his cost and on the exact efficiency he wants to ensure.

2) Regulating for Revenue: As illustrated in Table II, the revenue of the operators increases proportionally to α for $\alpha \in A_1$, and proportionally to $\log \alpha$ for $\alpha \in A_2$, while it remains constant when $\alpha \in A_3$. Notice that the revenue, unlike the market efficiency, depends on the value of α and not on the specific combination of W and U_0. These results are presented in Figure 6 for a market with $I = 3$ operators and $N = 1000$ users. In the upper plot, it is $U_0 = 0.1$

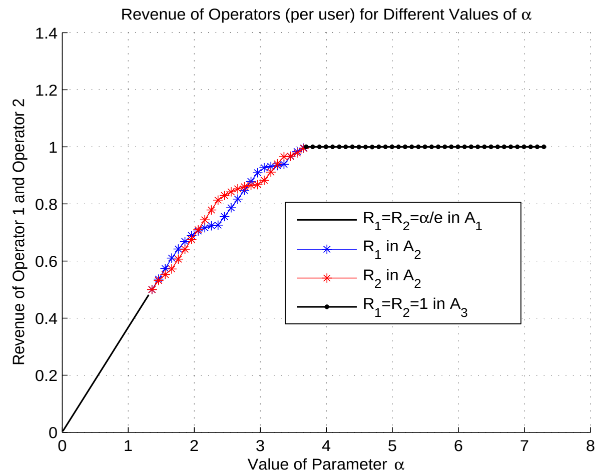

Fig. 3. The outcome of the operator competition ($\mathcal{G}_{\mathcal{P}}$ equilibrium) for different values of parameter α, i.e. in different intervals.

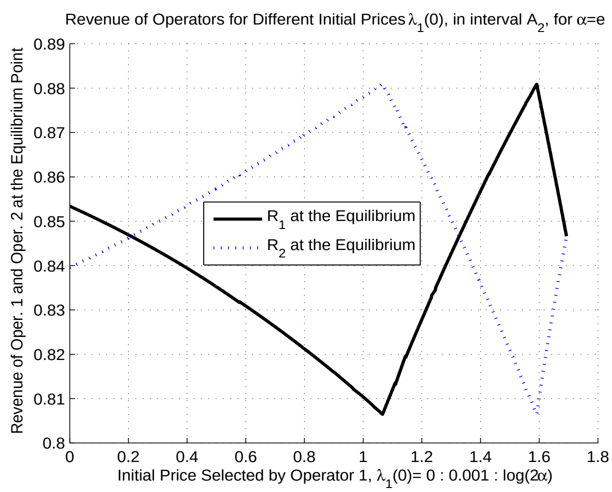

Fig. 4. The outcome of the competition of two operators, where $\alpha = e \in A_2$ and $N = 1000$. Operator 1 is assumed to set his price ($\lambda_1(0)$) first. Operator revenues at the equilibrium, R_1^* and R_2^*, depend on $\lambda_1(0)$.

and the regulator increases the value of α by increasing W. The aggregate utility is constant and equal to $U_{agg} = NU_0 = 100$ for $\alpha < e^{3/(3-1)}/3 \approx 1.5$ while it increases proportionally to $\log(W)$ for $\alpha > 1.5$. Obviously, increasing the spectrum of operators improves both their revenue and the efficiency of the market.

In the lower plot, the spectrum at the disposal of each operator is constant, $W = 5000$, and the regulator increases the value of α by decreasing U_0. In this case, the total revenue increases but at the expense of market efficiency. When $\alpha \in A_1 \cup A_2 = (0, e^{1.5}/3]$, the aggregate utility

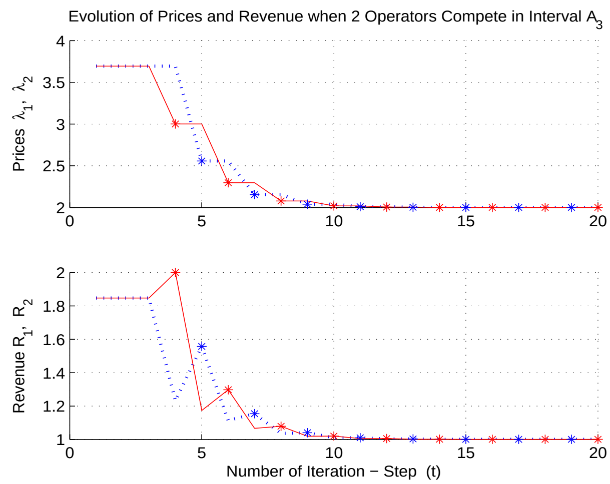

Fig. 5. Evolution of operator competition for $\alpha = e^3 \in A_3$.

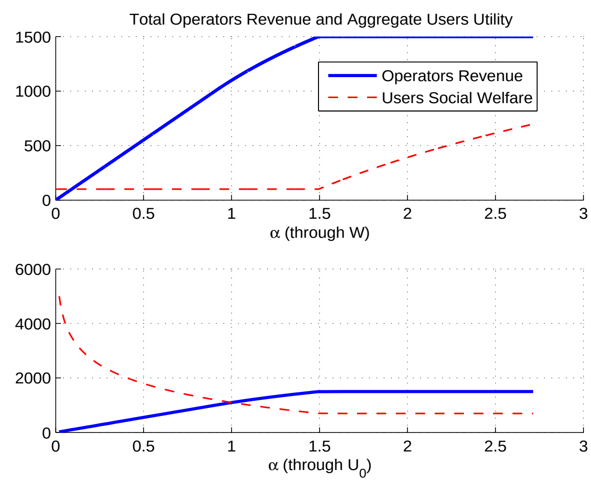

Fig. 6. Total revenue of operators and aggregate utility of the users for different values of parameter α. In the upper plot, the value of α changes through W while in the lower plot it changes through the tuning of U_0.

U_{agg} is reduced as U_0 decreases but for $\alpha > e^{1.5}/3$ it remains constant. It is interesting to notice that for very small values of α, U_{agg} is large. However, only a small portion of users are served by the market, x_s^* and the rest of them select P_0. This incurs additional cost to the regulator who has to provide the necessary resources (spectrum) in order to satisfy the demand of P_0.

VI. Conclusions

In this paper, we studied the operators price competition in a wireless services market where users have a certain reservation utility U_0. We modeled the users interaction as an evolutionary game and the competition of the operators as a non cooperative game which is proved to be a potential game. The two games are interrelated and both of them depend on the reservation utility U_0 and the amount of spectrum W each operator has at his disposal. Accordingly, we considered a regulating agency and discussed how he can intervene and change the outcome of the market by tuning either U_0 or W. Different regulation methods yield differen market outcomes and yield different cost for the regulator. There exist many directions for future work. For example, it is interesting to consider the possibility that operators cooperate and make peering agreements for jointly serving the users or collude and set their prices without competing. More interestingly, we can analyze the price competition game not only for the stationary point of the market but even before the users dynamics reach a stable point. This will allows us to understand how the operators should select their optimal pricing policy in a real time fashion.

Appendix

A. Derivation of Evolutionary Dynamics

Here, we derive the new differential equations that describe the evolution of the market of the users under the new introduced revision protocol. Recall that, the latter is given by the following equations:

$$p_{ij}(t) = x_j(t)[U_j(t) - U_i(t)]_+, \forall i, j \in \mathcal{I} \tag{30}$$

$$p_{i0}(t) = \gamma[U_0 - U_i(t)]_+, \forall i \in \mathcal{I} \tag{31}$$

$$p_{0i}(t) = x_i(t)[U_i(t) - U_0]_+, \forall i \in \mathcal{I} \tag{32}$$

where $p_{ij}(t)$ is the probability that a user which is associated with the i^{th} operator will move to operator j in time slot t, $p_{i0}(t)$ is the probability of selecting the neutral operator and $p_{0i}(t)$ the probability of returning from the neural operator back to the i^{th} operator. The constant value $\gamma \in R$ represents the frequency of the direct selection.

It is known that for imitation-based revision protocols, the dynamics of the system can be described with the well-known replicator dynamics [5]. The hybrid revision protocol defined with equations (30), (31) and (32) is in part imitation-based ($p_{ij}(t)$ and $p_{0i}(t)$) and in part a probabilistic direct selection of the neutral operator ($p_{i0}(t)$). Therefore, the respective evolutionary dynamics of the system cannot be described by the replicator dynamics equations which correspond to the pure imitation scenario. We have to stress here that the hybrid protocol we introduce here, differs from the hybrid protocol in [5] in that the users, in our model, select directly only the neutral operator and not the other I operators.

The number of users x_i who are associated with operator i changes from time t to the time $t + \delta t$, according to the following equation:

$$x_i(t+\delta t) = x_i(t) - x_i(t)\delta t \sum_{j \neq 0} x_j(t)(U_j - U_i)_+ - x_i(t)\delta t \gamma (U_0 - U_i)_+ + \sum_{j=0}^{I} \delta t x_j(t) x_i(t)(U_i - U_j)_+ \quad (33)$$

equivalently,

$$\frac{\mathrm{dx_i(t)}}{\mathrm{dt}} = x_i(t)[\sum_{j \neq 0} x_j(t) U_i(t) - \sum_{j \neq 0} x_j(t) U_j(t) - \gamma(U_0 - U_i)_+ + x_0(t)(U_i - U_0)_+] \quad (34)$$

or, if we omit the time index and rewrite the equation:

$$\frac{\mathrm{dx_i(t)}}{\mathrm{dt}} = x_i[U_i - U_{avg} - x_0(U_i - U_0) - \gamma(U_0 - U_i)_+ + x_0(U_i - U_0)_+] \quad (35)$$

which can be analyzed in:

$$\frac{\mathrm{dx_i(t)}}{\mathrm{dt}} = x_i(U_i - U_{avg}), if i \in \mathcal{I}^+ \quad (36)$$

$$\frac{\mathrm{dx_j(t)}}{\mathrm{dt}} = x_j[U_j - U_{avg} - (\gamma - x_0)(U_0 - U_j)], if j \in \mathcal{I}^- \quad (37)$$

where $\mathcal{I}^+$ is the set of operators offering utility $U_i(t) \geq U_0$, and $\mathcal{I}^-$ is the set of operators offering utility $U_j(t) < U_0$.

The dynamics of the population x_0 can be derived in a similar way:

$$x_0(t+\delta t) = x_0(t) - x_0(t)\delta t \sum_{i\neq 0} x_i(t)(U_i - U_0)_+ + \sum_{i\neq 0} x_i(t)\delta t \gamma (U_0 - U_i)_+ \tag{38}$$

which can be written as:

$$\frac{\mathrm{dx_0(t)}}{\mathrm{dt}} = (x_0 \sum_{i\in I^+} x_i(U_0 - U_i) + \gamma \sum_{j\in\mathcal{I}^-} x_j(U_0 - U_j)) \tag{39}$$

Hence, equations (36), (37) and (39) describe the evolutionary dynamics of game $\mathcal{G}_{\mathcal{U}}$.

B. Analysis of Stationary Points

Despite the system's different evolution, still it has the same stationary points as a system which is based solely on imitation revision strategies. In detail, the market state vector at a fixed point, $\mathbf{x}^* = (x_i^*, x_j^*, x_0^*$: $i \in \mathcal{I}^+$, $j \in \mathcal{I}^-)$, can be found by the following set of equations:

$$\frac{\mathrm{dx_i(t)}}{\mathrm{dt}} = \frac{\mathrm{dx_j(t)}}{\mathrm{dt}} = \frac{\mathrm{dx_0(t)}}{\mathrm{dt}} = 0 \tag{40}$$

Lemma A.1. *The stationary points of the evolutionary dynamics defined in equations (36), (37) and (39) are identical to the stationary points of the ordinary replicator dynamics [5] given by:*

$$\dot{x}_i(t) = 0 \Rightarrow x_i(t)[U_i(t) - U_{avg}(t)] = 0,\ i \in \mathcal{I} \tag{41}$$

and

$$\dot{x}_0(t) = 0 \Rightarrow x_0(t)[U_0 - U_{avg}(t)] = 0 \tag{42}$$

Proof: First we prove that, in any stationary point, x_j^* should be equal to zero. Since $U_{avg} \geq U_0 > U_j$, $x_j^* > 0$ implies that there should be at least one operator i with $U_i > U_{avg}$ and $x_i^* > 0$. Therefore $(U_i - U_{avg})$ cannot be equal to zero $\forall i \in \mathcal{I}^+$, and $\dot{x}_i$ cannot be equal to zero at least for one operator. Therefore (40) cannot be satisfied, if $x_j^* \neq 0$.

When $x_j = 0$, the evolutionary dynamics given by eq. (36), (37) and (39) reduce to ordinary replicator dynamics:

$$\dot{x}_i(t) = x_i(t)[U_i(t) - U_{avg}(t)] \;\forall i \in \mathcal{I}, \quad \dot{x}_0(t) = x_0(t)[U_0 - U_{avg}(t)] \tag{43}$$

Hence stationary points are also identical to the stationary points of ordinary replicator dynamics. ■

Due to the above lemma, the stationary points for the users population associated with each operator $i \in \mathcal{I}$ should satisfy one of the following conditions: (i) $x_i^* = 0$, or (ii) $x_i^* > 0$ and $U_i^* = U_{avg}$. Similarly, for the neutral operator P_0, eq. (42), it must hold: (i) $x_0^* = 0$ and $U_0^* > U_{avg}$, (ii) $x_0^* > 0$ and $U_0^* = U_{avg}$ or (iii) $x_0^* = 0$ and $U_0^* = U_{avg}$. The case $x_i^* = 0$ implies zero revenue for the i^{th} operator and hence case (i) does not constitute a valid choice. Therefore, there exist in total 3 possible combinations (cases) that will satisfy the stationarity properties given by eq. (41) and (42):

- **Case A**: $x_i^*,\, x_0^* > 0$ and $U_i^* = U_0$, $i \in \mathcal{I}$.
- **Case B**: $x_i^*,\, x_j^* > 0$, $x_0^* = 0$ and $U_i^* = U_j^*$, with $U_i^*,\, U_j^* > U_0, \forall\, i,\, j \in \mathcal{I}$.
- **Case C**: $x_i^*,\, x_j^* > 0$, $x_0^* = 0$ and $U_i^* = U_j^* = U_0$, $\forall\, i,\, j \in \mathcal{I}$.

Let us now find the exact value of the market state vector at the equilibrium (stationary point) $\mathbf{x}^*$ for each case. First, let us define for every operator $i \in \mathcal{I}$ the scalar parameter $\alpha_i = W_i/Ne^{U_0}$ and the respective vector $\alpha = (\alpha_i \,:\, i \in \mathcal{I})$.

We can find the stationary points for **Case A** by using the equation $U_i(W_i, x_i^*, \lambda_i) = U_0$ and imposing the constraint $x_0^* > 0$:

$$U_i(W_i, x_i^*, \lambda_i) = \log\frac{W_i}{Nx_i^*} - \lambda_i = U_0 \Rightarrow x_i^* = \frac{W_i}{Ne^{\lambda_i + U_0}} = \frac{\alpha_i}{e^{\lambda_i}} \tag{44}$$

and

$$x_0^* > 0 \Rightarrow 1 - \sum_{i=1}^{I}\frac{\alpha_i}{e^{\lambda_i}} > 0 \Rightarrow \sum_{i=1}^{I}\frac{\alpha_i}{e^{\lambda_i}} < 1 \tag{45}$$

Apparently, the state vector $\mathbf{x}^*$ depends on the operators' price vector λ. Therefore, we define the set of all possible **Case A** stationary points, X_A, as follows:

$$X_A = \left\{ x_i^* = \frac{\alpha_i}{e^{\lambda_i}}, \forall i \in \mathcal{I}, x_0^* = 1 - \sum_{i=1}^{I} \frac{\alpha_i}{e^{\lambda_i}} \,:\, \lambda \in \Lambda_A \right\} \tag{46}$$

where Λ_A is the set of prices for which a stationary point in X_A is reachable, i.e. $x_0^* > 0$:

$$\Lambda_A = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_I) \,:\, \sum_{i=1}^{I} \frac{\alpha_i}{e^{\lambda_i}} < 1 \right\} \tag{47}$$

Similarly, for **Case B**, we calculate the stationary points by using the set of equations $U_i(W_i, x_i^*, \lambda_i) = U_j(W_j, x_j^*, \lambda_j)$, $\forall\, i, j \in \mathcal{I}$, which yields:

$$\log\left(\frac{W_1}{Nx_1^*}\right) - \lambda_1 = \log\left(\frac{W_2}{Nx_2^*}\right) - \lambda_2 = \ldots = \log\left(\frac{W_i}{Nx_i^*}\right) - \lambda_i \tag{48}$$

or, equivalently:

$$x_j^* = x_i^* \frac{e^{\lambda_i}\alpha_j}{e^{\lambda_j}\alpha_i} \quad \forall i, j \in \mathcal{I} \tag{49}$$

Moreover since $x_0^* = 0$ for **Case B**, the following holds:

$$\sum_{i \in \mathcal{I}} x_i^* = 1 \tag{50}$$

Using (49) and (50),

$$x_i^* = \frac{\alpha_i}{e^{\lambda_i} \sum_{j \in \mathcal{I}} \frac{\alpha_j}{e^{\lambda_j}}} \tag{51}$$

Additionally, $U_i > U_0$ implies that:

$$\log\left(\frac{W_i}{Nx_i^*}\right) - \lambda_i > U_0 \Rightarrow x_i^* < \frac{\alpha_i}{e^{\lambda_i}} \tag{52}$$

Using (50) and (52),

$$\sum_{i=1}^{I} x_i^* < \sum_{i=1}^{I} \frac{\alpha_i}{e^{\lambda_i}} \Rightarrow \sum_{i=1}^{I} \frac{\alpha_i}{e^{\lambda_i}} > 1 \tag{53}$$

Therefore, according to (51) and (53), we define the set of all possible **Case B** stationary points,

X_B, as follows:

$$X_B = \left\{ x_i^* = \frac{\alpha_i}{e^{\lambda_i} \sum_{j=1}^{I} \alpha_j / e^{\lambda_j}}, \forall i \in \mathcal{I}, x_0^* = 0 : \lambda \in \Lambda_B \right\} \tag{54}$$

where Λ_B is the set of prices for which a stationary point in X_B is feasible, $U_i^* > U_0$:

$$\Lambda_B = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_I) : \sum_{i=1}^{I} \frac{\alpha_i}{e^{\lambda_i}} > 1 \right\} \tag{55}$$

Finally, the stationary points for the **Case C** solution must satisfy the following:

$$U_i = U_0, \quad x_0^* = 0 \tag{56}$$

which yields:

$$x_i^* = \frac{\alpha_i}{e^{\lambda_i}}, \quad \sum_{i=1}^{I} \frac{\alpha_i}{e^{\lambda_i}} = 1 \tag{57}$$

Therefore, we define the set of all possible **Case C** stationary points, X_C, as follows:

$$X_C = \left\{ x_i^* = \frac{\alpha_i}{e^{\lambda_i}}, \forall\, i \in \mathcal{I}, x_0^* = 0 : \lambda \in \Lambda_C \right\} \tag{58}$$

with

$$\Lambda_C = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_I) : \sum_{i=1}^{I} \frac{\alpha_i}{e^{\lambda_i}} = 1 \right\} \tag{59}$$

First, we show that the revenue of each operator $i \in \mathcal{I}$ is a continuous and a quasi-concave function. Secondly, we analyze best response pricing in game $\mathcal{G}_{\mathcal{P}}$. Then, we derive the Nash equilibriums (NEs) of the game using iterated strict dominance. Finally, we prove convergence to these equilibriums by showing that the $\mathcal{G}_{\mathcal{P}}$ is a potential game.

C. Properties of the Revenue Function

The revenue function of each operator i is given by the following equation:

$$R_i(\lambda_i, \lambda_{-i}) = \begin{cases} \dfrac{\alpha_i \lambda_i N}{e^{\lambda_i} \sum_{j=1}^{I} (\alpha_j / e^{\lambda_j})} & \text{if } \lambda_i < l_0, \\ \dfrac{\alpha_i \lambda_i N}{e^{\lambda_i}} & \text{if } \lambda_i \geq l_0. \end{cases} \tag{60}$$

where $l_0 = \log(\alpha_i/(1 - \sum_{j\neq i} \alpha_j/e^{\lambda_j}))$.

Each component (for each case) is a positive function which is also *log-concave*. This means that it is a quasiconcave function and hence uniqueness of optimal solution is ensured. Namely, it is:

$$R_i^A(\lambda_i) = \log(\frac{\alpha_i\lambda_i N}{e^{\lambda_i}}) = \log(\alpha_i\lambda_i N) - \lambda_i \tag{61}$$

and

$$R_i^A(\lambda_i)^{(1)} = \frac{1}{\lambda_i} - 1 \Rightarrow R_i^A(\lambda_i)^{(2)} = \frac{-1}{\lambda_i^2} < 0 \tag{62}$$

So, $f_A(\cdot)$ which is the log-function of R_i^A, is concave which means that the later is log-concave and since it is $R_i^A(\lambda_i) > 0$, it is also quasi-concave. Similarly, for the other component of the revenue function:

$$R_i^B(\lambda_i, \lambda_{-i}) = \log(\frac{\alpha_i\lambda_i N}{\alpha_i + \beta e^{\lambda_i}}) = \log(\alpha_i\lambda_i N) - \log(\alpha_i + \beta e^{\lambda_i}) \tag{63}$$

where $\beta = \sum_{j\neq i} \frac{\alpha_j}{e^{\lambda_i}}$. The second derivative is:

$$R_i^B(\lambda_i, \lambda_{-i})^{(2)} = \frac{-1}{\lambda_i^2} - \frac{\alpha_i\beta e^{\lambda_i}}{(\alpha_i + \beta e^{\lambda_i})^2} < 0 \tag{64}$$

Hence, $R_i^B(\cdot)$ is also quasiconcave. Finally, it is easy to see that the function is continuous:

$$R_i^A(l_0, \lambda_{-i}) = R_i^B(l_0, \lambda_{-i}) = N(1-\beta)\log(\frac{\alpha_i}{1-\beta}) \tag{65}$$

D. Best Response Pricing in $\mathcal{G}_{\mathcal{P}}$ Game

Each operator i finds his best response price λ_i^* for each set of the $I-1$ prices of the rest operators by solving the following optimization problems. For the case the price vector belongs to the set Λ_A, $\lambda \in \Lambda_A$, ($\mathbf{P}_i^A$):

$$\max_{\lambda_i \geq 0} \frac{\alpha_i\lambda_i N}{e^{\lambda_i}} \tag{66}$$

s.t.

$$\sum_{j=1}^{I} \frac{\alpha_j}{e^{\lambda_j}} < 1 \tag{67}$$

where the constraint can be rewritten as follows:

$$\lambda_i > \log\left(\frac{\alpha_i}{1 - \sum_{j \neq i} \alpha_j / e^{\lambda_j}}\right) \tag{68}$$

This problem is quasi-concave and hence it has a unique optimal solution, [16] which we denote λ_i^A and it is:

$$\lambda_i^A = 1, \text{ or } \lambda_i^A = \log\left(\frac{\alpha_i}{1 - \sum_{j \neq i} \alpha_j / e^{\lambda_j}}\right) + \epsilon \tag{69}$$

with $\epsilon > 0$. The value $\lambda_i^A = 1$ is the optimal solution of the respective unconstrained problem, with respective optimal revenue $R_i^A = \alpha N / e$, and it is feasible if $\lambda = (1, \lambda_{-i}) \in \Lambda_A$. Otherwise, since $R_i^A(\cdot)$ is a decreasing function of λ_i, operator i can only select the minimum price λ_i^A such that $(\lambda_i^A, \lambda_{-i}) \in \Lambda_A$.

Similarly, for the case the price vector belongs to the set Λ_B, $\lambda \in \Lambda_B$, it is ($\mathbf{P}_i^B$):

$$\max_{\lambda_i \geq 0} \frac{\lambda_i \alpha_i N}{e^{\lambda_i} \sum_{j \in \mathcal{I}} \frac{\alpha_j}{e^{\lambda_j}}} \tag{70}$$

s.t.

$$\sum_{j \in \mathcal{I}} \frac{\alpha_j}{e^{\lambda_j}} > 1 \ \Rightarrow \ \lambda_i < \log\left(\frac{\alpha_i}{1 - \sum_{j \neq i} \alpha_j / e^{\lambda_j}}\right) \tag{71}$$

This is also a concave problem which has unique solution and can be either the optimal solution of the respective unconstrained problem, λ_i^* if $(\lambda_i^*, \lambda_{-i}) \in \Lambda_B$, or the maximum price for which the price vector belongs to Λ_B ($R_i^B(\cdot)$ increases with λ_i:

$$\lambda_i^B = \mu_i^*, \text{ or } \lambda_i^B = \log\left(\frac{\alpha_i}{1 - \sum_{j \neq i} \alpha_j / e^{\lambda_j}}\right) - \epsilon \tag{72}$$

Finally, for the special case that $\lambda \in \Lambda_C$, the price of each operator i is directly determined by the prices that the other operators have selected. Namely:

$$\lambda_i^C = \log(\frac{\alpha_i}{1 - \sum_{j \neq i} \alpha_j / e^{\lambda_j}}) \tag{73}$$

Whether each operator i will agree and adopt this price or not, depends on the respective accrued revenue, $R_i^C(\lambda_i^C, \lambda_{-i})$.

In the sequel, we examine and analyze jointly the solutions of the above optimization problems and derive the exact best response of the i^{th} operator for each vector of the $I-1$ prices λ_{-i}.

Lemma A.2. *For an operator i, if $(1, \lambda_{-i}) \notin \Lambda_A$, then there is no best response price λ_i^*, such that $(\lambda_i^*, \lambda_{-i}) \in \Lambda_A$.*

Proof: Given that the price vector $\lambda \in \mathbf{\Lambda_A}$, best response price is, [**?**]:

$$\lambda_i^A = \begin{cases} 1 & \text{if } (1, \lambda_{-i}) \in \Lambda_A, \\ l_0 + \epsilon & \text{if } (1, \lambda_{-i}) \notin \Lambda_A. \end{cases} \tag{74}$$

and when $\lambda \in \mathbf{\Lambda_C}$, price of operator i is $\lambda_i^C = l_0$.

It is known that if $(1, \lambda_{-i}) \notin \Lambda_A$, then $l_0 + \epsilon > 1$ since otherwise price vector $(l_0 + \epsilon, \lambda_{-i})$ cannot be in Λ_A. Therefore, R_i^A is a decreasing function at the point $\lambda_i = l_0 + \epsilon$ due to quasi-concavity property. So, if $(1, \lambda_{-i}) \notin \Lambda_A$, then $R_i^C(l_0) = R_i^A(l_0) > R_i^A(l_0 + \epsilon)$ which means that λ_i^C always gives better response than λ_i^A. ∎

Lemma A.3. *Suppose that μ_i^* is the optimal solution of the unconstraint R_i^B. For an operator i, if $(\mu_i^*, \lambda_{-i}) \notin \Lambda_B$, then there is no best response price λ_i^*, such that $(\lambda_i^*, \lambda_{-i}) \in \Lambda_B$.*

Proof: Given that the price vector $\lambda \in \mathbf{\Lambda_B}$, best response price is:

$$\lambda_i^B = \begin{cases} \mu_i^* & \text{if } (\mu_i^*, \lambda_{-i}) \in \Lambda_B, \\ l_0 - \epsilon & \text{if } (\mu_i^*, \lambda_{-i}) \notin \Lambda_B. \end{cases} \tag{75}$$

and recall that $\lambda_i^C = l_0$.

It is known that if $(\mu_i^*, \lambda_{-i}) \notin \Lambda_B$, then $l_0 - \epsilon < \mu_i^*$ since otherwise price vector $(l_0 - \epsilon, \lambda_{-i})$ cannot be in Λ_B. Therefore, R_i^B is an increasing function at the point $\lambda_i = l_0 - \epsilon$ due to quasi-concavity property. So, if $(\mu_i^*, \lambda_{-i}) \notin \Lambda_B$, then $R_i^C(l_0) = R_i^B(l_0) > R_i^B(l_0 - \epsilon)$ which means that λ_i^C always gives better response than λ_i^B. ■

Theorem A.4. *The best response price of an operator i is:*

$$\lambda_i^* = \begin{cases} 1 & \textit{if } (1, \lambda_{-i}) \in \Lambda_A, \\ \mu_i^* & \textit{if } (\mu_i^*, \lambda_{-i}) \in \Lambda_B, \\ \lambda_i^C = l_0 & \textit{otherwise}. \end{cases} \tag{76}$$

Proof: First we prove that $(1, \lambda_{-i}) \in \Lambda_A$ and $(\mu_i^*, \lambda_{-i}) \in \Lambda_B$ cannot be true at the same time. Since μ_i^* is the optimal solution of unconstraint R_i^B:

$$\frac{\mathrm{dR_i^B(\lambda_i)}}{\mathrm{d\lambda_i}} = 0 \Rightarrow e^{\mu_i^*}(\mu_i^* - 1) = \frac{\alpha_i}{\sum_{j \neq i} \frac{\alpha_j}{e^{\lambda_j}}} \tag{77}$$

It is obvious that equation (77) can only hold when $\mu_i^* > 1$. Note that if $(\mu_i^*, \lambda_{-i}) \in \Lambda_B$, $(l, \lambda_{-i}) \in \Lambda_B$ is also valid for any price $l < \mu_i^*$. Hence, $(1, \lambda_{-i}) \in \Lambda_B$ should also be true. With a similar reasoning, when $(1, \lambda_{-i}) \in \Lambda_A$, $(l, \lambda_{-i}) \in \Lambda_A$ is valid for any price $l > 1$, hence $(\mu_i^*, \lambda_{-i}) \in \Lambda_A$.

It is also evident that if $(1, \lambda_{-i}) \in \Lambda_A$, λ_i^C cannot be a best response, because $R_i^A(1) > R_i^A(\lambda_i^C) = R_i^C(\lambda_i^C)$. Similarly, $(\mu_i^*, \lambda_{-i}) \in \Lambda_B$, λ_i^C cannot be a best response. Hence, first two cases of the theorem is proven.

Finally, from Lemma A.2 and Lemma A.3, we can say that λ_i^C dominates all other prices if $(1, \lambda_{-i}) \notin \Lambda_A$ and $(\mu_i^*, \lambda_{-i}) \notin \Lambda_B$. Hence the third case of the theorem is also proven. ■

E. Existence and Convergence Analysis of Nash Equilibriums

In the previous section, we derived the best response strategy for each player of the game $\mathcal{G}_{\mathcal{P}}$. The next important steps are **(i)** to explore the existence of Nash Equilibriums (NE) for

$\mathcal{G}_{\mathcal{P}}$, and **(ii)** to study if the convergence to them is guaranteed? In [17], it is proven that if the game can be modeled as a potential game, not only the existence of pure NEs are ensured, but also convergence to them is guaranteed under any finite improvement path. In other words, a potential game always converges to pure NE when the players adjust their strategies based on accumulated observations as game unfolds. In this section, after giving definition of ordinal potential games, we prove that the game $\mathcal{G}_{\mathcal{P}}$ is an ordinal potential game.

Definition A.5. *A game* $(\mathcal{I}, \lambda, \{R_i\})$ *is an* ***ordinal potential game****, if there is a potential function* $\mathcal{P} : [0, \lambda_{max}] \to \mathbb{R}$ *such that the following condition holds:*

$$\mathrm{sgn}(\mathcal{P}(\lambda_i, \lambda_{-i}) - \mathcal{P}(\lambda'_i, \lambda_{-i})) = \mathrm{sgn}(R_i(\lambda_i, \lambda_{-i}) - R_i(\lambda'_i, \lambda_{-i})) \text{ for any } i \in \mathcal{I} \text{ and } \lambda_i, \lambda'_i \in [0, \lambda_{max}] \tag{78}$$

where sgn(·) *is the sign function.*

Lemma A.6. *The game* $\mathcal{G}_{\mathcal{P}}$ *is an ordinal potential game.*

Proof: We define the potential function as:

$$\mathcal{P}(\lambda) = \begin{cases} \sum_{j=1}^{I} (\log \lambda_j - \lambda_j) & \text{if } \sum_{j=1}^{I} \alpha_j / e^{\lambda_j} \leq 1, \\ \sum_{j=1}^{I} (\log \lambda_j - \lambda_j) - \log \left(\sum_{j=1}^{I} \alpha_j / e^{\lambda_j} \right) & \text{if } \sum_{j=1}^{I} \alpha_i / e^{\lambda_j} > 1. \end{cases} \tag{79}$$

Therefore,

$$\mathcal{P}(\lambda_i, \lambda_{-i}) - \mathcal{P}(\lambda'_i, \lambda_{-i}) = \begin{cases} \log (\lambda_i / e^{\lambda_i}) - \log (\lambda'_i / e^{\lambda'_i}) & \text{if } \lambda_i, \lambda'_i \geq l_0 \\ \log \dfrac{\lambda_i}{e^{\lambda_i} (\frac{\alpha_i}{e^{\lambda_i}} + \sum_{j \neq i} \frac{\alpha_j}{e^{\lambda_j}})} - \log \dfrac{\lambda'_i}{e^{\lambda'_i} (\frac{\alpha_i}{e^{\lambda'_i}} + \sum_{j \neq i} \frac{\alpha_j}{e^{\lambda_j}})} & \text{if } \lambda_i, \lambda'_i < l_0 \\ \log \dfrac{\lambda_i}{e^{\lambda_i} (\frac{\alpha_i}{e^{\lambda_i}} + \sum_{j \neq i} \frac{\alpha_j}{e^{\lambda_j}})} - \log \dfrac{\lambda'_i}{e^{\lambda'_i}} & \text{if } \lambda_i < l_0,\ \lambda'_i \geq l_0 \\ \log \dfrac{\lambda_i}{e^{\lambda_i}} - \log \dfrac{\lambda'_i}{e^{\lambda'_i} (\frac{\alpha_i}{e^{\lambda'_i}} + \sum_{j \neq i} \frac{\alpha_j}{e^{\lambda_j}})} & \text{if } \lambda_i \geq l_0,\ \lambda'_i < l_0 \end{cases} \tag{80}$$

where $l_0 = \log(\alpha_i / (1 - \sum_{j \neq i} (\alpha_j / e^{\lambda_j})))$.

Moreover, using (81),

$$\log(R_i(\lambda_i,\lambda_{-i})) = \begin{cases} \log\left(\lambda_i/e^{\lambda_i}\right) + \log(\alpha_i N) & \text{if } \lambda_i \geq l_0, \\ \log \dfrac{\lambda_i}{e^{\lambda_i}(\frac{\alpha_i}{e^{\lambda_i}} + \sum_{j\neq i}\frac{\alpha_j}{e^{\lambda_j}})} + \log(\alpha_i N) & \text{if } \lambda_i < l_0. \end{cases} \tag{81}$$

Now, it is straightforward to show that $\mathcal{P}(\lambda_i,\lambda_{-i}) - \mathcal{P}(\lambda_i',\lambda_{-i}) = \log(R_i(\lambda_i,\lambda_{-i})) - \log(R_i(\lambda_i',\lambda_{-i}))$ for any operator $i \in \mathcal{I}$ and for any $\lambda_i, \lambda_i' \in [0, \lambda_{max}]$. Since $\log(R_i(\lambda_i,\lambda_{-i})) - \log(R_i(\lambda_i',\lambda_{-i}))$ has always same sign as $R_i(\lambda_i,\lambda_{-i}) - R_i(\lambda_i',\lambda_{-i})$, condition given in (78) is satisfied, and game $\mathcal{G}_{\mathcal{P}}$ is an ordinal potential game. ■

F. Detailed Analysis of the Nash Equilibriums

In the previous section, we proved the existence of pure NE and convergence to them. In this section, we extend our analysis further in order to find these NEs. For the sake of simplicity, we consider the case where all the operators have same amount of available spectrum $W_i = W, \forall i$ and hence $\alpha_i = \alpha$.

Before starting our analysis, we rewrite constraint of set Λ_A given in eq. (47) as follows:

$$\alpha \leq \frac{1}{\sum_{j\in\mathcal{I}} \frac{1}{e^{\lambda_j}}} = \frac{H(\{e^{\lambda_j} | j \in \mathcal{I}\})}{I} \tag{82}$$

where $H(\cdot)$ is the harmonic mean function of the variables $(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_I}) = (\{e^{\lambda_j} | j \in \mathcal{I}\})$. Therefore, if $\lambda \in \Lambda_A$, it is:

$$H(\{e^{\lambda_j} | j \in \mathcal{I}\}) \geq \alpha I \tag{83}$$

Similarly, according to (55), if $\lambda \in \Lambda_B$ then:

$$H(\{e^{\lambda_j} | j \in \mathcal{I}\}) \leq \alpha I \tag{84}$$

and finally, if $\lambda \in \Lambda_C$:

$$H(\{e^{\lambda_j} | j \in \mathcal{I}\}) = \alpha I \tag{85}$$

Next, we define a new variable, h as the natural logarithm of the harmonic mean:

$$h = \log(H(\{e^{\lambda_j}|j \in \mathcal{I}\})) \tag{86}$$

Note that, since e^h is the harmonic mean of $\{e^{\lambda_j}|j \in \mathcal{I}\}$, we can say that one of the following should hold:

1) Every operator $i \in \mathcal{I}$ adopts the same price $\lambda_i = h$.
2) If one operator $j \in \mathcal{I}$ selects a price $\lambda_j < h$, then there must be at least one other operator $k \in \mathcal{I}$ who will adopt a price $\lambda_k > h$.

Additionally, we define the variable h_{-i} which is similar to h except that price of the i^{th} operator is excluded. That is:

$$h_{-i} = \log(H(\{e^{\lambda_j}|j \in \mathcal{I} \setminus i\})) \tag{87}$$

It is obvious that if $\lambda_i > h$, then $h_{-i} < h$, if $\lambda_i < h$, then $h_{-i} > h$, and if $\lambda_i = h$, then $h_{-i} = h$.

Lemma A.7. *If $\alpha \in A_1 = (0, e/I)$, there is a unique NE $\lambda^* \in \Lambda_A$, with $\lambda^* = (\lambda_i^* = 1 : i \in \mathcal{I})$*

Proof: First, we prove that the NE cannot be in Λ_B or Λ_C ($\lambda^* \notin \Lambda_B \cup \Lambda_C$) if $\alpha \in A_1 = (0, e/I)$. Notice that, when the price vector is not in Λ_A, $h \leq \log(\alpha I) < 1$ for given α values. Therefore there exists at least one operator with price less than one. Since R_i^B is an increasing function between $\lambda_i \in (0, 1)$, operators with $\lambda_i < 1$ would gain more revenue by unilaterally increasing their prices. Therefore λ^* can only be in Λ_A. According to Theorem A.4, given that the price vector is in Λ_A, optimal price for any operator i can only be $\lambda_i^A = 1$ if $(1, \lambda_{-i}) \in \Lambda_A$. Since $\lambda^* = (\lambda_i^* = 1 : i \in \mathcal{I}) \in \Lambda_A$ when $\alpha \in A_1$, it is a feasible and unique solution. ■

Lemma A.8. *μ_i^* is always between $\frac{I}{I-1}$ and h_{-i}*

Proof: We can rewrite equation (77) as follows:

$$e^{\mu_i^*}(\mu_i^* - 1) = \frac{e^{h_{-i}}}{I - 1} \tag{88}$$

where h_{-i} is defined in equation (87). Now, if $h_{-i} < \frac{I}{I-1}$, or equivalently if $h_{-i}-1 < \frac{1}{I-1}$, then λ_i^* should be greater than h_{-i} in order to satisfy (88). Moreover, if $\lambda_i^* > h_{-i}$, then λ_i^*-1 should be less than $\frac{1}{I-1}$ in order to satisfy (88). Therefore, $h_{-i} < \lambda_i^* < \frac{I}{I-1}$. Similarly, if $h_{-i} \geq \frac{I}{I-1}$, then $\frac{I}{I-1} \leq \lambda_i^* \leq h_{-i}$, which proves the lemma. ∎

Lemma A.9. *If* $\alpha \in A_3 = (e^{I/(I-1)}, \infty)$*, there is a unique NE* $\lambda^* \in \Lambda_B$*, with* $\lambda^* = (\lambda_i^* = I/(I-1) : i \in \mathcal{I})$

Proof: First we prove that there is no NE in Λ_A if $\alpha \geq e/I$ (i.e. if $\alpha \in A_2 \cup A_3$). According to Theorem A.4, optimal price for any operator i can only be $\lambda_i^A = 1$ if $(1, \lambda_{-i}) \in \Lambda_A$. Otherwise λ_i^C dominates λ_i^A. Since $\lambda^* = (\lambda_i^* = 1 : i \in \mathcal{I}) \notin \Lambda_A$ when $\alpha \in A_2 \cup A_3$, there is no NE in Λ_A.

Secondly, we prove that there is no NE in Λ_C if $\alpha \in A_3 = (e^{I/(I-1)}, \infty)$. Recall that, when the price vector is in Λ_C, $h = \log(\alpha I) > I/(I-1)$, which means that there exists at least one operator with price $\lambda_i^C > I/(I-1)$ and $\lambda_i^C \geq h$. Remember that if $\lambda_i \geq h$, then $h_{-i} \leq h$, so $\lambda_i \geq h_{-i}$. Therefore, for an operator i, λ_i^C is greater than both h_{-i} and $I/(I-1)$. According to Theorem A.4 and Lemma A.8, when $(\mu_i^*, \lambda_{-i}) \in \Lambda_B$, best response price of operator i is μ_i^* which is between h_{-i} and $I/(I-1)$. This means that for at least one operator, λ_i^C is greater than μ_i^*, which implies that $(\mu_i^*, \lambda_{-i}) \in \Lambda_B$. This operator can increase his revenue by reducing his price to μ_i^*. Therefore, there is no NE in λ_C for the given α values, and we proved that the NE can only be in Λ_B.

Finally, we prove that the only NE is $\lambda^* = (\lambda_i^* = I/(I-1) : i \in \mathcal{I})$, if $\alpha \in A_3$. According to Lemma A.8, μ_i^* is between h_{-i} and $I/(I-1)$ for all operators. h can be greater than or less than $I/(I-1)$. If $h \geq I/(I-1)$, unless all of the operators set their prices to $I/(I-1)$, there exists at least one operator i with price λ_i greater than both h_{-i} and $I/(I-1)$. Hence, λ_i is also greater than μ_i^* and this operator can increase his revenue by reducing his price to μ_i^*. Similarly, if $h < I/(I-1)$, there exists at least one operator with price λ_i less than both h_{-i} and $I/(I-1)$. This operator can increase his revenue by increasing his price. If all the operators set their prices to $\lambda_i = I/(I-1)$, the price vector is in Λ_B and none of the operators can increase his revenue by unilaterally changing his price. Therefore, the only NE is $\lambda^* = (\lambda_i^* = I/(I-1) : i \in \mathcal{I})$.

Hence the lemma is proved. ■

Lemma A.10. *If $\alpha \in A_2 = [e/I, e^{I/(I-1)}]$, a NE can only be in Λ_C.*

Proof: In the proof of the Lemma A.9, we showed that there is no NE in Λ_A, if $\alpha \in A_2$. We can also prove that there is no NE in Λ_B for the given range of α values. If the price vector is in Λ_B and $\alpha < \frac{e^{I/(I-1)}}{I}$, then $h < I/(I-1)$. Therefore, there is at least one operator with price $\lambda_i \leq h_{-i}$ and $\lambda_i < I/(I-1)$, who can increase his revenue by increasing his price. So we conclude that, if $\alpha \in A_2 = [e/I, e^{I/(I-1)}]$, there is no NE in Λ_A or Λ_B. ■

Lemma A.11. *If $\alpha \in A_2 = [e/I, e^{I/(I-1)}]$, $\lambda^* \in \Lambda_C$ with $\lambda^* = (\lambda_i^* = \log(I\alpha) : i \in \mathcal{I})$ is a NE.*

Proof: When all the operators set the same price $\lambda_i^* = \log(I\alpha)$ and $\alpha \in A_2 = [e/I, e^{I/(I-1)}]$, $\log(I\alpha)$ is between 1 and μ_i^* for all operators (this can be verified through equation (77)). Therefore, for any operator i, $(1, \lambda_{-i}) \notin \Lambda_A$ and $(\mu_i^*, \lambda_{-i}) \notin \Lambda_B$. Then, according to Theorem A.4, best response price is λ_i^C which is equal to $\log(I\alpha)$. Hence, no operator can gain more revenue by unilaterally changing his price, and $\lambda^* = (\lambda_i^* = \log(I\alpha) : i \in \mathcal{I})$ is a NE. ■

Finally we analyze the NE for boundary values of A_2, i.e. for $\alpha = e/I$ and $\alpha = e^{I/(I-1)}/I$. In the previous lemma, it is proven that $\lambda^* = (\lambda_i^* = \log(I\alpha) : i \in \mathcal{I})$ is a NE if $\alpha \in A_2$. We can also prove that it is the only NE for these boundary values. In Lemma A.10, it is proven that any NE is in Λ_C for these α values. So, when $\alpha = e/I$, $h = \log(I\alpha) = 1$, which means that unless all of the users set their prices to one, there exists some operators with $\lambda_i < 1$. These operators would gain more revenue by setting their prices to one. Hence, the only NE is $\lambda_i^* = \log(I\alpha) = 1$. Similarly, when $\alpha = e^{I/(I-1)}/I$, $h = \log(I\alpha) = I/(I-1)$. This means that unless all of the users set their prices to $I/(I-1)$, there exists some operators with λ_i greater than both h_{-i} and $M/(M-1)$. These operators would gain more revenue by reducing their prices. Hence, the only NE is $\lambda_i^* = \log(I\alpha) = I/(I-1)$.

We also show that when $\alpha \in (e/I, e^{I/(I-1)})$, there can be infinitely many NEs, all in Λ_C, via numerical simulations. For different initial price settings, the game converges to different NE.

Theorem A.12. *The game $\mathcal{G}_\mathcal{P}$ attains a pure NE which depends on the value of parameter α as follows:*

- *If $\alpha \in A_1 = (0, e/I)$, there is a unique NE $\lambda^* \in \Lambda_A$, with $\lambda^* = (\lambda_i^* = 1 : i \in \mathcal{I})$ and respective market equilibrium $\mathbf{x}^* \in X_A$.*
- *If $\alpha \in A_3 = (e^{\frac{I}{I-1}}/I, \infty)$, there is a unique NE $\lambda^* \in \Lambda_B$, with $\lambda^* = (\lambda_i^* = \frac{I}{I-1} : i \in \mathcal{I})$, which induces a respective market equilibrium $\mathbf{x}^* \in X_B$.*
- *If $\alpha \in A_2 = [e/I, e^{\frac{I}{I-1}}/I]$, there exist infinitely many NEs, $\lambda^* \in \Lambda_C$, and each one of them yields a respective market stationary point $\mathbf{x}^* \in X_C$.*

Proof: Lemma A.6 proves that the game $\mathcal{G}_\mathcal{P}$ is a finite ordinal potential game. Therefore, it always attains a pure NE. Lemma A.7 proves the case for $\alpha \in A_1 = (0, e/I)$. Lemma A.9 proves the case for $\alpha \in A_3 = (e^{\frac{I}{I-1}}/I, \infty)$. The case for $\alpha \in A_2 = [e/I, e^{\frac{I}{I-1}}/I]$ is proven in Lemma A.10 and Lemma A.11. ■

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